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On Some Difference Sequence Sets and Their Topological Properties

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Abstract. The idea of difference sequence sets, $X(\Delta)=\{x=(x_k):\Delta x\in X\}$, where $X=\ell_\infty$, c and c_0 was introduced by Kızmaz [4], and then this subject has been studied and generalized by various mathematicians. In this study, we define a new sequence space denoted by $m(\phi,p)(\Delta^r)$ and give some properties of this sequence space. The obtained results generalize some known results.

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1. Introduction

Let w denote the space of all complex sequences and ℓ_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x=(x_k)$ with complex terms, respectively, normed by $\|x\|_\infty=\sup_k|x_k|$, where $k\in\mathbb{N}=\{1,2,\ldots\}$.

A sequence space X is called a K-space provided each of the maps $\tau_k: X\to\mathbb{C}$ (= complex numbers) defined by $\tau_k(x)=x_k$ $(k=1,2,\ldots)$ is continuous. A K-space X is called a BK-space provided X is a Banach space.

The difference sequence spaces were introduced by Kızmaz [4]. The notion of difference sequence spaces was generalized by Et and Çolak [3] as follows:

$$X(\Delta^r)=\{x\in w:\Delta^r x\in X\}, \tag{1}$$

for $X=\ell_\infty$, c and c_0, where $r\in\mathbb{N}$, $\Delta^0x=x$, $\Delta x=(x_k-x_{k+1})$, $\Delta^r x=\left(\Delta^{r-1}x_k-\Delta^{r-1}x_{k+1}\right)$, and so $\Delta^r x_k=\sum_{v=0}^{r}(-1)^v\binom{r}{v}x_{k+v}$. These sequence spaces are BK-spaces with the norm $\|x\|_\Delta=\sum_{i=1}^{r}|x_i|+\|\Delta^r x\|_\infty$.

Subsequently difference sequence spaces have been studied by Çolak and Et [1], Et and Basarır [2], Malkowsky and Parashar [6], Mursaleen [7] and many others.

Let X be a sequence space, then X is called perfect if $X=X^{\alpha\alpha}$; solid (or normal) if $(\alpha_k x_k)\in X$ whenever, $(x_k)\in X$ for all sequences (α_k) of scalars with $|\alpha_k|\le 1$ for all $k\in\mathbb{N}$ and symmetric if $(x_k)\in X$ implies $\left(x_{\pi(k)}\right)\in X$, where $\pi(k)$ is a permutation of $\mathbb{N}$, and $X^{\alpha\alpha}=(X^\alpha)^\alpha$ denotes the second α-dual of X (for definition of X^α see [1]).

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2. Main results

In this section we introduce a new class of sequences and establish some inclusion relations. Also we show that this space is not perfect and not normal. The obtained results are more general than those of Sargent [8] and Tripathy and Sen [10].

Throughout this section φ_s denotes the class of all subsets of $\mathbb{N}$, those do not contain more than s elements. Let (ϕ_n) be a non-decreasing sequence of positive numbers such that $n\phi_{n+1} \leq (n+1)\,\phi_n$ for all $n \in \mathbb{N}$. The class of all sequences (ϕ_n) is denoted by Φ. The sequence space $m(\phi)$ was introduced by Sargent [8] and he studied some of its properties and obtained some relations with the space l^p. Later on it was investigated by Tripathy [9], Tripathy and Sen [10] and Malkowsky and Mursaleen [5] from the point of view of sequence spaces.

Let r be a fixed positive integer and $0 \leq p < \infty$. Now we define the sequence space $m(\phi, p)(\Delta^r)$ as follows:

$$m(\phi,p)(\Delta^r) = \left\{ x \in w : \sup_{s\geq 1,\ \sigma\in\varphi_s} \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^r x_n|^p < \infty \right\}.$$

From the definition it is clear that $m(\phi,p)\left(\Delta^0\right) = m(\phi,p)$ and $m(\phi,1)\left(\Delta^0\right) = m(\phi)$. In the case $p=1$, we shall write $m(\phi)(\Delta^r)$ instead of $m(\phi,1)(\Delta^r)$.

