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Value Distribution of Meromorphic Functions Concerning Differential Polynomial

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Abstract. In this paper, we study the value distribution of meromorphic functions concerning differential polynomial. Moreover, some criteria for normality of families of meromorphic functions are obtained, which extend results respectively established by Chen and Fang, Pang and Zalcman.

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1. Introduction

Let f be a nonconstant meromorphic function. In this paper, we use the following standard notation of value distribution theory,

$$T(r,f),\ m(r,f),\ N(r,f), \overline{N}(r,f),\ N\left(r,\frac{1}{f}\right),\cdots$$

(see Schiff [6], Yang [9]). We denote by $S(r,f)$ any function satisfying

$$S(r,f)=o\{T(r,f)\},$$

as $r\to\infty$, possibly outside of a set with finite measure.

Let k be a positive integer. We denote by $N_{k)}(r,1/f)$ the counting function of those zeros of f whose multiplicity are less than or equal to k; by $N_{(k+1}(r,1/f)$ the counting function of those zeros of f whose multiplicity are greater than k. Moreover, we call a meromorphic function $\varphi(z)(\not\equiv 0,\infty)$ a small function with respect to f provided that $T(r,\varphi)=S(r,f)$.

Let D be a domain in $\mathbf{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined on D. $\mathcal{F}$ is said to be normal on D, in the sence of Montel, if for every sequence f_n there exists a subsequence f_{n_j}, such that f_{n_j} spherically converges, locally uniformly in D, to a meromorphic function or ∞. (See [6].)

In [3], Hayman posed the following conjecture.

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Conjecture 1.1 (Hayman conjecture)**.** *Let $\mathcal{F}$ be a family of meromorphic functions on D, let n be a positive integer and a be a nonzero complex number, if $f^n f' \neq a$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.*

A number of researchers have worked on this problem. In 1995, Chen Huaihui and Fang Mingliang [2] confirmed the conjecture by proving

Theorem 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions on D, let n be a positive integer and a be a nonzero complex number, if $f^n f' \neq a$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.*

In [5], Pang Xue Cheng and Lawrence Zalcman obtain a criteria for normality of families of holomorphic functions, they proved that

Theorem 1.2. *Let $\mathcal{F}$ be a family of holomorphic functions on D, all of whose zeros have multiplicity at least k, and there exists $n \geq 1$ and $a \in C$, $a \neq 0$, such that $f^n f^{(k)} \neq a$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is a normal family.*

In this paper, we shall generalize and improve above results and prove

Theorem 1.3. *Let $\mathcal{F}$ be a family of meromorphic functions on D, all of whose zeros have multiplicity at least k, let $n(\geq 2), k$ be positive integers, $h(z) \neq 0$ for $z \in D$, $h(z), a_1, a_2 \cdots, a_k$ be holomorphic functions on D. We define*

$$E(f) = \left\{ z : f^n \{ f^{(k)}(z) + \sum_{i=1}^{k} a_i(z) f^{(k-i)}(z) \} = h(z), \quad z \in D \right\}.$$

If there exists a constant $M > 0$ such that for any $f \in \mathcal{F}$ and $z \in E(f)$, we have

$$|f^{(k)}(z)| \leq M,$$

then $\mathcal{F}$ is normal on D.

Remark 1.1. The condition that the zeros of f in $\mathcal{F}$ have multiplicity at least k is necessary, as is shown by the following example.

Let $\mathcal{F}$=$\{mz^{k-1} : m = 1, 2, \cdots\}$. Let $D = \Delta$, the unit disk. Let $k \geq 2$, n be positive integers. Then for any $f \in \mathcal{F}$, $f^n f^{(k)} \neq 1$. Hence f satisfies $|f^{(k)}(z)| \leq M$ whenever $f^n f^{(k)} = 1$, where $M > 0$. But $\mathcal{F}$ is not normal at a point 0 on Δ.

