BULLETIN *of the*
Malaysian Mathematical
Sciences Society
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **28***(2) (2005), 107–116*

Some Properties of Almost Contra-Precontinuous Functions

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Abstract. Ekici [8] introduced and investigated the notion of almost contra-precontinuous functions as a generalization of contra-precontinuous functions defined by Jafari and Noiri [11]. In this paper, we obtain the further characterizations and properties of almost contra-precontinuous functions. We also show that (s, p)-continuity due to Jafari [9] is equivalent to almost contra-precontinuity.

2000 Mathematics Subject Classification: 54C10

Key words and phrases: Regular closed, (s, p)-continuous, contra-precontinuous, contra-continuous, almost contra-precontinuous.

1. Introduction

In 1996, Dontchev [6] introduced contra-continuous functions. Jafari and Noiri [11] introduced and studied contra-precontinuous functions. Recently Ekici [8] introduced the notion of almost contra-precontinuous functions. On the other hand, in 1980, Joseph and Kwack [13] introduced the notion of (θ, s)-continuous functions. In 1999, Jafari [9] introduced the notion of (p, s)-continuous functions.

The purpose of the present paper is to show that the concepts of almost contra-precontinuity and (p, s)-continuity are equivalent of each other and to obtain the further characterizations and properties of almost contra-precontinuous functions.

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular closed* (resp. *regular open*) if $\mathrm{Cl}(\mathrm{Int}(A)) = A$ (resp. $\mathrm{Int}(\mathrm{Cl}(A)) = A$).

Definition 2.1. *Let (X, τ) be a topological space. A subset A of X is said to be* semi-open [16] *(resp.* preopen [17], α-open [21], β-open [1] *or* semi-preopen [3]*) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$).*

Received: March 8, 2005; *Revised:* May 3, 2005.

The family of all semi-open (resp. preopen, α-open, β-open) sets in (X, τ) is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$ or τ^α, $\beta(X)$). The family of all semi-open (resp. preopen) sets containing x is denoted by $\mathrm{SO}(X, x)$ (resp. $\mathrm{PO}(X, x)$).

Definition 2.2. *The complement of a semi-open (resp. preopen, α-open, β-open) set is said to be* semi-closed [4] *(resp.* preclosed [17], α-closed [19], β-closed [1] *or* semi-preclosed [3]*).*

Definition 2.3. *The intersection of all semi-closed (resp. preclosed, α-closed, β-closed) sets of X containing A is called the* semi-closure [4] *(resp.* preclosure [7], α-closure [19], β-closure [2] *or* semi-preclosure [3]*) of A and is denoted by sCl(A) (resp. pCl(A), αCl(A), ${}_\beta$Cl(A) or spCl(A)).*

Definition 2.4. *The union of all semi-open (resp. preopen, α-open, β-open) sets of X contained in A is called the* semi-interior *(resp.* preinterior, α-interior, β-interior *or* semi-preinterior*) of A and is denoted by sInt(A) (resp. pInt(A), αInt(A), ${}_\beta$Int(A) or spInt(A)).*

Throughout the present paper, (X, τ) and (Y, σ) denote topological spaces and $f : (X, \tau) \to (Y, \sigma)$ presents a (single valued) function.

Lemma 2.1. (El-Deeb et al. [7]). *Let A be a subset of a topological space (X, τ) and $x \in X$. For the preclosure of A, the following properties hold:*
(1) $x \in \mathrm{pCl}(A)$ if and only if $A \cap U \neq \emptyset$ for each $U \in \mathrm{PO}(X, x)$,
(2) A is preclosed if and only if $\mathrm{pCl}(A) = A$.

The *θ-semi-closure* of A, denoted by $\mathrm{sCl}_\theta(A)$, is defined as the set of all points $x \in X$ such that $\mathrm{Cl}(V) \cap A \neq \emptyset$ for every $V \in \mathrm{SO}(X, x)$. A subset A is said to be *θ-semi-closed* if $A = \mathrm{sCl}_\theta(A)$ [13]. The complement of a θ-semi-closed set is said to be *θ-semi-open*.

