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$(\delta - \text{pre}, s)$ - Continuous Functions

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Abstract. The aim of this paper is to introduce and study $(\delta - \text{pre}, s)$ - continuous functions as a new weaker form of almost contra-precontinuous functions due to Erdal Ekici. Basic characterizations, preservation theorems and several properties concerning $(\delta - \text{pre}, s)$ - continuous functions are obtained. The relationships between $(\delta - \text{pre}, s)$ - continuous functions and graphs are also discussed.

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1. Introduction

One of important and basic topics in the general topology and several branches of mathematics, which have been researched by many authors, is continuity of functions. In recent years, Ekici [10] studied the notion of almost contra-precontinuous functions and Jafari and Noiri [12] introduced contra-precontinuous functions. The purpose of this paper is to study $(\delta - \text{pre}, s)$ - continuous functions and to obtain several characterizations and properties of $(\delta - \text{pre}, s)$ - continuous functions. Moreover, the relationships between $(\delta - \text{pre}, s)$ - continuous functions and graphs are also investigated.

2. Preliminaries

In this paper, spaces (X, τ) and (Y, υ) (or simply *X* and *Y*) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let *A* be a subset of a space *X*. For a subset *A* of (X, τ), $cl(A)$ and $\text{int}\,(A)$ represent the closure of *A* with respect to τ and the interior of *A* with respect to τ, respectively.

A subset *A* of a space *X* is said to be regular open (respectively regular closed) if $A = \text{int}\,(cl(A))$ (respectively $A = cl\,(\text{int}(A))$) [26]. The family of all regular open (respectively regular closed) sets of X is denoted by $RO(X)$ (respectively $RC(X)$). We put $RO(X, x) = \{U \in RO(X) : x \in U\}$ and $RC(X, x) = \{F \in RC(X) : x \in F\}$.

The δ-interior [28] of a subset A of X is the union of all regular open sets of X contained in A is denoted by $\delta-\text{int}(A)$. A subset A is called δ-open [28] if $A = \delta\text{-}\text{int}(A)$, i.e., a set is δ-open if it is the union of regular open sets. The complement of δ-open set is called δ-closed. Alternatively, a set A of (X, τ) is called δ-closed [28] if $A = \delta - cl(A)$, where $\delta - cl(A) = \{x \in X : A \cap \text{int}(cl(U)) \neq \emptyset$, $U \in \tau$ and $x \in U\}$.

A subset A of a space X is said to be preopen [17] (resp. δ-preopen [21]) if $A \subset \text{int}(cl(A))$ (resp. $A \subset \text{int}(\delta - cl(A))$). The family of all preopen (resp. δ-preopen) sets of X containing a point $x \in X$ is denoted by $PO(X, x)$ (resp. $\delta PO(X, x)$).

The complement of a preopen set is said to be preclosed [9].

A subset A is said to be semi-open [15] if $A \subset cl(\text{int}(A))$. The complement of a semi-open set is called semi-closed [4]. The intersection of all semi-closed sets containing A is called the semi-closure [4] of A and is denoted by $s - cl(A)$. The semi-interior of A is defined by the union of all semi-open sets contained in A and is denoted by $s - \text{int}(A)$. The family of all semi-open sets of X is denoted by $SO(X)$. We set $SO(X, x) = \{U : x \in U \in SO(X)\}$.

The complement of a δ-preopen set is said to be δ-preclosed. The intersection of all δ-preclosed sets of X containing A is called the δ-preclosure [21] of A and is denoted by $\delta - pcl(A)$. The union of all δ-preopen sets of X contained A is called δ-preinterior of A and is denoted by $\delta - \text{pint}(A)$ [21]. A subset U of X is called a δ-preneighborhood [21] of a point $x \in X$ if there exists a δ-preopen set V such that $x \in V \subset U$. Note that $\delta - pcl(A) = A \cup cl(\delta - \text{int}(A))$ and $\delta - \text{pint}(\text{A}) = \text{A} \cap \text{int}(\delta - cl(\text{A}))$.

A point $x \in X$ is said to be a θ-semi-cluster point [13] of a subset A of X if $cl(U) \cap A \neq \emptyset$ for every $U \in SO(X, x)$. The set of all θ-semi-cluster points of A is called the θ-semi-clusure of A and is denoted by $\theta - s - cl(A)$. A subset A is called θ-semi-closed [13] if $A = \theta - s - cl(A)$. The complement of a θ-semi-closed set is called θ-semi-open.

