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# Special Classes of Univalent Functions with Missing Coefficients and Integral Transforms

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**Abstract.** Let $\mathcal{A}_n$ be the class of all analytic functions $f$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad z \in \Delta,$$

where $n \in \mathbb{N}$ is fixed. For $\lambda > 0$ and $\alpha < 1$, define

$$\mathcal{U}_n(\lambda) = \left\{ f \in \mathcal{A}_n : \left| \left(\frac{z}{f(z)}\right)^{n+1} f'(z) - 1 \right| < \lambda,\ z \in \Delta \right\}$$

and

$$\mathcal{S}_\alpha^* = \left\{ f \in \mathcal{S}^*(\alpha) : \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha,\ z \in \Delta \right\}.$$

In this paper, we find suitable conditions on $\lambda$ and $\alpha$ so that $\mathcal{U}_n(\lambda)$ is included in $\mathcal{S}_\alpha$ and $\mathcal{S}^*(\alpha)$. Here $\mathcal{S}_\alpha$ and $\mathcal{S}^*(\alpha)$ denote the usual classes of strongly starlike and starlike of order $\alpha$, respectively. We determine necessary conditions so that $f \in \mathcal{U}_n(\lambda)$ implies that

$$\left| \frac{zf'(z)}{f(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta}, \quad z \in \Delta,$$

or

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta}, \quad |z| < r,$$

where $r = r(\lambda, n)$ will be specified. For $c + 1 - n > 0$, define

$$[I(f)](z) = F(z) = z \left[ \frac{c+1-n}{z^{c+1-n}} \int_0^z \left(\frac{t}{f(t)}\right)^n t^{c-n}\, dt \right]^{1/n}.$$

We also find conditions on $\lambda$, $\alpha$ and $c$ so that $I(\mathcal{U}_n(\lambda)) \subset \mathcal{S}_\alpha^*$.

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## 1. Introduction and preliminaries

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ denote the class of all functions $f$ in $\mathcal{H}$ such that $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \Delta\}$ and (see [3])

$$\mathcal{S}^* = \{f \in \mathcal{A} : \ f(\Delta) \text{ is starlike}\} \equiv \left\{f \in \mathcal{S} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \ z \in \Delta\right\}.$$

Also, we let $\mathcal{S}^*(\alpha)$, $\alpha < 1$, to be the family of starlike functions of order $\alpha$. It is well-known that $f \in \mathcal{S}^*(\alpha)$ iff $\operatorname{Re}(zf'(z)/f(z)) > \alpha$ for $z \in \Delta$; $\mathcal{S}^*(\alpha) \subsetneq \mathcal{S}^*$ for $0 < \alpha < 1$. For $0 < \alpha \leq 1$, a function $f \in \mathcal{A}$ is called strongly starlike of order $\alpha$ iff $f$ satisfies the condition

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}, \quad z \in \Delta,$$

where $\prec$ denotes the usual subordination (see [3]). The class of all strongly starlike functions of order $\alpha$ is denoted by $\mathcal{S}_\alpha$. Clearly, $\mathcal{S}_1 \equiv \mathcal{S}^*$ and if $0 < \alpha < 1$, then the class $\mathcal{S}_\alpha$ is completely contained in the class of all bounded starlike functions [2]. For $\mu < 0$, define

$$\mathcal{B}(\mu) = \left\{f \in \mathcal{A} : \operatorname{Re}\left(f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1}\right) > 0, \ z \in \Delta\right\}.$$

It is shown in [1] that $\mathcal{B}(\mu)$ is a subclasses of the class of Bazilveič functions that is contained in the class $\mathcal{S}$.