Definition 2.5. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be* contra-continuous [6] *(resp.* contra-precontinuous [11]*) if $f^{-1}(V)$ is closed (resp. preclosed) in X for each open set V of Y.*

Definition 2.6. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be* almost contra-precontinuous [8] *if $f^{-1}(V)$ is preclosed in X for each regular open set V of Y.*

Definition 2.7. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be (p, s)-continuous* [9] *(resp.* (θ, s)-continuous [13]*) if for each $x \in X$ and each $V \in \mathrm{SO}(X, f(x))$, there exists $U \in \mathrm{PO}(X, x)$ (resp. $U \in \tau$ containing x) such that $f(U) \subset \mathrm{Cl}(V)$.*

3. Characterizations

In this section we obtain the further characterizations of almost contra-precontinuous functions.

Theorem 3.1. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*
(1) f is (p, s)-continuous;
(2) f is almost contra-precontinuous;
(3) $f^{-1}(V)$ is preopen in X for each θ-semi-open set V of Y;
(4) $f^{-1}(F)$ is preclosed in X for each θ-semi-closed set F of Y.

Proof. (1) $\Rightarrow$ (2): Let F be any regular closed set of Y and $x \in f^{-1}(F)$. Then $f(x) \in F$ and F is semi-open. Since f is (p,s)-continuous, there exists $U \in \mathrm{PO}(X,x)$ such that $f(U) \subset \mathrm{Cl}(F) = F$. Hence we have $x \in U \subset f^{-1}(F)$ which implies that $x \in \mathrm{pInt}(f^{-1}(F))$. Therefore, $f^{-1}(F) \subset \mathrm{pInt}(f^{-1}(F))$ and hence $f^{-1}(F) = \mathrm{pInt}(f^{-1}(F))$. This shows that $f^{-1}(F) \in \mathrm{PO}(X)$. It follows from Theorem 10 of [8] that f is almost contra-precontinuous.
(2) $\Rightarrow$ (3): This follows from the fact that every θ-semi-open set is the union of regular closed sets.
(3) $\Leftrightarrow$ (4): This is obvious.
(4) $\Rightarrow$ (1): Let $x \in X$ and V be any semi-open set of Y containing $f(x)$. Since $\mathrm{Cl}(V)$ is regular closed, it is θ-semi-open. Now, put $U = f^{-1}(\mathrm{Cl}(V))$. Then U is a preopen set of X containing x and $f(U) \subset \mathrm{Cl}(V)$. This shows that f is (p,s)-continuous. □

Theorem 3.2. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*
(1) f is almost contra-precontinuous;
(2) $f(\mathrm{pCl}(A)) \subset \mathrm{sCl}_\theta(f(A))$ for every subset A of X;
(3) $\mathrm{pCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{sCl}_\theta(B))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (2): Let A be any subset of X. Suppose that $x \in \mathrm{pCl}(A)$ and G be any semi-open set of Y containing $f(x)$. Since f is almost contra-precontinuous, by Theorem 3.1 there exists $U \in \mathrm{PO}(X,x)$ such that $f(U) \subset \mathrm{Cl}(G)$. Since $x \in \mathrm{pCl}(A)$, by Lemma 2.1 $U \cap A \neq \emptyset$ and hence $\emptyset \neq f(U) \cap f(A) \subset \mathrm{Cl}(G) \cap f(A)$. Therefore, we obtain $f(x) \in \mathrm{sCl}_\theta(f(A))$ and hence $f(\mathrm{pCl}(A)) \subset \mathrm{sCl}_\theta(f(A))$.
(2) $\Rightarrow$ (3): Let B be any subset of Y. Then

$$f(\mathrm{pCl}(f^{-1}(B))) \subset \mathrm{sCl}_\theta(f(f^{-1}(B))) \subset \mathrm{sCl}_\theta(B)$$