As an immediate consequence of Theorem 1.3, we have the

Corollary 1.1. *Let $\mathcal{F}$ be a family of meromorphic functions on D, let $n(\geq 2), k$ be positive integers, $h(z) \neq 0$ for $z \in D$, $h(z), a_1, a_2, \cdots, a_k$ be holomorphic functions on D. We define*

$$E(f) = \left\{ z : f^n \{ f^{(k)}(z) + \sum_{i=1}^{k} a_i(z) f^{(k-i)}(z) \} = h(z), \quad z \in D \right\}.$$

If there exists a constant $M > 0$ such that for any $f \in \mathcal{F}$, $f \neq 0$ and $z \in E(f)$, we have

$$|f^{(k)}(z)| \leq M,$$

then $\mathcal{F}$ is normal on D.

Corollary 1.2. *Let $\mathcal{F}$ be a family of meromorphic functions on D, all of whose zeros have multiplicity at least k, let $n(\geq 2), k$ be positive integers, let $h(z)(\neq 0), a_1, a_2, \cdots, a_k$ be holomorphic functions on D. If for any $f \in \mathcal{F}$,*

$$f^n\{f^{(k)}(z) + \sum_{i=1}^{k} a_i(z) f^{(k-i)}(z)\} \neq h(z), \quad z \in D,$$

then $\mathcal{F}$ is normal on D.

Corollary 1.3. *Let $\mathcal{F}$ be a family of meromorphic functions on D, let $n(\geq 2), k$ be positive integers, let $h(z)(\neq 0), a_1, a_2, \cdots, a_k$ be holomorphic functions on D. If for any $f \in \mathcal{F}$, $f \neq 0$,*

$$f^n\{f^{(k)}(z) + \sum_{i=1}^{k} a_i(z) f^{(k-i)}(z)\} \neq h(z), \quad z \in D,$$

then $\mathcal{F}$ is normal on D.

Corollary 1.4. *Let $\mathcal{F}$ be a family of meromorphic functions on D, all of whose zeros have multiplicity at least k, let k and $n(\geq 2)$ be positive integers, let $h(z)(\neq 0)$ be a holomorphic function on D. If for any $f \in \mathcal{F}$,$f^n f^{(k)} \neq h(z)$ on D, then $\mathcal{F}$ is normal on D.*

Requiring that f have pole points with multiplicity at least 3, we have

Theorem 1.4. *Let $\mathcal{F}$ be a family of meromorphic functions on D, all of poles are of multiplicity at least 3 and whose zeros are of multiplicity at least k, let n,k be positive integers and $h(z) \neq 0$ for $z \in D$, $h(z), a_1, a_2, \cdots, a_k$ be holomorphic functions on D. We define*

$$E(f) = \left\{ z : f^n\{f^{(k)}(z) + \sum_{i=1}^{k} a_i(z) f^{(k-i)}(z)\} = h(z), \quad z \in D \right\}.$$

If there exists a constant $M > 0$ such that for any $f \in \mathcal{F}$ and $z \in E(f)$, we have

$$|f^{(k)}(z)| \leq M$$

then $\mathcal{F}$ is normal on D.

From Theorem 1.4, we immediately have the following result:

Theorem 1.5. *Let $\mathcal{F}$ be a family of holomorphic functions on D, all of zeros are of multiplicity at least k, let n,k be positive integers and $h(z) \neq 0$ for $z \in D$, $h(z), a_1, a_2, \cdots, a_k$ be holomorphic functions on D. We define*

$$E(f) = \left\{ z : f^n\{f^{(k)}(z) + \sum_{i=1}^{k} a_i(z) f^{(k-i)}(z)\} = h(z), \quad z \in D \right\}.$$

If there exists a constant $M > 0$ such that for any $f \in \mathcal{F}$ and $z \in E(f)$, we have

$$|f^{(k)}(z)| \leq M,$$

then $\mathcal{F}$ is normal on D.

Remark 1.2. It is easily seen that Theorem 1.2 is a consequence of Theorem 1.5.

2. Some lemmas

For the proof of our results, we need the following lemmas.

Lemma 2.1. [1, 8] *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ all of whose zeros have multiplicity at least k, let k be a positive integer and let α be a real number satisfying $0 \le \alpha < k$. Then $\mathcal{F}$ is not normal on Δ if and only if there exists*

(a) *a number r, $0 < r < 1$;*
(b) *points z_j, $|z_j| < r$;*
(c) *functions $f_j \in \mathcal{F}$;*
(d) *positive numbers $\rho_j \to 0$;*

such that

$$\frac{f_j(z_j + \rho_j \xi)}{\rho_j^{\alpha}} \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbf{C}$ such that $g^{\sharp}(\xi) \le g^{\sharp}(0) = 1$.