The family of all δ-open (resp. δ-preopen, δ-preclosed) sets of X containing a point $x \in X$ is denoted by $\delta O(X, x)$ (resp. $\delta PO(X, x)$, $\delta PC(X, x)$), i.e., $\delta O(X, x) = \{U \in \delta O(X) : x \in U\}$ (respectively $\delta PO(X, x) = \{U \in \delta PO(X) : x \in U\}$, $\delta PC(X, x) = \{F \in \delta PC(X) : x \in F\}$).

The family of all δ-open (resp. δ-preopen, δ-preclosed) sets of X is denoted by $\delta O(X)$ (resp. $\delta PO(X)$, $\delta PC(X)$).

Definition 1. *A function $f : X \to Y$ is said to be*

(1) *perfectly continuous* [18] *if $f^{-1}(V)$ is clopen in X for every open set V of Y,*

(2) *contra-continuous* [5] *if $f^{-1}(V)$ is closed in X for every open set V of Y,*

(3) *regular set-connected* [6] *if $f^{-1}(V)$ is clopen in X for every $V \in RO(Y)$,*

(4) *s-continuous* [3] *if for each* $x \in X$ *and each* $V \in SO(Y, f(x))$, *there exists an open set U in X containing x such that* $f(U) \subset V$,

(5) *almost s-continuous* [20] *if for each* $x \in X$ *and each* $V \in SO(Y, f(x))$, *there exists an open set U in X containing x such that* $f(U) \subset s - cl(V)$,

(6) *contra-precontinuous* [12] *if* $f^{-1}(V) \in PC(X)$ *for each open set V of Y,*

(7) *almost contra-precontinuous* [10] *if* $f^{-1}(V) \in PC(X)$ *for each* $V \in RO(Y)$.

3. $(\delta - \text{pre}, s)$ - continuous functions

In this section, the notion of $(\delta - \text{pre}, s)$ - continuous functions is introduced and characterizations and some properties of $(\delta - \text{pre}, s)$ - continuous functions are investigated.

Definition 2. *A function* $f : X \to Y$ *is called*

(1) $(\delta - pre, s)$ *- continuous at a point* $x \in X$ *if for each* $V \in SO(Y f(x))$, *there exists a δ-preopen set U in X containing x such that* $f(U) \subset cl(V)$,

(2) $(\delta - pre, s)$ *- continuous if it has this property at each point of X.*

Theorem 1. *The following are equivalent for a function* $f : X \to Y$:

(1) f *is* $(\delta - pre, s)$ *- continuous;*

(2) $f^{-1}(V) \subset \delta - \text{p}\operatorname{int}(f^{-1}(cl(V)))$ *for every* $V \in SO(Y)$;

(3) *the inverse image of a regular closed set of Y is δ-preopen;*

(4) *the inverse image of a regular open set of Y is δ-preclosed;*

(5) *the inverse image of a θ-semi-open set of Y is δ-preopen;*

(6) $f^{-1}(\operatorname{int}(cl(G)))$ *is δ-preclosed for every open subset G of Y;*

(7) $f^{-1}(cl(\operatorname{int}(F)))$ *is δ-preopen for every closed subset F of Y.*

Proof. (1)⇒(2): Let $V \in SO(Y)$ and $x \in f^{-1}(V)$. Then $f(x) \in V$. Since f is $(\delta - \text{pre}, s)$ - continuous, there exists a δ-preopen set U in X containing x such that $f(U) \subset cl(V)$. It follows that $x \in U \subset f^{-1}(cl(V))$. Hence $x \in \delta -$ $\text{pint}(f^{-1}(cl(V)))$. We have $f^{-1}(V) \subset \delta - \text{pint}(f^{-1}(cl(V)))$.

(2)⇒(3): Let F be any regular closed set of Y. Since $F \in SO(Y)$, then by (2), it follows that $f^{-1}(F) \subset \delta - \text{pint}(f^{-1}(F))$. This shows that $f^{-1}(F)$ is δ-preopen in X.

(3)⇔(4): This is obvious.

(3)⇒(5): This follows from the fact that any θ-semi-open set is a union of regular closed sets.
(5)⇒(1): Let $x \in X$ and $V \in SO(Y)$. Since $cl(V)$ is θ-semi-open in Y, by (5) there exists a δ-preopen set U in X containing x such that $x \in U \subset f^{-1}(cl(V))$. Hence $f(U) \subset cl(V)$. This show that f is $(\delta\text{-pre}, s)$- continuous.
(4) ⇔ (6): Let G be an open subset of Y. Since $\text{int}(cl(G))$ is regular open, then by (4), it follows that $f^{-1}(\text{int}(cl(G)))$ is δ-preclosed.
The converse can be shown easily.
(3) ⇔ (7): It can be obtained similar as (4) ⇔ (6).