and hence $\mathrm{pCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{sCl}_\theta(B))$.
(3) $\Rightarrow$ (1): Let V be any semi-open set of Y containing $f(x)$. Since $\mathrm{Cl}(V) \cap (Y - \mathrm{Cl}(V)) = \emptyset$, we have $f(x) \notin \mathrm{sCl}_\theta(Y - \mathrm{Cl}(V))$ and hence $x \notin f^{-1}(\mathrm{sCl}_\theta(Y - \mathrm{Cl}(V)))$. By (3), $x \notin \mathrm{pCl}(f^{-1}(Y - \mathrm{Cl}(V)))$. By Lemma 2.1, there exists $U \in \mathrm{PO}(X,x)$ such that $U \cap f^{-1}(Y - \mathrm{Cl}(V)) = \emptyset$; hence $f(U) \cap (Y - \mathrm{Cl}(V)) = \emptyset$. This shows that $f(U) \subset \mathrm{Cl}(V)$. Therefore, by Theorem 3.1 f is almost contra-precontinuous. □

Theorem 3.3. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*
(1) f is almost contra-precontinuous;
(2) $f^{-1}(\mathrm{Cl}(V))$ is preopen in X for every $V \in \beta(Y)$;
(3) $f^{-1}(\mathrm{Cl}(V))$ is preopen in X for every $V \in \mathrm{SO}(Y)$;
(4) $f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))$ is preclosed in X for every $V \in \mathrm{PO}(Y)$.

Proof. (1) $\Rightarrow$ (2): Let V be any β-open set of Y. It follows from Theorem 2.4 of [3] that $\mathrm{Cl}(V)$ is regular closed. Then by Theorem 1 of [8] $f^{-1}(\mathrm{Cl}(V)) \in \mathrm{PO}(X)$.
(2) $\Rightarrow$ (3): This is obvious since $\mathrm{SO}(Y) \subset \beta(Y)$.
(3) $\Rightarrow$ (4): Let $V \in \mathrm{PO}(Y)$. Then $Y - \mathrm{Int}(\mathrm{Cl}(V))$ is regular closed and hence it is semi-open. Then, we have $X - f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)) = f^{-1}(Y - \mathrm{Int}(\mathrm{Cl}(V))) = f^{-1}(\mathrm{Cl}(Y - \mathrm{Int}(\mathrm{Cl}(V)))) \in \mathrm{PO}(X)$. Hence $f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))$ is preclosed in X.
(4) $\Rightarrow$ (1): Let V be any regular open set of Y. Then $V \in \mathrm{PO}(Y)$ and hence $f^{-1}(V) = f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))$ is preclosed in X. □

Lemma 3.1. (Noiri [24]). *For a subset V of a topological space (Y, σ), the following properties hold:*
(1) $\alpha\mathrm{Cl}(V) = \mathrm{Cl}(V)$ *for every* $V \in \beta(Y)$,
(2) $\mathrm{pCl}(V) = \mathrm{Cl}(V)$ *for every* $V \in \mathrm{SO}(Y)$,
(3) $\mathrm{sCl}(V) = \mathrm{Int}(\mathrm{Cl}(V))$ *for every* $V \in \mathrm{PO}(Y)$.

Corollary 3.1. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*
(1) f is almost contra-precontinuous;
(2) $f^{-1}(\alpha\mathrm{Cl}(V))$ *is preopen in X for every* $V \in \beta(Y)$;
(3) $f^{-1}(\mathrm{pCl}(V))$ *is preopen in X for every* $V \in \mathrm{SO}(Y)$;
(4) $f^{-1}(\mathrm{sCl}(V))$ *is preclosed in X for every* $V \in \mathrm{PO}(Y)$.

Proof. This is an immediate consequence of Theorem 3.3 and Lemma 3.1. □

4. Relations to weak forms of continuity

In 1982, Janković [12] introduced the notion of almost weakly continuous functions. It is shown in [30] that almost weak continuity is equivalent to quasi-continuity due to Paul and Bhattacharyya [27]. The properties of almost weakly continuous functions are studied in [10], [22], [27], [28] and [30].