Lemma 2.2. [4] *Let f be a transcendental meromorphic function on C, let $F = f^n f^{(k)} - c$, $c \neq 0, \infty$ be a complex number, then for any positive integer $n(\ge 2)$, there exists $M > 0$ such that*

$$T(r, f) < M\overline{N}(r, \frac{1}{F}) + S(r, f).$$

Lemma 2.3. *Let f be a meromorphic function in C. Let n be a positive integer, let a be a nonzero finite complex number, if $f^n f' \neq a$, then f is a constant.*

Proof. Suppose that f is not a constant. Then we can find $z_0 \in \mathbf{C}$ such that $f^{\sharp}(z_0) \neq 0$. Let $f_j(z) = j^{-\frac{1}{n+1}} f(z_0 + jz)$ for $z \in \Delta$. Clearly, $f_j'(z) f_j^n(z) \neq a$ on Δ, so the functions f_j belong to the family of Theorem 1.1. By Marty's theorem, there exists M such that $f_j^{\sharp}(0) \le M$ for all j. But $f_j^{\sharp}(0) \ge j^{\frac{n}{1+n}} f^{\sharp}(z_0)$, which tends to ∞ with j, a contradiction. Thus Lemma 2.3 is proved. □

Lemma 2.4. [7] *Let f be a transcendental meromorphic function and n, k be positive integers, and let $c(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that all poles of f have multiplicity at least $s = 2 + [1/n]$ and that $uN_{1)}(r, 1/f) \le \lambda T(r, f)$, then $f^n(z) f^{(k)}(z) - c(z)$ has infinitely many zeros, where $u, \lambda(< 1/3)$ are constants, and $u = 0$ if $n \ge 2, u = 1$ otherwise.*

Remark 2.1. From Lemma 2.4, we can see that the following results is obviously true: Let f be a transcendental meromorphic function and $k(\ge 2)$ be a positive integer, let a be a nonzero complex number. Suppose that all poles of f have multiplicity at least 3, and all zeros of f have multiplicity at least k, then $ff^{(k)} - a$ has infinitely many zeros.

Lemma 2.5. *Let f be a rational function on C, let n, k be positive integers, and the zeros of f be of multiplicity at least k. If $f^n f^{(k)} \neq a$, where $a(\neq 0)$ is a complex number, then f is a constant.*

Proof. Suppose f is a nonconstant rational function, we assume $f(z) = Q(z)/P(z)$, where $Q(z)$ and $P(z)$ are two coprime polynomials. Set $p = \deg(P)$, $q = \deg(Q)$. Since $f^n f^{(k)} \neq a$, we can deduce that there exists polynomial $h(z)$ such that

$$f^n f^{(k)} = a + \frac{1}{h(z)} = \frac{ah(z)+1}{h(z)}. \tag{2.1}$$

So we have

$$\deg(ah(z)+1) - \deg h(z) = (n+1)(q-p) - k.$$

From (2.1), we have $k = (n+1)(q-p)$, and $q - p \geq 1$.

Set $m = q - p$, then

$$f(z) = a_0 z^m + \cdots + a_m + \frac{R(z)}{P(z)},$$

where $R(z)$ and $P(z)$ be two coprime polynomials and

$$\deg(P) - \deg(R) > 0.$$

Obviously,

$$f^{(k)}(z) = \left(\frac{R(z)}{P(z)}\right)^{(k)},$$

then from (2.1), we can obtain that

$$\deg(P) - \deg(R) = -m,$$

which contradicts that $\deg(P) - \deg(R) > 0$. Thus the proof of this lemma is completed. □

3. Proof of the theorems

Proof of Theorem 1.3. Without loss of generality, we may assume that $D = \Delta$. Suppose that $\mathcal{F}$ is not normal at a point $z_0 \in \Delta$, then by Lemma 2.1, there exists $\{f_j\} \in \mathcal{F}$,$z_j \to z_0$ and $\rho_j \to 0^+$ such that

$$g_j(\xi) = \frac{f_j(z_j + \rho_j \xi)}{\rho_j^{\frac{k}{n+1}}} \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a non-constant meromorphic function satisfying $g^\sharp(\xi) \leq g^\sharp(0) = 1$. Since $g_j(\xi)$ has only zeros of multiplicity at least k, by Hurwitz's theorem, the zeros of $g(\xi)$ are of multiplicity at least k. Obviously, $h(z_0) \neq 0, \infty$.

