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Approximating Fisher's Information for the Replicated Linear Circular Functional Relationship Model

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Abstract. The problem that this paper attempting to solve is the derivation of Fisher's information matrix using four parameters which are two error concentration parameters of variables, intercept and slope parameter for the replicated linear circular functional relationship model. The model is formulated assuming both variables are circular, subject to errors and there is a linear relationship between them. The maximum likelihood estimation have been used to estimate all the parameters. It is shown that estimate of Fisher's information can be obtained by using various theories of matrices and approximation of the asymptotic properties of Bassel function.

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1. Introduction

The functional relationship model is an extension of a regression model which allows for sampling variability in the measurements of both the response and explanatory variables. This relationship arises from data or observation that have been obtained or measured from continuous linear variables subject to some sort of errors, e.g. observational or individual error variability. This model have been explored since the later part of the 19^{th} century when Adcock [1, 2] investigated estimation properties under somewhat restrictive but realistic assumption in ordinary linear regression model when both variables are subject to errors. Since then, several authors have worked on the problem of estimating the parameters, in particular for the unreplicated linear functional relationship model. The work of Fuller [3] represent the most comprehensive single source of information on functional model to date. This book covers the topics of functional extending ordinary linear regression models, multivariate linear regression and non-linear regression models.

The functional model can also be extended to the case when both the response and explanatory variables are circular instead of continuous linear. Circular random variable is one which takes values on the circumference of a circle, i.e. they are angles

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in the range $(0, 2\pi)$ radians or $(0^0, 360^0)$. This random variable must analysed by techniques differing from those appropriate for the usual Euclidean type variables because the circumference is a bounded closed space, for which the concept of origin is arbitrary or undefined. A continuous linear variable is a random variable with realisations on the straight line which may be analysed straightforwardly by usual techniques.

By "linear circular functional relationship model" we denote the model which has an (unknown) linear relationship and generally given by $x_i = X_i + \delta_i$ and $y_i = Y_i + \epsilon_i$, where $Y_i = \alpha + \beta X_i \pmod{2\pi}$, for $i = 1, 2, \ldots, n$. In this, observations are made of two circular variables on range $(0, 2\pi)$ subject to error. One example of the application of the linear circular functional relationship model is in an analysis of the wind direction data, specifically an instrument calibration problem where the aim is to compare the accuracy of the new instruments (high frequency radar) with that of a standard instrument (anchored wave buoy) for measuring a wind direction, (Sova, [4]).

The errors δ_i and ϵ_i are assumed to be mutually independently distributed with von Mises distributions, that is $\delta_i \sim VM(0, \kappa)$ and $\epsilon_i \sim VM(0, \nu)$ respectively. The von Mises distribution denoted by $VM(\mu, \kappa)$ was first introduced by von Mises in 1918 to study the deviations of measured atomic weights from integral values, i.e. the remainders when divided by some quantum value. This is a symmetric distribution which is the most common model for unimodal samples of circular data and in many respects this distribution is also the natural analogue on the circle of the Normal distribution on the real line. The parameter μ is the mean direction, while the parameter κ is described as the concentration parameter. The distribution is symmetric about μ and the mode is also μ. For large κ the distribution is clustered about the mean direction. The smaller the value of κ, the more spread the distribution. Depending on the values of κ, the von Mises distribution can be approximated by other distribution such as the Uniform, the Cardioid or the Wrapped Normal. The probability density function for the von Mises distribution is given by

$$f(\theta; \mu, \kappa) = \frac{1}{2\pi I_0(\kappa)} \exp\left\{\kappa \cos(\theta - \mu)\right\}, \quad 0 < \theta \le 2\pi, 0 \le \mu < 2\pi, \kappa > 0,$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero, i.e.

$$I_0(\kappa) = \sum_{r=0}^{\infty} \frac{1}{r!^2} \left(\frac{1}{2}\kappa\right)^{2r}.$$

We will consider primarily the linear circular functional relationship model above i.e. when the variables have a simple linear relationship of $Y = \alpha + \beta X \pmod{2\pi}$. Strictly, the term "wrapped linear functional relationship model" might be preferable to express the relationship between X and Y but for conciseness we will omit "wrapped" from the terminology. Without loss of generality we restrict α to the range $(0, 2\pi)$. In the next section we propose the model for the replicated linear circular functional relationship and establish notation. Maximum likelihood estimation of the parameters is discussed as well as the Fisher information matrix together with the numerical example.