Definition 4.1. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be*
(1) almost precontinuous [20], [31] *(resp.* precontinuous [17]*)* $f^{-1}(V)$ *is preopen in X for every regular open (resp. open) set V of Y,*
(2) almost weakly continuous [12] $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))))$ *for each open set V of Y,*
(3) weakly continuous [15] *if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subset \mathrm{Cl}(V)$.*

Lemma 4.1. (Popa and Noiri [30]). *A function $f : (X, \tau) \to (Y, \sigma)$ is almost weakly continuous if and only if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \mathrm{PO}(X, x)$ such that $f(U) \subset \mathrm{Cl}(V)$.*

Theorem 4.1. *If a function $f : (X, \tau) \to (Y, \sigma)$ is almost contra-precontinuous, then it is almost weakly continuous.*

Proof. Let $x \in X$ and V be any open set of Y containing $f(x)$. Then $\mathrm{Cl}(V)$ is a regular closed set of Y containing $f(x)$. Since f is almost contra-precontinuous, by Theorem 1 of [8] there exists $U \in \mathrm{PO}(X, x)$ such that $f(U) \subset \mathrm{Cl}(V)$. By Lemma 4.1, f is almost weakly continuous. □

It follows that a function $f : (X, \tau) \to (Y, \sigma)$ is (θ, s)-continuous if and only if for each θ-semi-open set V of Y, $f^{-1}(V)$ is open in X. Since every regular closed set is θ-semi-open, by Theorem 4.1 we have the following relations for the functions defined above:

$$\begin{array}{ccccc}
\text{contra-continuity} & \Rightarrow & (\theta, s)\text{-continuity} & \Rightarrow & \text{weak continuity} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{contra-precontinuity} & \Rightarrow & \text{almost contra-precontinuity} & \Rightarrow & \text{almost weak continuity}
\end{array}$$

In the above diagram, the converses of each implications need not be true as shown by the following examples.

Example 4.1. Let $X = \{a,b,c\}, \tau = \{X, \emptyset, \{a,b\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{b,c\}\}$. Then, the identity function $f : (X,\tau) \to (X,\sigma)$ is contra-precontinuous but it is not weakly continuous.

Example 4.2. Let $X = \{a,b,c\}, \tau = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{a,b\}\}$. Then, the identity function $f : (X,\tau) \to (X,\sigma)$ is (θ, s)-continuous but it is not contra-precontinuous.

Example 4.3. Let $\mathbf{R}$ be the reals with the usual topology and $f : \mathbf{R} \to \mathbf{R}$ the identity function. Then f is continuous and hence weakly continuous and almost precontinuous. But f is not almost contra-precontinuous.

Remark 4.1. (1) By Lemma 4.1 and Theorem 4.1, almost weak continuity is implied by both almost contra-precontinuity and almost precontinuity.
(2) It turns out from Example 4.3 and the below example that each one of the following three pairs of functions are independent of each other:
(i) continuity and contra-continuity;
(ii) precontinuity and contra-precontinuity;
(iii) almost precontinuity and almost contra-precontinuity.

Example 4.4. Let $X = \{a,b,c\},\ \tau = \{X, \emptyset, \{a\}, \{b\},\ \{a,b\}, \{a,c\}\}$ and $\sigma = \{X, \emptyset, \{b\}, \{c\}, \{b,c\}\}$. Then, the identity function $f : (X,\tau) \to (X,\sigma)$ is contra-continuous but it is not almost precontinuous.

Corollary 4.1. (Jafari and Noiri [11]). *If a function $f : (X,\tau) \to (Y,\sigma)$ is contra-precontinuous, then f is almost weakly continuous.*

A topological space (X,τ) is said to be *preconnected* [29] if X cannot be written as the union of two nonempty disjoint preopen sets.

Corollary 4.2. (Ekici [8]). *If $f : (X,\tau) \to (Y,\sigma)$ is an almost contra-precontinuous surjection and (X,τ) is preconnected, then (Y,σ) is connected.*

Proof. Since f is almost contra-precontinuous, by Theorem 4.1 f is almost weakly continuous. It follows from Theorem 4.7 of [28] that (Y,σ) is connected. □

A function $f : (X,\tau) \to (Y,\sigma)$ is said to be *M-preopen* [18] if the image of each preopen set of X is preopen in Y. In [31], M-preopen functions are called *almost regular open.*

Corollary 4.3. *If a function $f : (X,\tau) \to (Y,\sigma)$ is M-preopen and almost contra-precontinuous, then f is almost precontinuous.*

Proof. Since f is almost contra-precontinuous, by Theorem 4.1 f is almost weakly continuous. It follows from Theorem 14 of [31] that f is almost precontinuous. □

Corollary 4.4. (Jafari and Noiri [11]). *If a function $f : (X,\tau) \to (Y,\sigma)$ is M-preopen and contra-precontinuous, then f is almost precontinuous.*

A topological space (X,τ) is said to be *almost regular* [32] if for each regular closed set F of X and each $x \in X - F$, there exists disjoint open sets U and V of X such that $x \in U$ and $F \subset V$.