Theorem 2.1. *For any $\phi \in \Phi$ the space $m(\phi,p)(\Delta^r)$ is a Banach space with the norm*

$$\|x\|_{\Delta_1} = \sum_{i=1}^{r} |x_i| + \sup_{s\geq 1,\ \sigma\in\varphi_s} \frac{1}{\phi_s} \left(\sum_{n\in\sigma} |\Delta^r x_n|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \tag{2}$$

and a complete p-normed space by p-norm

$$\|x\|_{\Delta_2} = \sum_{i=1}^{r} |x_i|^p + \sup_{s\geq 1,\ \sigma\in\varphi_s} \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^r x_n|^p, \quad 0 < p < 1. \tag{3}$$

Proof. It is a routine verification that $m(\phi,p)(\Delta^r)$ is a normed linear space normed by (2) for $1 \leq p < \infty$ and a p-normed space by p-norm (3) for $0 < p < 1$. To show that $m(\phi,p)(\Delta^r)$ is complete, let $\left(x^l\right)$ be a Cauchy sequence in $m(\phi,p)(\Delta^r)$, $(1 \leq p < \infty)$, where $x^l = \left(x_k^l\right)_k = \left(x_1^l, x_2^l, \ldots\right) \in m(\phi,p)(\Delta^r)$ for each $l \in \mathbb{N}$. Then given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\left\|x^l - x^t\right\|_{\Delta_1} = \sum_{i=1}^{r} \left|x_i^l - x_i^t\right| + \sup_{s\geq 1,\ \sigma\in\varphi_s} \frac{1}{\phi_s} \left(\sum_{k\in\sigma} \left|\Delta^r \left(x_k^l - x_k^t\right)\right|^p \right)^{\frac{1}{p}} < \varepsilon \tag{4}$$

for all $l, t > n_0$. Hence we obtain

$$\left|x_k^l - x_k^t\right| \to 0, \text{ as } l,t \to \infty, \text{for each } k \in \mathbb{N}.$$

Therefore $\left(x_k^l\right)_l = \left(x_k^1, x_k^2, \ldots\right)$ is a Cauchy sequence in $\mathbb{C}$ for each k. Since $\mathbb{C}$ is complete, it is convergent

$$\lim_l x_k^l = x_k$$

say, for each $k \in \mathbb{N}$. Taking limit as $t \to \infty$ in (4), we get

$$\sum_{i=1}^{r} \left| x_i^l - x_i \right| + \sup_{s \geq 1,\ \sigma \in \varphi_s} \frac{1}{\phi_s} \left(\sum_{k \in \sigma} \left| \Delta^r \left(x_k^l - x_k \right) \right|^p \right)^{\frac{1}{p}} < \varepsilon$$

for all $l > n_0$. Hence $\left(x_k^l - x_k \right) \in m(\phi, p)(\Delta^r)$. Since $m(\phi, p)(\Delta^r)$ is a linear space and $\left(x_k^l \right), \left(x_k^l - x_k \right)$ are in $m(\phi, p)(\Delta^r)$, it follows that

$$(x_k) = \left(x_k^l \right) - \left(x_k^l - x_k \right) \in m(\phi, p)(\Delta^r).$$

Therefore $m(\phi, p)(\Delta^r)$ is complete.

It can similarly be shown that $m(\phi, p)(\Delta^r)$ is complete space p-normed by (3) for $0 < p < 1$. □

Theorem 2.2. *For any $\phi \in \Phi$ the space $m(\phi, p)(\Delta^r)$ is a K-space.*

Proof. Omitted. □

Theorem 2.3. *$m(\phi)(\Delta^r) \subset m(\phi, p)(\Delta^r)$, for any $\phi \in \Phi$.*