Remark 1. The following diagram holds:

perfectly continuous	⇒	contra-continuous	⇒	contra-precontinuous
⇓				⇓
regular set-connected		⇒		almost contra-precontinuous
⇑				⇓
almost s-continuous				$(\delta - \text{pre}, s)$- continuous
⇑				
s-continuous				

None of these implications is reversible.

Example 1. Let R be the set of real numbers and τ be the countable extension topology on R, i.e., the topology with subbase $\tau_1 \cup \tau_2$ where τ_1 is the Euclidean topology of R and τ_2 is the topology of countable complements of R and σ be the discrete topology of R. Define a function $f : (R, \tau) \to (R, \sigma)$ as follows:

$$f(x) = \begin{cases} 1 \ , & \text{if } x \text{ is rational,} \\ 2 \ , & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is $(\delta - \text{pre}, s)$- continuous but not almost contra-precontinuous, since $\{1\}$ is regular closed in (R, σ) and $f^{-1}(\{1\}) = Q$ where Q is the set of rationals is not preopen in (R, τ).

The other implications are not reversible as shown in [6, 7, 10, 12, 20].

Definition 3. *Let (X, τ) be a topological space. The collection of all regular open sets forms a base for a topology τ^*. It is called the semiregularization.*

In case when $\tau = \tau^*$, the space (X, τ) is called semi-regular [26].

Theorem 2. *Let $f : X \to Y$ be a function from a semi-regular topological space (X, τ) to a topological space (Y, v). f is $(\delta - pre, s)$-continuous if and only if f is almost contra-precontinuous.*

Proof. Obvious.

Definition 4. *A space is said to be P_{Σ}* [29] *or strongly s-regular* [11] *if for any open set V of X and each $x \in V$, there exists $F \in RC(X, x)$ such that $x \in F \subset V$.*

Definition 5. *A function $f : X \to Y$ is said to be δ-almost continuous* [21] *if $f^{-1}(V)$ is δ-preopen in X for every open set V in Y.*

Theorem 3. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then, if f is $(\delta - pre, s)$-continuous and Y is P_{Σ}, then f is δ-almost continuous.*

Proof. Let V be any open set of Y. Since Y is P_{Σ}, there exists a subfamily Ψ of $RC(Y)$ such that $V = \cup\{F : F \in \Psi\}$. Since f is $(\delta - pre, s)$-continuous, $f^{-1}(F)$ is δ-preopen in X for each $F \in \Psi$ and $f^{-1}(V)$ is δ-preopen in X. Therefore, f is δ-almost continuous.

Definition 6. *A space is said to be weakly P_{Σ}* [19] *if for any $V \in RO(X)$ and each $x \in V$, there exists $F \in RC(X, x)$ such that $x \in F \subset V$.*

Definition 7. *A function $f : X \to Y$ is said to be almost δ-precontinuous if $f^{-1}(V)$ δ-preopen in X for every $V \in RO(Y)$.*

Theorem 4. *Let $f : (X, \tau) \to (Y, \sigma)$ be a $(\delta - pre, s)$-continuous function. If Y is weakly P_{Σ}, then f is almost δ-precontinuous.*

Proof. Let V be any regular open set of Y. Since Y is weakly P_{Σ}, there exists a subfamily Ψ of $RC(Y)$ such that $V = \cup\{F : F \in \Psi\}$. Since f is $(\delta - \text{pre}, s)$-continuous, $f^{-1}(F)$ is δ-preopen in X for each $F \in \Psi$ and $f^{-1}(V)$ is δ-preopen in X. Hence, f is almost δ-precontinuous.

Theorem 5. *Let $f : X \to Y$ be a function and let $g : X \to X \times Y$ be the graph function of f, defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is $(\delta - pre, s)$-continuous, then f is $(\delta - pre, s)$-continuous.*

Proof. Let $F \in RC(Y)$, then $X \times F = X \times cl(\text{int}(F)) = cl(\text{int}(X)) \times cl(\text{int}(F)) = cl(\text{int}(X \times F))$. Therefore, $X \times F \in RC(X \times Y)$. It follows from Theorem 1 that $f^{-1}(F) = g^{-1}(X \times F) \in \delta PO(X)$. Thus, f is $(\delta - \text{pre}, s)$- continuous.