Corollary 4.5. *If a function $f : (X,\tau) \to (Y,\sigma)$ is almost contra-precontinuous and (Y,σ) is almost regular, then f is almost precontinuous.*

Proof. Since f is almost contra-precontinuous, by Theorem 4.1 f is almost weakly continuous. It follows from Corollary 4 of [31] that f is almost precontinuous. □

The *θ-closure* of A, denoted by $\mathrm{Cl}_\theta(A)$, is defined as the set of all points $x \in X$ such that $\mathrm{Cl}(V) \cap A \neq \emptyset$ for every open set V containing x. A subset A is said to be *θ-closed* if $A = \mathrm{Cl}_\theta(A)$ [34]. The complement of a θ-closed set is said to be *θ-open*. It is shown in [34] that $\mathrm{Cl}_\theta(V) = \mathrm{Cl}(V)$ for every open set V of X and $\mathrm{Cl}_\theta(S)$ is closed in X for every subset S of X.

Definition 4.2. *A function $f : (X,\tau) \to (Y,\sigma)$ is said to be* faintly precontinuous [25] *if for each $x \in X$ and each θ-open set V of Y containing $f(x)$, there exists $U \in \mathrm{PO}(X,x)$ such that $f(U) \subset V$.*

Lemma 4.2. (Noiri and Popa [25]). *For a function $f : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*
(1) f is faintly precontinuous;
(2) $f^{-1}(V) \in \mathrm{PO}(X)$ for every θ-open set V of Y;
(3) $f^{-1}(K)$ is preclosed in X for every θ-closed set K of Y.

Theorem 4.2. *Let (Y,σ) be a regular space. Then for a function $f : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent:*
(1) f is precontinuous;
(2) f is almost precontinuous;
(3) f is almost weakly continuous;
(4) f is faintly precontinuous.

Proof. (1) ⇒ (2) and (2) ⇒ (3): The proofs are obvious.
(3) ⇒ (4): Let F be any θ-closed set of Y. It follows from Theorem 3.3 of [30] that $\mathrm{pCl}(f^{-1}(F)) \subset f^{-1}(\mathrm{Cl}_\theta(F)) = f^{-1}(F)$. This implies that $\mathrm{pCl}(f^{-1}(F)) = f^{-1}(F)$. By Lemma 2.1, $f^{-1}(F)$ is preclosed in X. Therefore, by Lemma 4.2 f is faintly precontinuous.
(4) ⇒ (1): Let V be any open set of Y. Since Y is regular, V is θ-open. By Lemma 4.2, $f^{-1}(V) \in \mathrm{PO}(X)$ and hence f is precontinuous. □

Remark 4.2. If (Y,σ) is regular, then by Corollary 4.5 and Theorem 4.2, contra-precontinuity implies precontinuity. Howevery, by Remark 3.1 of [11] every precontinuous function is not always contra-precontinuous even if (Y,σ) is regular.

We recall that a topological space (X,τ) is said to be *extremally disconnected* (briefly E.D.) if the closure of every open set of X is open in X.