Proof. Let $x \in m(\phi)(\Delta^r)$. Then there is a positive number K such that

$$\sum_{n \in \sigma} |\Delta^r x_n| \leq K\phi_s, \quad \sigma \in \varphi_s$$

for each fixed s. Hence $\sum_{n \in \sigma} |\Delta^r x_n|^p < K\phi_s$, for each $p > 0$ and $\sigma \in \varphi_s$. Thus $x \in m(\phi, p)(\Delta^r)$. □

Theorem 2.4. *For any two sequences (ϕ_s) and (ψ_s) of real numbers*

$$m(\phi, p)(\Delta^r) \subset m(\psi, p)(\Delta^r)$$

if and only if

$$\sup_{s \geq 1} \left(\frac{\phi_s}{\psi_s} \right) < \infty.$$

Proof. Let $x \in m(\phi, p)(\Delta^r)$. Then

$$\sup_{s \geq 1,\ \sigma \in \varphi_s} \frac{1}{\phi_s} \sum_{n \in \sigma} |\Delta^r x_n|^p < \infty.$$

Suppose that $\sup_{s \geq 1} \left(\frac{\phi_s}{\psi_s} \right) < \infty$. Then $\phi_s \leq K\psi_s$ and so that $\frac{1}{\psi_s} \leq \frac{K}{\phi_s}$ for some positive number K and for all s. Therefore we have

$$\frac{1}{\psi_s} \sum_{n \in \sigma} |\Delta^r x_n|^p \leq \frac{K}{\phi_s} \sum_{n \in \sigma} |\Delta^r x_n|^p$$

for each s. Now taking supremum over $s \geq 1$ and $\sigma \in \varphi_s$ we get

$$\sup_{s \geq 1,\ \sigma \in \varphi_s} \frac{1}{\psi_s} \sum_{n \in \sigma} |\Delta^r x_n|^p \leq K \sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\phi_s} \sum_{n \in \sigma} |\Delta^r x_n|^p$$

and hence

$$\sup_{s \geq 1,\ \sigma \in \varphi_s} \frac{1}{\psi_s} \sum_{n \in \sigma} |\Delta^r x_n|^p < \infty.$$

Therefore $x \in m(\psi, p)(\Delta^r)$.

Conversely let $m(\phi, p)(\Delta^r) \subset m(\psi, p)(\Delta^r)$ and suppose that $\sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) = \infty$. Then there exists an increasing sequence (s_i) of naturals numbers such that $\lim_i \left(\frac{\phi_{s_i}}{\psi_{s_i}}\right) = \infty$. Now for every $B \in \mathbb{R}_+$, the set of positive real numbers, there exists $i_0 \in \mathbb{N}$ such that $\frac{\phi_{s_i}}{\psi_{s_i}} > B$ for all $s_i \geq i_0$. Hence $\frac{1}{\psi_{s_i}} > \frac{B}{\phi_{s_i}}$ and so that

$$\frac{1}{\psi_{s_i}} \sum_{n\in\sigma} |\Delta^r x_n|^p > \frac{B}{\phi_{s_i}} \sum_{n\in\sigma} |\Delta^r x_n|^p$$

for all $s_i \geq i_0$. Now taking supremum over $s_i \geq i_0$ and $\sigma \in \varphi_s$ we get

$$\sup_{s_i\geq i_0,\ \sigma\in\varphi_s} \frac{1}{\psi_{s_i}} \sum_{n\in\sigma} |\Delta^r x_n|^p > B \sup_{s_i\geq i_0,\ \sigma\in\varphi_s} \frac{1}{\phi_{s_i}} \sum_{n\in\sigma} |\Delta^r x_n|^p . \tag{5}$$

Since (5) holds for all $B \in \mathbb{R}_+$ (we may take the number B sufficiently large) we have

$$\sup_{s_i\geq i_0,\ \sigma\in\varphi_s} \frac{1}{\psi_{s_i}} \sum_{n\in\sigma} |\Delta^r x_n|^p = \infty$$

when $x \in m(\phi, p)(\Delta^r)$ with

$$0 < \sup_{s\geq 1,\ \sigma\in\varphi_s} \frac{1}{\phi_s} \sum_{n\in\sigma} |\Delta^r x_n|^p < \infty.$$

Therefore $x \notin m(\psi, p)(\Delta^r)$. This contradicts to $m(\phi, p)(\Delta^r) \subset m(\psi, p)(\Delta^r)$. Hence $\sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty$. □

From Theorem 2.4, we get the following result.