Lemma 1. *Let A and X_0 be subsets of a space (X, τ). If $A \in \delta PO(X)$ and $X_0 \in \delta O(X)$, then $A \cap X_0 \in \delta PO(X_0)$* [21].

Lemma 2. *Let $A \subset X_0 \subset X$. If $X_0 \in \delta O(X)$ and $A \in \delta PO(X_0)$, then $A \in \delta PO(X)$* [21].

Theorem 6. *If $f : X \to Y$ is a $(\delta - pre, s)$- continuous function and A is any δ-open subset of X, then the restriction $f|_A : A \to Y$ is $(\delta$- $pre, s)$- continuous.*

Proof. Let $F \in RC(Y)$. Then, by Theorem 1, $f^{-1}(F) \in \delta PO(X)$. Since A is δ-open in X, it follows from Lemma 1 that $(f|A)^{-1}(F) = A \cap f^{-1}(F) \in \delta PO(A)$. Therefore, $f|_A$ is a $(\delta - \text{pre}, s)$- continuous function.

Theorem 7. *Let $f : X \to Y$ be a function and $\{U_\alpha : \alpha \in I\}$ be a δ-open cover of X. If for each $\alpha \in I$, $f|_{U\alpha}$ is $(\delta - pre, s)$- continuous, then $f : X \to Y$ is a $(\delta - pre, s)$- continuous function.*

Proof. Let $V \in RC(Y)$. Since $f|_{U\alpha}$ is $(\delta - \text{pre}, s)$- continuous for each $\alpha \in I$, $(f|_{U\alpha})^{-1}(V) \in \delta PO(U_\alpha)$. Since $U_\alpha \in \delta O(X)$, by the Lemma 2, $(f|_{U\alpha})^{-1}(V) \in \delta PO(X)$ for each $\alpha \in I$. Then $f^{-1}(V) = \cup_{\alpha \in I} [(f|_{U\alpha})^{-1}(V)] \in \delta PO(X)$. This gives f is a $(\delta - \text{pre}, s)$- continuous function.

Definition 8. *A filter base Λ is said to be δ-preconvergent (resp. rc-convergent) to a point x in X if for any $U \in \delta PO(X)$ containing x (resp. $U \in RC(X)$ containing x), there exists a $B \in \Lambda$ such that $B \subset U$.*

Theorem 8. *If a function $f : X \to Y$ is $(\delta - pre, s)$- continuous, then for each point $x \in X$ and each filter base Λ in X which is δ-preconvergent to x, the filter base $f(\Lambda)$ is rc-convergent to $f(x)$.*

Proof. Let $x \in X$ and Λ be any filter base in X which is δ-preconvergent to *x*. Since f is $(\delta - \text{pre}, s)$- continuous, then for any $V \in RC(Y)$ containing $f(x)$, there exists $U \in \delta PO(X)$ containing x such that $f(U) \subset V$. Since Λ is δ-preconvergent to x, there exists a $B \in \Lambda$ such that $B \subset U$. This means that $f(B) \subset V$ and therefore the filter base $f(\Lambda)$ is rc-convergent to $f(x)$.

Theorem 9. *Let $f : X \to Y$ be a function and $x \in X$. If there exists $U \in \delta O(X)$ such that $x \in U$ and the restriction of f to U is a $(\delta - pre, s)$ - continuous function at x, then f is $(\delta - pre, s)$ - continuous at x.*

Proof. Suppose that $F \in SO(Y)$ containing $f(x)$. Since $f|_U$ is $(\delta - \text{pre}, s)$ - continuous at x, there exists $V \in \delta PO(U)$ containing x such that $f(V) = (f|_U)(V) \subset cl(F)$. Since $U \in \delta O(X)$ containing x, it follows from Lemma 2 that $V \in \delta PO(X)$ containing x. This shows clearly that f is $(\delta - \text{pre}, s)$ - continuous at x.