Next we claim $g^n g^{(k)} - h(z_0) = 0$ has a solution in C.

If g is a rational function and $g^n g^{(k)} \neq h(z_0)$, by Lemma 2.5 g is a constant, which is a contradiction. If g is a transcendental meromorphic function, then by Lemma 2.2, there exists ξ_0 such that $g^n(\xi_0) g^{(k)}(\xi_0) = h(z_0)$. For otherwise $T(r, g) < S(r, g)$, which contradicts that g is a nonconstant transcendental meromorphic function. Thus $g^n g^{(k)} - h(z_0) = 0$ has a solution in C. Without loss of generality, we may assume there exists a solution ξ_0 such that $g^n(\xi_0) g^{(k)}(\xi_0) = h(z_0)$.

Since $g(\xi_0) \neq \infty$, hence there exists $\delta > 0$ such that $g(\xi)$ is analytic on $D_{2\delta}(\xi : |\xi - \xi_0| < 2\delta)$. Thus $g_j^{(i)}(\xi)$ are analytic on $D_\delta(\xi : |\xi - \xi_0| < \delta)$ for sufficiently large j

and $g_j^{(i)}(\xi)$ converges uniformly to $g^{(i)}(\xi)$ on $D_\delta(\xi : |\xi-\xi_0|<\delta)(i=0,1,2,\cdots,k)$. As

$$\begin{aligned}
&g_j^n(\xi)g_j^{(k)}(\xi)-h(z_j+\rho_j\xi)\\
&=f_j^n(z_j+\rho_j\xi)\{f_j^{(k)}(z_j+\rho_j\xi)+a_1(z_j+\rho_j\xi)f_j^{(k-1)}(z_j+\rho_j\xi)\\
&\qquad\qquad+\cdots+a_k(z_j+\rho_j\xi)f_j(z_j+\rho_j\xi)\}-h(z_j+\rho_j\xi)\\
&\quad-f_j^n(z_j+\rho_j\xi)\{a_1(z_j+\rho_j\xi)f_j^{(k-1)}(z_j+\rho_j\xi)+\cdots+a_k(z_j+\rho_j\xi)f_j(z_j+\rho_j\xi)\}\\
&=f_j^n(z_j+\rho_j\xi)\{f_j^{(k)}(z_j+\rho_j\xi)+a_1(z_j+\rho_j\xi)f_j^{(k-1)}(z_j+\rho_j\xi)\\
&\qquad\qquad+\cdots+a_k(z_j+\rho_j\xi)f_j(z_j+\rho_j\xi)\}-h(z_j+\rho_j\xi)\\
&\quad-\sum_{m=0}^{k-1}a_{k-m}(z_j+\rho_j\xi)\rho_j^{k-m}g_j^n(\xi)g_j^{(m)}(\xi).
\end{aligned}$$

Considering $a_{k-m}(z)$ $(m=0,1,\cdots k-1)$ are analytic on Δ, $z_j\to z_0$ and $\rho_j\to 0^+$, $g_j^{(i)}(\xi)$ are analytic on $D_\delta(\xi:|\xi-\xi_0|<\delta)$ for sufficiently large j, we have

$$|a_{k-m}(z_j+\rho_j\xi)|\le M(z_0)<+\infty \quad (m=0,1,\cdots k-1)$$

for j large enough, where $M(z_0)$ is a constant, and we can deduce that

$$\sum_{m=0}^{k-1}a_{k-m}(z_j+\rho_j\xi)\rho_j^{k-m}g_j^n(\xi)g_j^{(m)}(\xi)$$

converges uniformly to 0 on $D_{\frac{1}{2}\delta}(\xi:|\xi-\xi_0|<\frac{1}{2}\delta)$. Thus we know that