2. The replicated linear circular functional relationship model

The unreplicated linear circular functional relationship assumed that the circular observation (x_i, y_i) measures, inexactly, the parameters (X_i, Y_i), where $Y_i = \alpha + \beta X_i \pmod{2\pi}$. However, corresponding to a particular pair (X_i, Y_i) there may be replicated observations of X_i and Y_i occurring in p groups. Measurements x_{ij}, $(j = 1, \ldots, m_i)$ are made on X_i and measurements y_{ik}, $(k = 1, \ldots, n_i)$ on Y_i, where $0 \le x_{ij}, y_{ik} < 2\pi$. Note that there is no inherent pairing of any x_{ij} with y_{ik}, indeed m_i may not equal n_i. We also assume that the observations on X_i and Y_i have been measured with errors δ_i and ϵ_i respectively.

The full model for the replicated linear circular functional relationship is therefore

$$\begin{aligned} x_{ij} = X_i + \delta_{ij} \text{ and } y_{ik} = Y_i + \epsilon_{ik}, \text{ where } Y_i = \alpha + \beta X_i \pmod{2\pi}, \\ \text{for } i = 1, \ldots, p, \quad j = 1, \ldots, m_i \text{ and } k = 1, \ldots n_i. \end{aligned} \tag{2.1}$$

The errors δ_{ij} and ϵ_{ik} are homogeneous and independently distributed with a von Mises distributions of zero mean circular, i.e. $\delta_{ij} \sim VM(0, \kappa)$ and $\epsilon_{ik} \sim VM(0, \nu)$. There are $(p+4)$ parameters to be estimated, i.e. α, β, κ, ν and the incidental parameters X_i, $i = 1, \ldots, p$ by the maximum likelihood method. Suppose L is the log likelihood function of model (2.1). Then

$$L(\alpha, \beta, \kappa, \nu, X_i, \ldots, X_p; x_{11}, \ldots, x_{pm_p}, y_{11}, \ldots, y_{pn_p}) = -NM \log(2\pi) - N \log I_0(\kappa)$$
$$-M \log I_0(\nu) + \sum\sum \kappa \cos(x_{ij} - X_i) + \sum\sum \nu \cos(y_{ik} - \alpha - \beta X_i)$$

where $N = \sum_{i=1}^{p} n_i$, $M = \sum_{i=1}^{p} m_i$ and as usual $I_0(\nu)$ and $I_0(\kappa)$ are the modified Bessel functions of the first kind and order zero. Differentiating $\log L$ with respect to α, β, κ, ν and X_i we obtain the likelihood equation for parameters which may be solved iteratively given some suitable initial values at the estimate and given by

$$\hat{\alpha} = \begin{cases} \tan^{-1}\left(\frac{S}{C}\right), & S > 0, C > 0 \\ \tan^{-1}\left(\frac{S}{C}\right) + \pi, & C < 0 \\ \tan^{-1}\left(\frac{S}{C}\right) + 2\pi, & S < 0, C > 0 \end{cases}$$

where $S = \sum \sin(y_i - \hat{\beta} x_i)$ and $C = \sum \cos(y_i - \hat{\beta} x_i)$. Also

$$\hat{X}_{i1} \approx \hat{X}_{i0} + \frac{\sum_j \sin(x_{ij} - \hat{X}_{i0}) + \frac{\hat{\nu}}{\hat{\kappa}} \hat{\beta} \sum_k \sin(y_{ik} - \hat{\alpha} - \hat{\beta} \hat{X}_{i0})}{\sum_j \cos(x_{ij} - \hat{X}_{i0}) + \frac{\hat{\nu}}{\hat{\kappa}} \hat{\beta}^2 \sum_k \cos(y_{ik} - \hat{\alpha} - \hat{\beta} \hat{X}_{i0})}, \tag{2.2}$$

where $\hat{X}_{i1}$ is an improvement estimate of $\hat{X}_{i0}$ which is the initial estimate of $\hat{X}_i$.