Definition 9. *A function $f : X \to Y$ is said to be*

(1) *θ-irresolute* [14] *if for each $x \in X$ and each $V \in SO(Y, f(x))$, there exists $U \in SO(X, x)$ such that $f(cl(U)) \subset cl(V)$,*

(2) *δ-preirresolute if for each $x \in X$ and each $V \in \delta PO(Y, f(x))$, there exists a δ-preopen set U in X containing x such that $f(U) \subset V$.*

Theorem 10. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then, the following properties hold:*

(1) *If f is δ-preirresolute and g is $(\delta - pre, s)$ - continuous, then $g \, o \, f : X \to Z$ is $(\delta - pre, s)$ - continuous.*

(2) *If f is $(\delta - pre, s)$ - continuous and g is θ-irresolute, then $g \, o \, f : X \to Z$ is $(\delta - pre, s)$ - continuous.*

Proof. (1) Let $x \in X$ and $W \in SO(Z, (g \, o \, f)(x))$. Since g is $(\delta - \text{pre}, s)$ - continuous, a δ-preopen set V in Y containing $f(x)$ such that $g(V) \subset cl(W)$. Since f is δ-preirresolute, there exists a δ-preopen set U in X containing x such that $f(U) \subset V$. This shows that $(g \, o \, f)(U) \subset cl(W)$. Therefore, $g \, o \, f$ is $(\delta\text{-pre}, s)$ - continuous.
(2) Let $x \in X$ and $W \in SO(Z, (g \, o \, f)(x))$. Since g is θ-irresolute, there exists $V \in SO(Y, f(x))$ such that $g(cl(V)) \subset cl(W)$. Since f is $(\delta - \text{pre}, s)$ - continuous, there exists a δ-preopen set U in X containing x such that $f(U) \subset cl(V)$. Hence $(g \, o \, f)(U) \subset cl(W)$. This shows that $g \, o \, f$ is $(\delta - \text{pre}, s)$ - continuous.

Definition 10. *A function $f : X \to Y$ is called δ-preopen if image of each δ-preopen set is δ-preopen.*

Theorem 11. *If $f : X \to Y$ is a surjective δ-preopen function and $g : Y \to Z$ is a function such that $g\, o\, f : X \to Z$ is $(\delta - pre, s)$-continuous, then g is $(\delta - pre, s)$-continuous.*

Proof. Suppose that x and y are two points in X and Y, respectively, such that $f(x) = y$. Let $V \in SO(Z, (g\, o\, f)(x))$. Then there exists a δ-preopen set U in X containing x such that $g(f(U)) \subset cl(V)$. Since f is δ-preopen, $f(U)$ is a δ-preopen set in Y containing y such that $g(f(U)) \subset cl(V)$. This implies that g is $(\delta - \text{pre}, s)$-continuous.

Corollary 1. *Let $f : X \to Y$ be a surjective δ-preirresolute and δ-preopen function and let $g : Y \to Z$ be a function. Then, $g\, o\, f : X \to Z$ is $(\delta - pre, s)$-continuous if and only if g is $(\delta - pre, s)$-continuous.*

Proof. It can be obtained from Theorem 10 and Theorem 11.

Definition 11. *A function $f : X \to Y$ is called weakly $(\delta - pre, s)$-continuous if for each $x \in X$ and each $V \in SO(Y, f(x))$, there exists a δ-preopen set U in X containing x such that $\text{int}(f(U)) \subset cl(V)$.*

Definition 12. *A function $f : X \to Y$ is called δ-pre-semi-open if image of each δ-preopen set is semi-open.*

Theorem 12. *If a function $f : X \to Y$ is weakly $(\delta - pre, s)$-continuous and δ-pre-semi-open, then f is $(\delta - pre, s)$-continuous.*

Proof. Let $x \in X$ and $V \in SO(Y, f(x))$. Since f is weakly $(\delta - \text{pre}, s)$-continuous, there exists a δ-preopen set U in X containing x such that $\text{int}(f(U)) \subset cl(V)$. Since f is δ-pre-semi-open, $f(U) \in SO(Y)$ and $f(U) \subset cl(\text{int}(f(U))) \subset cl(V)$. This shows that f is $(\delta - \text{pre}, s)$-continuous.

Definition 13. *A space X is said to be*

(1) *s-Urysohn* [2] *if for each pair of distinct points x and y in X, there exist $U \in SO(X, x)$ and $V \in SO(X, y)$ such that $cl(U) \cap cl(V) = \varnothing$,*

(2) *weakly Hausdorff* [24] *if each element of X is an intersection of regular closed sets,*

(3) *δ-pre-Hausdorff if for each pair of distinct points x and y in X, there exist $U \in \delta PO(X, x)$ and $V \in \delta PO(X, y)$ such that $U \cap V = \varnothing$,*

(4) *δ - pre- T_1 if for each pair of distinct points in X, there exist δ-preopen sets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$.*

Remark 2. The following implications are hold for a topological space *X*:

(1) $T_1 \Rightarrow \delta$ - pre- T_1,

(2) $T_2 \Rightarrow \delta$ - pre- T_2.