Corollary 2.1. *$m(\phi, p)(\Delta^r) = m(\psi, p)(\Delta^r)$ if and only if*

$$0 < \inf_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) \leq \sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right) < \infty.$$

Theorem 2.5. *$m(\phi, p)\left(\Delta^{r-1}\right) \subset m(\phi, p)(\Delta^r)$ and the inclusion is strict.*

Proof. Proof follows from the following inequality and Minkowski's inequality

$$|\Delta^r x| = \left|\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1}\right| \leq \left|\Delta^{r-1} x_k\right| + \left|\Delta^{r-1} x_{k+1}\right|.$$

To show the inclusion is strict consider the following example.

Example 2.1. Let $\phi_n = 1$, for all $n \in \mathbb{N}$ and $x = \left(k^{r-1}\right)$, then

$$x \in \ell_p(\Delta^r) \backslash \ell_p\left(\Delta^{r-1}\right).$$

□

Theorem 2.6. *The sequence space $m(\phi, p)(\Delta^r)$ is not sequence algebra, is not solid and is not symmetric, for $r \geq 1$.*

Proof. For the proof of this theorem, consider the following examples:

Example 2.2. It is obvious that $x = \left(k^{r-1}\right) \in m(\phi, p)(\Delta^r)$, but

$$\alpha x = (\alpha_k x_k) \notin m(\phi, p)(\Delta^r) \quad \text{for} \quad \alpha = (\alpha_k) = \left((-1)^k\right).$$

Hence $m(\phi, p)(\Delta^r)$ is not solid.

Example 2.3. Let us consider the sequence $x = \left(k^{r-1}\right) \in m(\phi, p)(\Delta^r)$. Let (y_k) be a rearrangement of (x_k) which is defined as follows:

$$y_k = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, \ldots\}.$$

Then $y \notin m(\phi, p)(\Delta^r)$. Hence $m(\phi, p)(\Delta^r)$ is not symmetric.

Example 2.4. Let $x = \left(k^{r-1}\right)$ and $y = \left(k^{r-1}\right)$. Then $x, y \in m(\phi, p)(\Delta^r)$, but $x.y \notin m(\phi, p)(\Delta^r)$. Hence $m(\phi, p)(\Delta^r)$ is not sequence algebra.

□

The following result is a consequence of Theorem 2.6.

Corollary 2.2. *The sequence space* $m(\phi, p)(\Delta^r)$ *is not perfect.*

Theorem 2.7. $\ell_p(\Delta^r) \subset m(\phi, p)(\Delta^r) \subset \ell_\infty(\Delta^r)$.

Proof. Since $m(\phi, p)(\Delta^r) = \ell_p(\Delta^r)$ for $\phi_n = 1$, for all $n \in \mathbb{N}$, then

$$\ell_p(\Delta^r) \subset m(\phi, p)(\Delta^r).$$

Now assume that $x \in m(\phi, p)(\Delta^r)$. Then we have

$$\sup_{s \geq 1,\ \sigma \in \varphi_s} \frac{1}{\phi_s} \sum_{n \in \sigma} |\Delta^r x_n|^p < \infty$$

and so $|\Delta^r x_n| < K\phi_1$, for all $n \in \mathbb{N}$ and for some positive number K. Thus $x \in \ell_\infty(\Delta^r)$. □

Corollary 2.3. *If* $0 < p < q$, *then* $m(\phi, p)(\Delta^r) \subset m(\phi, q)(\Delta^r)$.

Proof. Proof follows from the following inequality

$$\left(\sum_{k=1}^{n} |x_k|^q\right)^{\frac{1}{q}} \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}, \quad (0 < p < q).$$

□

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