$$\hat{\beta}_1 \approx \hat{\beta}_0 + \frac{\sum\sum X_i \sin(y_{ik} - \hat{\alpha} - \hat{\beta}_0 \hat{X}_i)}{\sum\sum X_i \cos(y_{ik} - \hat{\alpha} - \hat{\beta}_0 \hat{X}_i)}, \tag{2.3}$$

where $\hat{\beta}_1$ is an improvement estimate of $\hat{\beta}_0$ which is an initial estimate of $\hat{\beta}$. Further, the estimates of κ and ν are given by

$$\hat{\kappa} = A^{-1}\left(\frac{1}{N} \sum\sum \cos(x_{ij} - \hat{X}_i)\right)$$

and

$$\hat{\nu} = A^{-1}\left(\frac{1}{M}\sum\sum\cos(y_{ik} - \hat{\alpha} - \hat{\beta}\hat{X})\right)$$

respectively where

$$A(r) = \frac{I_1(r)}{I_0(r)} = 1 - \frac{1}{2r} - \frac{1}{8r^2} - \frac{1}{8r^3} + 0(r^{-4}).$$

Hence $\hat{\alpha}, \hat{\beta}, \hat{\kappa}, \hat{\nu}, \hat{X}_1, \ldots \hat{X}_p$ can be solved iteratively and possible initial estimates for the iteration are putting $\hat{\beta}_0 = 1.0$ in equation (2.3) and $\frac{\hat{\nu}}{\hat{\kappa}} = 1.0$ in equation (2.2). An initial estimate of X_i in equation (2.2) can be chosen from the mean direction of x_{ij}, that is

$$\hat{X}_{i0} = \begin{cases} \tan^{-1}\left(\frac{S_i}{C_i}\right), & S_i > 0, C_i > 0 \\ \tan^{-1}\left(\frac{S_i}{C_i}\right) + \pi, & C_i < 0 \\ \tan^{-1}\left(\frac{S_i}{C_i}\right) + 2\pi, & S_i < 0, C_i > 0 \end{cases}$$

where $S_i = \sum\limits_{j=1}^{m_i} \sin(x_{ij})$ and $C_i = \sum\limits_{j=1}^{m_i} \cos(x_{ij})$. Finally, the estimates of ν and κ can be obtain by using the approximation given by Dobson [5], that is

$$A^{-1}(w) \approx \frac{9 - 8w + 3w^2}{8(1-w)}.$$

Thus for replicated circular functional relationship, when the errors are distributed as a von Mises, all the parameters can be estimated.

3. Fisher information matrix of parameters

In this section we consider the Fisher information matrix of parameters. The first partial derivatives for log likelihood function are given by

$$\begin{aligned}
\frac{\partial L}{\partial \alpha} &= \sum\sum \nu \sin(y_{ik} - \alpha - \beta X_i) \\
\frac{\partial L}{\partial \beta} &= \sum\sum \nu X_i \sin(y_{ik} - \alpha - \beta X_i) \\
\frac{\partial L}{\partial X_i} &= \kappa \sum_j \sin(x_{ij} - X_i) + \nu\beta \sum_k \sin(y_{ik} - \alpha - \beta X_i) \\
\frac{\partial L}{\partial \kappa} &= -NA(\kappa) + \sum\sum \cos(x_{ij} - X_i) \\
\frac{\partial L}{\partial \nu} &= -MA(\nu) + \sum\sum \cos(y_{ik} - \alpha - \beta X_i).
\end{aligned}$$

The second derivatives for log likelihood function and their negative expected values are given by

$$\frac{\partial^2 L}{\partial X_i^2} = -\kappa \sum_j \cos(x_{ij} - X_i) - \nu\beta^2 \sum_k \cos(y_{ik} - \alpha - \beta X_i), \text{ hence}$$

$$E\left[-\frac{\partial^2 L}{X_i^2}\right] = \frac{\kappa N}{P} A(\kappa) + \frac{\nu\beta^2 M}{P} A(\nu).$$

$$\frac{\partial^2 L}{\partial X_i \partial X_j} = 0, \text{ hence } E\left[-\frac{\partial^2 L}{\partial X_i \partial X_j}\right] = 0, \text{ for } i \neq j.$$

$$\frac{\partial^2 L}{\partial X_i \partial \alpha} = -\nu\beta \sum_k \cos(y_{ik} - \alpha - \beta X_i), \text{ hence } E\left[-\frac{\partial^2 L}{\partial X_i \partial \alpha}\right] = \frac{\nu\beta M}{P} A(\nu).$$