None of these implications is reversible.

Example 2. Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \varnothing, \{a\}, \{b, c\}\}$. Then (X, τ) is δ - pre- T_2 but not T_1.

Lemma 3. *Let A be a subset of a space (x, τ). Then $A \in \delta PO(X)$ if and only if $A \cap U \in \delta PO(X)$ for each regular open (δ-open) set U of X* [21].

Definition 14. *A function $f : X \to Y$ is called strongly (θ, s) - continuous if for each $x \in X$ and each $V \in SO(Y, f(x))$, there exists a regular open set U in X containing x such that $f(U) \subset cl(V)$.*

Theorem 13. *If $f : X \to Y$ is strongly (θ, s) - continuous, $g : X \to Y$ is $(\delta - pre, s)$ - continuous and Y is s-Urysohn, then $E = \{x \in X : f(x) = g(x)\}$ is δ-preclosed in X.*

Proof. If $x \in X \setminus E$, then it follows that $f(x) \neq g(x)$. Since Y is s-Urysohn, there exist $V \in SO(Y, f(x))$ and $W \in SO(Y, g(x))$ such that $cl(V) \cap cl(W) = \varnothing$. Since f is strongly (θ, s) - continuous and g is $(\delta - \text{pre}, s)$ - continuous, there exist a regular open set U containing x and a δ-preopen set G containing x such that $f(U) \subset cl(V)$ and $g(G) \subset cl(W)$. Set $O = U \cap G$. By the previous lemma, O is δ-preopen in *X*. Therefore $f(O) \cap g(O) = \varnothing$ and it follows that $x \notin \delta\text{-}pcl(E)$. This shows that *E* is δ-preclosed in *X*.

Theorem 14. *If f is a $(\delta - pre, s)$ - continuous injection and Y is s-Urysohn, then X is δ-pre-Hausdorff.*

Proof. Suppose that Y is s-Urysohn. By the injectivity of *f*, it follows that $f(x) \neq f(y)$ for any distinct points x and y in *X*. Since Y is s-Urysohn, there exist $V \in SO(Y, f(x))$ and $W \in SO(Y, f(y))$ such that $cl(V) \cap cl(W) = \varnothing$. Since f is a

$(\delta - \text{pre}, s)$ - continuous, there exist δ-preopen sets U and G in X containing x and y, respectively, such that $f(U) \subset cl(V)$ and $f(G) \subset cl(W)$. Hence $U \cap G = \varnothing$. This shows that X is δ-pre-Hausdorff.

Theorem 15. *If f is a $(\delta - pre, s)$ - continuous injection and Y is weakly Hausdorff, then X is δ - pre - T_1.*

Proof. Suppose that Y is weakly Hausdorff. For any distinct points x and y in X, there exist V, $W \in RC(Y)$ such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$ and $f(y) \in W$. Since f is $(\delta - pre, s)$ - continuous, by Theorem 1, $f^{-1}(V)$ and $f^{-1}(W)$ are δ-preopen subsets of X such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that X is δ - pre- T_1.

4. Relationships

In this section, several relationships of $(\delta - \text{pre}, s)$ - continuity are investigated.

Definition 15. *A space X said to be*

(1) *δ-pre-compact if every δ-preopen cover of X has a finite subcover.*

(2) *countably δ-pre-compact if every countable cover of X by δ-preopen sets has a finite subcover.*

(3) *δ-pre-Lindelof if every δ-preopen cover of X has a countable subcover.*

(4) *S-closed* [27] *if every regular closed cover of X has a finite subcover.*

(5) *countably S-closed* [1] *if every countable cover of X by regular closed sets has a finite subcover.*

(6) *S-Lindelof* [16] *if every cover of X by regular closed sets has a countable subcover.*

Theorem 16. *Let $f : X \to Y$ be a $(\delta - pre, s)$ - continuous surjection. Then the following statements hold:*

(1) *if X is δ-pre-compact, then Y is S-closed.*

(2) *if X is δ-pre-Lindelof, then Y is S-Lindelof.*

(3) *if X is countably δ-pre-compact, then Y is countably S-closed.*

Proof. Let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of Y. Since f is $(\delta - \text{pre}, s)$-continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a δ-preopen cover of X and hence there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Therefore, we have $Y = \cup\{V_\alpha : \alpha \in I_0\}$ and Y is S-closed.
The proofs of (2) and (3) can be obtained similarly.