Lemma 4.3. (Popa et al. [31]). *A function $f : (X,\tau) \to (Y,\sigma)$ is* almost precontinuous *if and only if for each $x \in X$ and each regular open set V of Y containing $f(x)$, there exists $U \in \mathrm{PO}(X,x)$ such that $f(U) \subset V$.*

Theorem 4.3. *Let (Y, σ) be E.D. Then, a function $f : (X, \tau) \to (Y, \sigma)$ is almost contra-precontinuous if and only if it is almost precontinuous.*

Proof. Necessity. Let $x \in X$ and V be any regular open set of Y containing $f(x)$. Since (Y, σ) is E.D., by Lemma 5.6 of [26] V is clopen and hence V is regular closed. By Theorem 1 of [8], there exists $U \in \mathrm{PO}(X, x)$ such that $f(U) \subset V$. By Lemma 4.3, f is almost precontinuous.
Sufficiency. Let F be any regular closed set of Y. Since (Y, σ) is E.D., F is also regular open and $f^{-1}(F)$ is preopen in X. This shows that f is almost contra-precontinuous. □

5. Strongly contra-preclosed graphs

Definition 5.1. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to have a* strongly contra-preclosed graph *(resp.* contra-preclosed graph [11]*) if for each $(x, y) \in (X \times Y) - G(f)$ there exists $U \in \mathrm{PO}(X, x)$ and a regular closed (resp. closed) set V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.*

Lemma 5.1. A function $f : (X, \tau) \to (Y, \sigma)$ has a strongly contra-preclosed graph if and only if for each $(x, y) \in (X \times Y) - G(f)$ there exists $U \in \mathrm{PO}(X, x)$ and a regular closed set V of Y containing y such that $f(U) \cap V = \emptyset$.

Theorem 5.1. *If a function $f : (X, \tau) \to (Y, \sigma)$ is almost weakly continuous and (Y, σ) is Urysohn, then G(f) is strongly contra-preclosed.*

Proof. Suppose that $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since Y is Urysohn, there exist open sets V and W in Y containing y and $f(x)$, respectively, such that $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$. Since f is almost weakly continuous, by Theorem 3.1 of [30] there exists $U \in \mathrm{PO}(X, x)$ such that $f(U) \subset \mathrm{Cl}(W)$. This shows that $f(U) \cap \mathrm{Cl}(V) = \emptyset$ and hence by Lemma 5.1 $G(f)$ is strongly contra-preclosed. □

Corollary 5.1. *If a function $f : (X, \tau) \to (Y, \sigma)$ is almost contra-precontinuous and (Y, σ) is Urysohn, then G(f) is strongly contra-preclosed.*

Proof. This is an immediate consequence of Theorems 4.1 and 5.1. □

Corollary 5.2. (Jafari and Noiri [11]). *If a function $f : (X, \tau) \to (Y, \sigma)$ is contra-precontinuous and (Y, σ) is Urysohn, then G(f) is contra-preclosed.*

Theorem 5.2. *If a function $f : (X, \tau) \to (Y, \sigma)$ is almost precontinuous and (Y, σ) is Hausdorff, then G(f) is strongly contra-preclosed.*

Proof. Suppose that $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since Y is Hausdorff, there exist open sets V and W in Y containing y and $f(x)$, respectively, such that $V \cap W = \emptyset$; hence $\mathrm{Cl}(V) \cap \mathrm{Int}(\mathrm{Cl}(W)) = \emptyset$. Since f is almost precontinuous, by Lemma 4.3 there exists $U \in \mathrm{PO}(X, x)$ such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(W))$. This shows that $f(U) \cap \mathrm{Cl}(V) = \emptyset$ and hence by Lemma 5.1 $G(f)$ is strongly contra-preclosed. □

Definition 5.2. *A topological space (X, τ) is said to be* pre-T_2 [14] *if for each pair of distinct points x and y in X, there exist disjoint preopen sets containing x and y, respectively.*

Theorem 5.3. *If $f : (X, \tau) \to (Y, \sigma)$ is an injective almost contra-precontinuous function with the strongly contra-preclosed graph, then (X, τ) is pre-T_2.*

Proof. Let x and y be distinct points of X. Then, since f is injective, we have $f(x) \neq f(y)$. Then we have $(x, f(y)) \in (X \times Y) - G(f)$. Since $G(f)$ is strongly contra-preclosed, by Lemma 5.1 there exists $U \in \mathrm{PO}(X, x)$ and a regular closed set V containing $f(y)$ such that $f(U) \cap V = \emptyset$. Since f is almost contra-precontinuous, by Theorem 1 of [8] there exists $G \in \mathrm{PO}(X, y)$ such that $f(G) \subset V$. Therefore, we have $f(U) \cap f(G) = \emptyset$; hence $U \cap G = \emptyset$. This shows that (X, τ) is pre-T_2. □