$$\frac{\partial^2 L}{\partial X_i \partial \beta} = -\nu\beta X_i \sum_k \cos(y_{ik} - \alpha - \beta X_i), \text{ hence } E\left[-\frac{\partial^2 L}{\partial X_i \partial \beta}\right] = \frac{\nu\beta M}{P} X_i A(\nu).$$

$$\frac{\partial^2 L}{\partial X_i \partial \kappa} = \sum_j \sin(x_{ij} - X_i), \text{ hence } E\left[-\frac{\partial^2 L}{\partial X_i \partial \kappa}\right] = 0, \text{ and also } E\left[-\frac{\partial^2 L}{\partial X_i \partial \nu}\right] = 0.$$

$$\frac{\partial^2 L}{\partial \alpha^2} = -\sum\sum \nu \cos(y_{ik} - \alpha - \beta X_i), \text{ hence } E\left[-\frac{\partial^2 L}{\partial \alpha^2}\right] = \nu M A(\nu).$$

$$\frac{\partial^2 L}{\partial \alpha \partial \beta} = -\nu \sum\sum X_i \cos(y_{ik} - \alpha - \beta X_i), \text{ hence } E\left[-\frac{\partial^2 L}{\partial \alpha \partial \beta}\right] = \frac{M\nu}{P} A(\nu) \sum X_i.$$

$$\frac{\partial^2 L}{\partial \alpha \partial \kappa} = 0, \text{ hence } E\left[-\frac{\partial^2 L}{\partial \alpha \partial \kappa}\right] = 0.$$

$$\frac{\partial^2 L}{\partial \alpha \partial \nu} = \sum\sum \sin(y_{ik} - \alpha - \beta X_i), \text{ hence } E\left[-\frac{\partial^2 L}{\partial \alpha \partial \nu}\right] = 0.$$

$$\frac{\partial^2 L}{\partial \beta^2} = -\sum\sum \nu X_i^2 \cos(y_{ik} - \alpha - \beta X_i), \text{ hence } E\left[-\frac{\partial^2 L}{\partial \beta^2}\right] = \frac{M}{P} \nu A(\nu) \sum X_i^2.$$

$$\frac{\partial^2 L}{\partial \beta \partial \kappa} = 0, \text{ hence } E\left[-\frac{\partial^2 L}{\partial \beta \partial \kappa}\right] = 0.$$

$$\frac{\partial^2 L}{\partial \beta \partial \nu} = \sum\sum X_i \sin(y_{ik} - \alpha - \beta X_i), \text{ hence } E\left[-\frac{\partial^2 L}{\partial \beta \partial \nu}\right] = 0.$$

$$\frac{\partial^2 L}{\partial \kappa^2} = -NA'(\kappa), \text{ hence } E\left[\frac{\partial^2 L}{\partial \kappa^2}\right] = NA'(\kappa).$$

$$\frac{\partial^2 L}{\partial \kappa \partial \nu} = 0, \text{ hence} E\left[-\frac{\partial^2 L}{\partial \kappa \partial \nu}\right] = 0.$$

$$\frac{\partial^2 L}{\partial^2 \nu} = -MA'(\nu), \text{ hence} E\left[-\frac{\partial^2 L}{\partial \nu^2}\right] = MA'(\nu).$$

Next we find the estimated Fisher information matrix, F, for $\hat{X}_i, \ldots, \hat{X}_P$, $\hat{\kappa}$, $\hat{\nu}$, $\hat{\alpha}$ and $\hat{\beta}$ given by

$$F = \begin{bmatrix} B & 0 & E \\ 0 & C & 0 \\ E^T & 0 & D \end{bmatrix}$$

where B is a $P \times P$ matrix given by

$$B = \begin{bmatrix} \frac{\hat{\kappa}NA(\hat{\kappa})+\hat{\nu}\hat{\beta}^2MA(\hat{\nu})}{P} & & 0 \\ & \ddots & \\ 0 & & \frac{\hat{\kappa}NA(\hat{\kappa})+\hat{\nu}\hat{\beta}^2MA(\hat{\nu})}{P} \end{bmatrix},$$