Definition 16. *A space X said to be*

(1) *δ-preclosed-compact if every δ-preclosed cover of X has a finite subcover.*

(2) *countably δ-preclosed-compact if every countable cover of X by δ-preclosed sets has a finite subcover.*

(3) *δ-preclosed-Lindelof if every cover of X by δ-preclosed sets has a countable subcover.*

(4) *nearly compact* [22] *if every regular open cover of X has a finite subcover.*

(5) *nearly countably compact* [8, 23] *if every countable cover of X by regular open sets has a finite subcover.*

(6) *nearly Lindelof if every cover of X by regular open sets has a countable subcover.*

Theorem 17. *Let $f : X \to Y$ be a $(\delta - pre, s)$-continuous surjection. Then the following statements hold:*

(1) *if X is δ-preclosed-compact, then Y is nearly compact.*

(2) *if X is δ-preclosed-Lindelof, then Y is nearly Lindelof.*

(3) *if X is countably δ-preclosed-compact, then Y is nearly countably compact.*

Proof. Let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of Y. Since f is $(\delta - pre, s)$-continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a δ-preclosed cover of X. Since X is δ-preclosed-compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Thus, we have $Y = \cup\{V_\alpha : \alpha \in I_0\}$ and Y is nearly compact.
The proofs of (2) and (3) can be obtained similarly.

Definition 17. *A space X said to be*

(1) *mildly δ-pre-compact if every δ-preclopen cover of X has a finite subcover,*

(2) *mildly countably δ-pre-compact if every δ-preclopen countable cover of X has a finite subcover,*

(3) *mildly δ-pre-Lindelof if every δ-preclopen cover of X has a countable subcover.*

Theorem 18. *Let $f : X \to Y$ be a $(\delta - pre, s)$-continuous surjection. Suppose that f is a δ-almost continuous function. Then*

(1) *If X is mildly δ-pre-compact, then Y is nearly compact and S-closed.*

(2) *If X is mildly countably δ-pre-compact, then Y is nearly countably compact and countably S-closed.*

(3) *If X is mildly δ-pre-Lindelof, then Y is nearly Lindelof and S-Lindelof.*

Proof. (1) Let $V \in RC(Y)$. Then since f is $(\delta - \text{pre}, s)$-continuous and δ-almost continuous, $f^{-1}(V)$ is δ-preopen and δ-preclosed in X and hence $f^{-1}(V)$ is δ-preclopen.

Let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of Y. Then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a δ-preclopen cover of X and since X is mildly δ-pre-compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjective, we obtain $Y = \cup\{V_\alpha : \alpha \in I_0\}$. This shows that Y is S-closed.
It can be obtained similarly that Y is nearly compact.
The other proofs can be obtained similarly.

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 18. *A graph $G(f)$ of a function $f : X \to Y$ is said to be*

(1) *$(\delta - pre, s)$-closed if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist a δ-preopen set U in X containing x and $V \in SO(Y, y)$ such that $(U \times cl(V)) \cap G(f) = \varnothing$,*

(2) *δ-pre-regular-closed if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist a δ-preopen set U in X containing x and $V \in RO(Y, y)$ such that $(U \times V) \cap G(f) = \varnothing$.*

Lemma 4. *The following properties are equivalent for a graph $G(f)$ of a function f:*

(1) *$G(f)$ is $(\delta - pre, s)$-closed;*

(2) *for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist a δ-preopen set U in X containing x and $V \in SO(Y, y)$ such that $f(U) \cap cl(V) = \varnothing$;*

(3) *for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist a δ-preopen set U in X containing x and $F \in RC(Y, y)$ such that $f(U) \cap F = \varnothing$.*

Proof. (1)⇒(2): It is an immediate consequence of definition of $(\delta - pre, s)$-closed graph and the fact that for any subsets $A \subset X$ and $B \subset Y, (A \times B) \cap G(f) = \varnothing$ if and only if $f(A) \cap B = \varnothing$.

(2)⇒(3): It follows from the fact that $cl(V) \in RC(Y)$ for any $V \in SO(Y)$.

(3)⇒(1): It is obvious since every regular closed set is semi-open and closed.

