On Estimating Error Concentration Parameter for Circular Functional Model

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Abstract. This paper present the mathematical approach on how to find the estimation of concentration parameters for the unreplicated linear circular functional relationship model or also known as a circular functional model assuming the ratio of error concentration parameters, λ is known. We have also shown that by using the asymptotic properties of Bassel function we can find the estimate of error concentration parameter for any value of λ.

1. Introduction

The theory of classical linear regression analysis assumes that the explanatory variables are measured without errors. In practice, particularly in the social science and in the biological assay as well as in environmental science, this assumption is often violated. From the early of 19th century onwards, there has been a lot of work on the problem of parameter estimation when some of the explanatory variables contain errors of measurements. This is also known as the errors in variables or the measurement error model. As in the simple linear regression model, we consider two mathematical variables X and Y which are linearly related but observed with mutually independent normally distributed errors δ and ε, respectively. For the functional relationship model we would consider X_i as parameters, whereas for the structural relationship model, X_i are considered as independent observations from a distribution, usually a normal distribution with unknown parameters. The linear ultrastructural relationship model is the synthesis of the linear functional and structural relationship model.

In this article we consider the relationship when both variables are circular and subject to errors. What we meant by a circular variable is one which takes values on the circumference of a circle, i.e. they are angles in the range $(0, 2\pi)$ radians or $(0°, 360°)$. Some of the examples are the wind and wave direction data measured by two different methods, the anchored wave buoy and HF radar system as given by [1]. Since the variables are circular we refer the model as the *linear circular functional relationship model* or *circular functional model* and as an analogy to the linear functional relationship model, we assume the errors of circular variables X and Y are independently distributed

and follow the von Mises distribution with mean zero and concentration parameters κ and ν, respectively. For any circular random variable θ, it is said to have a von Mises distribution if its p.d.f. is given by $g(\theta; \mu_0, \kappa) = \frac{1}{2\pi I_o(\kappa)} \exp\{\kappa \cos(\theta - \mu_0)\}$, where $I_0(\kappa)$ is the modified Bassel function of the first kind and order zero [4]. The parameter μ_0 is the mean direction while the parameter κ is described as the concentration parameter. This paper will present the mathematical methodology on how to estimate the concentration parameter κ.

2. The model

Suppose x_i and y_i are observed values of the circular variables X and Y, respectively, thus $0 \le x_i, y_i < 2\pi$, for $i = 1, \cdots, n$. For any fixed X_i, we assume that the observations x_i and y_i (which are unreplicated) have been measured with errors δ_i and ε_i, respectively and thus the full model can be written as

$$x_i = X_i + \delta_i \text{ and } y_i = Y_i + \varepsilon_i, \text{ where}$$

$$Y_i = \alpha + \beta X_i \pmod{2\pi}, \text{ for } i = 1, 2, \cdots, n.$$

We also assume δ_i and ε_i are independently distributed with (potentially different) von Mises distributions, that is $\delta_i \sim VM(0, \kappa)$ and $\varepsilon_i \sim VM(0, \nu)$. Suppose we assume that the ratio of the error concentration parameters, that is $\frac{\nu}{\kappa} = \lambda$ is known. Then the log likelihood function is given by

$$\log L(\alpha, \beta, \kappa, X_1, \cdots, X_n; \lambda, x_1, \cdots, x_n, y_1, \cdots, y_n)$$

$$= -2n \log(2\pi) - n \log I_0(\kappa) - n \log I_0(\lambda \kappa)$$

$$+ \kappa \sum \cos(x_i - X_i) + \lambda \kappa \sum \cos(y_i - \alpha - \beta X_i).$$

In the next section we will show how to estimate the error concentration parameter κ when $\lambda = 1$. Further we will extend the case by using the asymptotic properties of the Bassel function to estimate κ for any value of λ.

3. Estimation of the concentration parameter κ

By setting $\frac{\partial \log L}{\partial \kappa} = 0,$ of the log likelihood function we get the equation

$$A(\kappa) + \lambda A(\lambda \kappa) = \frac{1}{n} \left\{ \sum \cos(x_i - \hat{X}_i) + \lambda \sum \cos(y_i - \hat{\alpha} - \hat{\beta}\hat{X}_i) \right\}. \tag{1}$$

The approximation given by Dobson [2], that is

$$A^{-1}(w) = \frac{9 - 8w + 3w^2}{8(1 - w)}, \tag{2}$$

to estimate κ in (1) can only be used for the case when $\lambda = 1.0$. In this section we show that by using the asymptotic properties of the Bassel function we can find an estimate of κ for any value of λ.

From the asymptotic power series for the Bassel functions $I_0(r)$ and $I_1(r)$ in [1], we have,

$$A(r) = \frac{I_1(r)}{I_0(r)} = 1 - \frac{1}{2r} - \frac{1}{8r^2} - \frac{1}{8r^3} + 0(r^{-4}). \tag{3}$$

Simplifying the equation (1) using (3) we have the expression approximately given by

$$8(1 + \lambda - c)\kappa^3 - 8\kappa^2 - \left(1 + \frac{1}{\lambda}\right)\kappa - \left(1 + \frac{1}{\lambda^2}\right) = 0, \tag{4}$$

where

$$c = \frac{1}{n} \left\{ \sum \cos(x_i - \hat{X}_i) + \lambda \sum \cos(y_i - \hat{\alpha} - \hat{\beta}\hat{X}_i) \right\}.$$

It can be shown that the above cubic equation in κ, i.e. equation (4) has only one positive real root and two complex roots, giving $\hat{\kappa}$ as the positive real root by using the following procedure [5].