6. Some other properties

Theorem 6.1. *If for each pair of distinct points x_1 and x_2 in a topological space (X, τ), there exists a function f of (X, τ) into a Urysohn space (Y, σ) such that $f(x_1) \neq f(x_2)$ and f is almost weakly continuous at x_1 and x_2, then (X, τ) is pre-T_2.*

Proof. Let $x_1 \neq x_2$. Then by the hypothesis there exists a function $f : (X, \tau) \to (Y, \sigma)$ which satisfies the condition of this theorem. Since (Y, σ) is Urysohn and $f(x_1) \neq f(x_2)$, there exist open sets V_1 and V_2 containing $f(x_1)$ and $f(x_2)$, respectively, such that $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Since f is almost weakly continuous at x_i, there exists $U_i \in \mathrm{PO}(X, x_i)$ such that $f(U_i) \subset \mathrm{Cl}(V_i)$ for $i = 1, 2$. Hence, we obtain $U_1 \cap U_2 = \emptyset$. Therefore, (X, τ) is pre-T_2. □

Corollary 6.1. *If $f : (X, \tau) \to (Y, \sigma)$ is an almost contra-precontinuous injection and (Y, σ) is Urysohn, then (X, τ) is pre-T_2.*

A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *almost α-continuous* [23] if $f^{-1}(V)$ is α-open in X for every regular open set V of Y. A subset A of a topological space (X, τ) is said to be *p-dense* in X if $\mathrm{pCl}(A) = X$.

Theorem 6.2. *Let $f, g : (X, \tau) \to (Y, \sigma)$ be functions, where (Y, σ) is Hausdorff. If*
(1) f is almost α-continuous,
(2) g is almost contra-precontinuous and
(3) $f = g$ on a p-dense set D of X,
then $f = g$ on X.

Proof. Since g is almost contra-precontinuous, by Theoem 4.1 g is almost weakly continuous. Let $A = \{x \in X : f(x) = g(x)\}$. Then by Theorem 5.3 of [30], A is preclosed in X. On the other hand, $f = g$ on D and hence $D \subset A$. Since D is p-dense in X, $X = \mathrm{pCl}(D) \subset \mathrm{pCl}(A) = A$. Therefore, $f = g$ on X. □

The *prefrontier* [31] of a subset A of a topological space (X, τ), $\mathrm{pFr}(A)$, is defined by $\mathrm{pFr}(A) = \mathrm{pCl}(A) \cap \mathrm{pCl}(X\text{-}A) = \mathrm{pCl}(A)\text{-}\mathrm{pInt}(A)$.

Theorem 6.3. *The set of all points $x \in X$ at which a function $f : (X, \tau) \to (Y, \sigma)$ is not almost contra-precontinuous is identical with the union of the prefrontier of the inverse images of regular closed sets of Y containing f(x).*

Proof. Suppose that f is not almost contra-precontinuous at $x \in X$. By Theorem 1 of [8], there exists a regular closed set F of Y containing $f(x)$ such that $f(U) \cap (Y - F) \neq \emptyset$ for every $U \in \mathrm{PO}(X, x)$. Then by Lemma 2.1, $x \in \mathrm{pCl}(f^{-1}(Y - F)) = \mathrm{pCl}(X - f^{-1}(F))$. On the other hand, we obtain $x \in f^{-1}(F) \subset \mathrm{pCl}(f^{-1}(F))$ and hence $x \in \text{pFr}(f^{-1}(F))$.

Conversely, suppose that f is almost contra-precontinuous at x and let F be any regular closed set of Y containing $f(x)$. By Theorem 1 of [8], there exists $U \in \text{PO}(X, x)$ such that $x \in U \subset f^{-1}(F)$. Therefore, $x \in \text{pInt}(f^{-1}(F))$. This contradicts that $x \in \text{pFr}(f^{-1}(F))$. Thus f is not almost contra-precontinuous. □

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