Theorem 19. *If $f : X \to Y$ is $(\delta - pre, s)$-continuous and Y is weakly Hausdorff, $G(f)$ is δ-pre-regular-closed graph in $X \times Y$.*

Proof. Suppose that Y is weakly Hausdorff. Let $(x, y) \in (X \times Y) \setminus G(f)$, then $f(x) \neq y$. Since Y is weakly Hausdorff, there exist $F \in RC(Y)$ such that $f(x) \in F$ and $y \notin F$. Since f is $(\delta - \text{pre}, s)$-continuous, there exists a δ-preopen set U in X containing x such that $f(U) \subset F$. Therefore, we obtain $y \in Y \setminus F \in RO(Y)$ and $f(U) \cap (Y \setminus F) = \varnothing$. This shows that $G(f)$ is δ-pre-regular-closed in $X \times Y$.

Theorem 20. *If $f : X \to Y$ is $(\delta - pre, s)$-continuous and Y is s-Urysohn, $G(f)$ is $(\delta - pre, s)$-closed graph in $X \times Y$.*

Proof. Suppose that Y is s-Urysohn. Let $(x, y) \in (X \times Y) \setminus G(f)$. It follows that $f(x) \neq y$. Since Y is s-Urysohn, there exist $V \in SO(Y, f(x))$ and $W \in SO(Y, y)$ such that $cl(V) \cap cl(W) = \varnothing$. Since f is $(\delta - \text{pre}, s)$-continuous, there exists a δ-preopen set U in X containing x such that $f(U) \subset cl(V)$. Therefore, $f(U) \cap cl(W) = \varnothing$ and $G(f)$ is $(\delta - \text{pre}, s)$-closed in $X \times Y$.

Theorem 21. *Let $f : X \to Y$ have a $(\delta - pre, s)$-closed graph. If f is injective, then X is δ-pre-T_1.*

Proof. Let x and y be any two distinct points of X. Then, we have $(x, f(y)) \in (X \times Y) \setminus G(f)$. By Lemma 4, there exist a δ-preopen set U in X containing x and $F \in RC(Y, f(y))$ such that $f(U) \cap F = \varnothing$; hence $U \cap f^{-1}(F) = \varnothing$. Therefore, we have $y \notin U$. This implies that X is δ-pre-T_1.

Theorem 22. *Let $f : X \to Y$ have a $(\delta - pre, s)$-closed graph. If f is surjective, then Y is weakly T_2.*

Proof. Let y_1 and y_2 be any distinct points of Y. Since f is surjective, $f(x) = y_1$, for some $x \in X$ and $(x, y_2) \in (X \times Y) \setminus G(f)$. By Lemma 4, there exist a δ-preopen set U in X containing x and $F \in RC(Y, y_2)$ such that $f(U) \cap F = \varnothing$; hence $y_1 \notin F$. This implies that Y is weakly T_2.

Definition 19. *A space X is called δ-preconnected provided that X is not the union of two disjoint nonempty δ-preopen sets.*

Theorem 23. *If $f : X \to Y$ is $(\delta - pre, s)$-continuous surjection and X is δ-preconnected, then Y is connected.*

Proof. Suppose that *Y* is not connected space. There exist nonempty disjoint open sets V_1 and V_2 such that $Y = V_1 \cup V_2$. Therefore, V_1 and V_2 are clopen in *Y*. Since *f* is $(\delta - \text{pre}, s)$-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are δ-preopen in *X*. Moreover, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. This shows that *X* is not δ-preconnected. This contradicts that *Y* is not connected assumed. Hence, *Y* is connected.

Definition 20. *A topological space is called*

(1) *δ-pre-ultra-connected if every two non-void δ-preclosed subsets of X intersect,*

(2) *hyperconnected* [25] *if every open set is dense.*

Theorem 24. *If X is δ-pre-ultra-connected and $f : X \to Y$ is $(\delta - pre, s)$-continuous and surjective, then Y is hyperconnected.*

Proof. Assume that *Y* is not hyperconnected. Then there exists an open set *V* such that *V* is not dense in *Y*. Then there exist disjoint non-empty regular open subsets B_1 and B_2 in *Y*, namely $\text{int}(cl(V))$ and $Y \setminus cl(V)$. Since *f* is (δ-pre,*s*)-continuous and onto, by Theorem 1, $A_1 = f^{-1}(B_1)$ and $A_2 = f^{-1}(B_2)$ are disjoint non-empty δ-preclosed subsets of *X*. By assumption, the δ-pre-ultra-connectedness of *X* implies that A_1 and A_2 must intersect. By contradiction, *Y* is hyperconnected.

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