Step 1. Find a suitable substitution such that the equation can be transformed into the form

$$x^3 + px + q = 0.$$

Step 2. Now define

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$$

and the rules are

(a) if $D > 0$, there exist one real and two complex roots,

(b) if $D = 0$, there exist three real roots (at least two are equal), and

(c) if $D < 0$, there exist three distinct real roots.

Step 3. For $D > 0$, let r_1, r_2 and r_3 be the roots, then

$$r_1 = u + v,$$

$$r_2 = -\left(\frac{u + v}{2}\right) + \left(\frac{u - v}{2}\right)\sqrt{3}\,i\,,$$

and
$$r_3 = -\left(\frac{u + v}{2}\right) - \left(\frac{u - v}{2}\right)\sqrt{3}\,i.$$

where

$$u = \left(\frac{-q}{2} + \sqrt{D}\right)^{1/3} \text{ and } v = \left(\frac{-q}{2} - \sqrt{D}\right)^{1/3}.$$

Our aim is to solve the following equation for κ

$$8(1 + \lambda - c)\kappa^3 - 8\kappa^2 - \left(1 + \frac{1}{\lambda}\right)\kappa - \left(1 + \frac{1}{\lambda^2}\right) = 0$$

or

$$a_0\kappa^3 + a_1\kappa^2 + a_2\kappa + a_3 = 0,$$

where

$$a_0 = 8(1 + \lambda - c) > 0, \text{ since } \lambda > 0$$

and
$$c = \frac{1}{n}\left(\sum \cos(x_i - \hat{X}_i)\right) + \lambda \sum \cos(y_i - \hat{\alpha} - \hat{\beta}\hat{X}) < (1 + \lambda)$$

$$a_1 = -8,$$

$$a_2 = -\left(1 + \frac{1}{\lambda}\right),$$

and
$$a_3 = -\left(1 + \frac{1}{\lambda^2}\right).$$

a suitable substitution is

$$x = y - \left(\frac{a_1}{3a_0}\right),$$

which gives

$$y^3 + py + q = 0,$$

where

$$p = \frac{3a_0 a_2 - a_1^2}{3a_0^2},$$

and
$$q = \frac{2a_1^3 - 9a_1 a_2 a_0 + 27a_3 a_0^2}{27a_0^3}.$$

These coefficients may be written as

$$p = -\left(\frac{3\Delta(1 + \lambda) + 8\lambda}{24\lambda\Delta^2}\right),$$

and

$$q = -\left(\frac{16\lambda^2 + 9\lambda(\lambda + 1)\Delta + 27(\lambda^2 + 1)\Delta^2}{216\lambda^2\Delta^3}\right),$$

where

$$\Delta = 1 + \lambda - c.$$

The next step is to show that

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 > 0.$$

By direct substitution for p and q, it can be shown that

$$D = \frac{3(3\Delta)^2}{\left(432\lambda^2\Delta^3\right)^2} \left\{ 2\lambda^3 + (1+\lambda)\left(31\lambda^2 + 18\lambda\Delta(1+\lambda) + 27\Delta^2\left(1+\lambda^2\right)\right) \right\},$$

which is always positive for all λ and Δ, or $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 > 0$.

Hence $D > 0$ is satisfied, and the equation has one real root and two complex roots. Our final step is to show that this real root is positive by showing

$$u + v = \left(\frac{-q}{2} + \sqrt{D}\right)^{1/3} + \left(\frac{-q}{2} - \sqrt{D}\right)^{1/3} > 0.$$

Let

$$u = \left(\frac{-q}{2} + \sqrt{D}\right)^{1/3} \quad \text{and} \quad v = \left(\frac{-q}{2} - \sqrt{D}\right)^{1/3}.$$

If $(u + v) > 0$, then $(u + v)^3 > 0$. We have

$$(u + v)^3 = u^3 + v^3 + 3uv(u + v)$$
$$= -q - p(u + v)$$

which is always positive since $p, q < 0$ and $u > v$.
Therefore, we conclude that the equation

$$8(1 + \lambda - c)\kappa^3 - 8\kappa^2 - \left(1 + \frac{1}{\lambda}\right)\kappa - \left(1 + \frac{1}{\lambda^2}\right) = 0$$

has only one positive real root, given by

$$\left(\frac{-q}{2} + \sqrt{D}\right)^{1/3} + \left(\frac{-q}{2} - \sqrt{D}\right)^{1/3}.$$

4. Conclusion

We have shown that by using the approximation given by Dobson, we can find the estimate of κ when $\lambda = 1$ and by using the asymptotic properties of the Bassel function we can find the estimate of κ for any value of λ. As an illustration we used the wind and wave direction data in [1] to show the approximation given by Dobson [2] (Method 1) and asymptotic properties of the Bassel function (Method 2) give almost the same value of $\hat{\kappa}$ when $\lambda = 1$.

Table below gives a comparison of $\hat{\kappa}$ and its standard deviation obtained by Method 1 (approximation given by Dobson [2]) and Method 2 (asymptotic power series for the Bassel function), in which we assumed the ratio of error concentration parameters, λ is equal to 1. The result suggest that the methods give similar estimates of κ and its standard error. Moreover by using the asymptotic properties of the Bassel function we can find the estimate of κ for any value of λ, which we cannot get by using the approximation given by Dobson [2] and hence it gives the estimates of the other parameters of the unreplicated linear circular functional relationship model for a known value of the ratio of error concentration parameters, λ.

Table. Comparison of estimating $\hat{\kappa}$ and its standard error by two different methods

	Method 1	Method 2
$\hat{\kappa}$	22.9842	22.9839
s.e. $(\hat{\kappa})$	2.0014	2.0020

(a) Wind direction data

	Method 1	Method 2
$\hat{\kappa}$	12.9619	12.9610
s.e. $(\hat{\kappa})$	1.4389	1.4402

(b) Wave direction data

References

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