E is a $P \times 2$ matrix given by

$$E = \begin{bmatrix} \frac{\hat{\nu}\hat{\beta}M}{P}A(\hat{\nu}) & \frac{\hat{\nu}\hat{\beta}M}{P}\hat{X}_iA(\hat{\nu}) \\ \vdots & \vdots \\ \frac{\hat{\nu}\hat{\beta}M}{P}A(\hat{\nu}) & \frac{\hat{\nu}\hat{\beta}M}{P}\hat{X}_iA(\hat{\nu}) \end{bmatrix},$$

C is a 2×2 matrix given by

$$C = \begin{bmatrix} NA'(\hat{\kappa}) & 0 \\ 0 & NA'(\hat{\nu}) \end{bmatrix},$$

D is a 2×2 matrix, given by

$$D = \begin{bmatrix} \hat{\nu}MA(\hat{\nu}) & \frac{M}{P}\hat{\nu}A(\hat{\nu})\sum\hat{X}_i \\ \frac{M}{P}\hat{\nu}A(\hat{\nu})\sum\hat{X}_i & \frac{M}{P}\hat{\nu}A(\hat{\nu})\sum\hat{X}_i^2 \end{bmatrix}.$$

Our main interest is the asymptotic covariance matrix of $\hat{\kappa}, \hat{\nu}, \hat{\alpha}$ and $\hat{\beta}$, which is the bottom right minor of order $4 \times P$ of the inverse of matrix F. From the theory of partitioned matrices, (Graybill [6]), this is given by

$$\widehat{Var}\begin{bmatrix} \hat{\kappa} \\ \hat{\nu} \\ \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} C^{-1} & 0 \\ 0 & (D - E^TB^{-1}E)^{-1} \end{bmatrix},$$

where

$$C^{-1} = \begin{bmatrix} (NA'(\hat{\kappa}))^{-1} & 0 \\ 0 & (MA'(\hat{\nu}))^{-1} \end{bmatrix}.$$

It can be shown that

$$\left(D - E^TB^{-1}E\right)^{-1} = H\begin{bmatrix} \frac{\sum\hat{X}_i^2}{P} & -\frac{\sum\hat{X}_i}{P} \\ -\frac{\sum\hat{X}_i}{P} & 1 \end{bmatrix},$$

where

$$H = \frac{P^2\left(\hat{\kappa}NA(\hat{\kappa}) + \hat{\nu}\hat{\beta}^2MA(\hat{\nu})\right)}{\hat{\nu}MA(\hat{\nu})\hat{\kappa}NA(\hat{\kappa})\left(P\sum\hat{X}_i^2 - (\sum\hat{X}_i)^2\right)}.$$

Therefore, the asymptotic covariance matrix for $\hat{\kappa}, \hat{\nu}, \hat{\alpha}$ and $\hat{\beta}$ is given by

$$\begin{bmatrix} (NA'(\hat{\kappa}))^{-1} & 0 & 0 & 0 \\ 0 & (MA'(\hat{\nu}))^{-1} & 0 & 0 \\ 0 & 0 & \frac{H\sum \hat{X}_i^2}{P} & -\frac{H\sum \hat{X}_i}{P} \\ 0 & 0 & -\frac{H\sum \hat{X}_i}{P} & H \end{bmatrix}$$

where

$$H = \frac{p^2\left(\hat{\kappa}NA(\hat{\kappa}) + \hat{\nu}\hat{\beta}^2 MA(\hat{\nu})\right)}{\hat{\nu}MA(\hat{\nu})\hat{\kappa}NA(\hat{\kappa})\left(p\sum \hat{X}_i^2 - (\sum \hat{X}_i)^2\right)}.$$