$$\begin{aligned}
&f_j^n(z_j+\rho_j\xi)\{f_j^{(k)}(z_j+\rho_j\xi)+a_1(z_j+\rho_j\xi)f_j^{(k-1)}(z_j+\rho_j\xi)+\\
&\qquad\cdots+a_k(z_j+\rho_j\xi)f_j(z_j+\rho_j\xi)\}-h(z_j+\rho_j\xi)\\
&\qquad\qquad=g_j^n(\xi)g_j^{(k)}(\xi)-h(z_j+\rho_j\xi)+\sum_{m=0}^{k-1}a_{k-m}(z_j+\rho_j\xi)\rho_j^{k-m}g_j^n(\xi)g_j^{(m)}(\xi)
\end{aligned}$$

converges uniformly to $g^n(\xi)g^{(k)}(\xi)-h(z_0)$ on $D_{\frac{1}{2}\delta}(\xi:|\xi-\xi_0|<\frac{1}{2}\delta)$.

Since $g^n(\xi_0)g^{(k)}(\xi_0)-h(z_0)=0$, if $g^n(\xi)g^{(k)}(\xi)\equiv h(z_0)$, then $g\neq 0$, so g is an entire function and hence of exponential type. Hence $g(\xi)=Ae^{c\xi}$, where $A\neq 0, c\neq 0$. But then $g^n(\xi)g^{(k)}(\xi)=c^kA^{n+1}e^{(n+1)c\xi}$, which contradicts $g^ng^{(k)}\equiv h(z_0)$. Thus $g^n(\xi_0)g^{(k)}(\xi_o)=h(z_0)$, and $g^n(\xi)g^{(k)}(\xi)-h(z_0)\not\equiv 0$. By Hurwitz Theorem, there exists $\xi_j,\xi_j\to\xi_0$, such that for j large enough,

$$\begin{aligned}
&f_j^n(z_j+\rho_j\xi_j)\{f_j^{(k)}(z_j+\rho_j\xi_j)+a_1(z_j+\rho_j\xi_j)f_j^{(k-1)}(z_j+\rho_j\xi_j)\\
&\qquad\qquad+\cdots+a_k(z_j+\rho_j\xi_j)f_j(z_j+\rho_j\xi_j)\}-h(z_j+\rho_j\xi_j)\\
&=g_j^{(k)}(\xi_j)g_j^n(\xi_j)+\sum_{m=0}^{k-1}a_{k-m}(z_j+\rho_j\xi)\rho_j^{k-m}g_j^n(\xi)g_j^{(m)}(\xi)-h(z_j+\rho_j\xi_j)=0
\end{aligned}$$

on $D_{\frac{1}{2}\delta}(\xi : |\xi - \xi_0| < \frac{1}{2}\delta)$. i.e. (for j large enough)

$$f_j^n(z_j+\rho_j\xi_j)\{f_j^{(k)}(z_j+\rho_j\xi_j)+a_1(z_j+\rho_j\xi_j)f_j^{(k-1)}(z_j+\rho_j\xi_j) \\ +\cdots+a_k(z_j+\rho_j\xi_j)f_j(z_j+\rho_j\xi_j)\}=h(z_j+\rho_j\xi_j)$$

on $D_{\frac{1}{2}\delta}(\xi : |\xi - \xi_0| < \frac{1}{2}\delta)$.

By assumption, we have

$$|g_j^{(k)}(\xi_j)| = \rho_j^{k-\frac{k}{n+1}}|f_j^{(k)}(z_j+\rho_j\xi_j)| \le \rho_j^{k-\frac{k}{n+1}}M$$

on $D_{\frac{1}{2}\delta}(\xi : |\xi - \xi_0| < \frac{1}{2}\delta)$. Thus

$$g^{(k)}(\xi_0) = \lim_{j\to\infty} g_j^{(k)}(\xi_j) = 0$$

on $D_{\frac{1}{2}\delta}(\xi : |\xi - \xi_0| < \frac{1}{2}\delta)$, which contradicts that $g^{(k)}(\xi_0)g^n(\xi_0) = h(z_0) \neq 0$. Thus the proof of Theorem 1.3 is complete. □

Theorem 1.4 can be proved by a similar manner using Lemmas 2.2, 2.3, 2.4 and 2.5. We omit the details.

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