4. Results

Using the Fisher information matrix derived above we have the following results:

$$V\hat{a}r(\hat{\kappa}) = \frac{\hat{\kappa}}{N\left(\hat{\kappa} - \hat{\kappa}A^2(\hat{\kappa}) - A(\hat{\kappa})\right)},$$

$$V\hat{a}r(\hat{\nu}) = \frac{\hat{\nu}}{M\left(\hat{\nu} - \hat{\nu}A^2(\hat{\nu}) - A(\hat{\nu})\right)},$$

$$V\hat{a}r(\hat{\alpha}) = \frac{p\left(\hat{\kappa}NA(\hat{\kappa}) + \hat{\nu}\hat{\beta}^2 MA(\hat{\nu})\right)\sum \hat{X}_i^2}{\hat{\nu}MA(\hat{\nu})\hat{\kappa}NA(\hat{\kappa})\left(p\sum \hat{X}_i^2 - (\sum \hat{X}_i)^2\right)},$$

$$V\hat{a}r(\hat{\beta}) = \frac{p^2\left(\hat{\kappa}NA(\hat{\kappa}) + \hat{\nu}\hat{\beta}^2 MA(\hat{\nu})\right)}{\hat{\nu}MA(\hat{\nu})\hat{\kappa}NA(\hat{\kappa})\left(p\sum \hat{X}_i^2 - (\sum \hat{X}_i)^2\right)}$$

and

$$C\hat{o}v\left(\hat{\alpha}, \hat{\beta}\right) = -\frac{p\left(\hat{\kappa}NA(\hat{\kappa}) + \hat{\nu}\hat{\beta}^2 MA(\hat{\nu})\right)\sum \hat{X}_i}{\hat{\nu}MA(\hat{\nu})\hat{\kappa}NA(\hat{\kappa})\left(p\sum \hat{X}_i^2 - (\sum \hat{X}_i)^2\right)}$$

Furthermore, for calculation purposes $A(\kappa)$ can be approximated (Mardia [7]), by

$$A(\kappa) = \frac{1}{2}\kappa\left(1 - \frac{1}{8}\kappa^2 + \frac{1}{48}\kappa^4 \ldots\right)$$

for small value of κ (less than 10) and

$$A(\kappa) = 1 - \frac{1}{2\kappa} - \frac{1}{8\kappa^2} - \frac{1}{8\kappa^3} \ldots$$

for large value of κ.

5. Numerical example and discussion

As an illustration we used the wave direction data in Sova [4] to show that, in particular by using various approximation and Fisher information matrix we can find the estimated of parameters as well as the standard error. We proposed the model for replicated data given by

Table 1. Parameter estimates for the wave direction data

Parameter	Estimate	Standard Error
α	3.827	0.442
β	1.039	0.113
κ	1.451	0.181
ν	1.361	0.165

Table 2. Parameter estimates for the wave direction data assuming $\beta = 1.0$

Parameter	Estimate	Standard Error
α	3.827	0.435
κ	1.451	0.181
ν	1.361	0.163

$$x_{ij} = X_i + \delta_{ij} \text{ and } y_{ik} = Y_i + \epsilon_{ik} \text{ where } Y_i = \alpha + \beta X_i \pmod{2\pi},$$
$$\text{for } i = 1, 2, \ldots, p, \ j = 1, 2, \ldots, m_i \text{ and } k = 1, 2, \ldots, n_i$$

and also $\delta_{ij} \sim VM(0, \kappa)$ and $\epsilon_{ik} \sim VM(0, \nu)$ where x_{ij} are the measurement for group i by radar with some random error δ_{ij} and y_{ik} is the measurement for group i by an anchored buoy with some random error ϵ_{ik}. X_i and Y_i are said to be the underlying or real directions measured by radar and anchored buoy respectively. The maximum likelihood estimates and standard errors are given in Table 1. The estimated ratio of error concentration parameters is given by $\hat{\lambda} = \frac{\hat{\nu}}{\hat{\kappa}} = 0.94$. However, the 95% confidence interval for $\hat{\beta}$ is given by (0.818, 1.260), which suggests that $\beta = 1.0$ is a reasonable value. Re-estimating the parameters by assuming β equal to 1.0, we obtain the estimates as shown in Table 2.

We found that at the 5% significance level, there is no difference from 1.0 in the estimates of the slope parameter, β, for the wave directions data. We found that there is a non-zero intercept, i.e $\hat{\alpha} = 3.827$. This suggests that there is almost no difference in the relative calibration between the measurements by radar and anchored buoy but that an additive correction is required to move from one measurement method to the other.

We also found that the ratio of error concentration parameters between anchored buoy and radar, λ, is less than 1.0, and also the estimated standard error for error concentration parameters of measurements by anchored buoy is less than the estimated standard error for error concentration parameters by radar which suggest that measurements by anchored buoy seems to be more precise.

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