In [9], Ponnusamy has considered a subclass of $\mathcal{B}(\mu)$ defined by

$$\mathcal{U}(\lambda, \mu) = \left\{f \in \mathcal{A} : \left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} - 1\right| < \lambda, \ z \in \Delta\right\}.$$

Clearly, $\mathcal{U}(\lambda, \mu)$ is contained in $\mathcal{S}$ for $\mu < 0$. In [9], Ponnusamy found conditions on $\lambda$ and $\mu < 0$ so that $\mathcal{U}(\lambda, \mu)$ is included in $\mathcal{S}^*$ or other well-known subclasses of $\mathcal{S}$. On the other hand, Nunokawa and Ozaki [8] has shown that $\mathcal{U}(\lambda, 1) \equiv \mathcal{U}(\lambda)$ is also included in $\mathcal{S}$ for $0 < \lambda \leq 1$. It is important to observe that the Koebe function $z/(1-z)^2$ belongs to $\mathcal{U}(1)$ but $\mathcal{U}(1)$ not included in $\mathcal{S}^*$, see [6]. In view of these observations, Ponnusamy and Vasundhra [13] found conditions on $\lambda_0$ so that $\mathcal{U}(\lambda) \subseteq \mathcal{S}^*$ for $0 < \lambda \leq \lambda_0$. Further, it is interesting to find the analog of the inclusion results (such as the containment theorems $\mathcal{U}(\lambda, \mu) \subset \mathcal{S}^*(\alpha)$ for $\mu < 0$) also for the case $0 < \mu < 1$. For $0 < \mu < 1$, the class $\mathcal{U}(\lambda, \mu)$ has been discussed by Obradović [5].

Let $\mathcal{A}_n$ denote the class of all functions $f \in \mathcal{A}$ such that $f$ has the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k,$$

where $n \in \mathbb{N}$ is fixed. Clearly, $\mathcal{A} := \mathcal{A}_1$. For $f \in \mathcal{A}_n$ such that $f(z)/z \neq 0$, we have

$$\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) = 1 + (n-\mu)a_{n+1}z^n + \cdots,$$

and so, it is essential to consider two cases, namely $\mu \in (0, n)$ and $\mu = n$, separately. For $\mu \in (0, n)$ and $\lambda > 0$, the class $\mathcal{U}_n(\lambda, \mu) \equiv \mathcal{A}_n \cap \mathcal{U}(\lambda, \mu)$ has been discussed by the authors in [10]. However, the case $\mu = n$, which does produce a slightly different implication, has not been discussed in [10]. For $\mu = n$, the class $\mathcal{U}_n(\lambda, \mu)$ will be denoted by $\mathcal{U}_n(\lambda)$ for convenience. Thus it is now natural to raise the following problem.

**Problem.** Find conditions on $\lambda$ and $\alpha$ such that $\mathcal{U}_n(\lambda)$ is included $\mathcal{S}^*(\alpha)$ or $\mathcal{S}_\alpha$.

The main aim of this paper is to answer this problem in a more general form. For the special case $n = 1$, this class has been studied by several authors [5, 6, 7, 12].

## 2. Basic properties of $\mathcal{U}_n(\lambda)$

By definition, each $f \in \mathcal{U}_n(\lambda)$ can be written as

$$\left(\frac{z}{f(z)}\right)^{n+1} f'(z) = 1 + \lambda w(z) = 1 + A_{n+1} z^{n+1} + \cdots, \tag{2.1}$$

for some $w \in \mathcal{B}_n$. Here,

$$\mathcal{B}_n = \{w \in \mathcal{H} : \; w(0) = w'(0) = \cdots = w^{(n)}(0) = 0, \quad \text{and } |w(z)| < 1 \text{ for } z \in \Delta\}$$

and throughout the paper $a_{n+1}$ is meant for $f^{(n+1)}(0)/(n+1)!$. If we set

$$p(z) = \left(\frac{z}{f(z)}\right)^n = 1 - n a_{n+1} z^n + \cdots,$$

then $p$ is analytic in $\Delta$, $p(0) = 1$ and $p^{(k)}(0) = 0$ for $k = 1, 2, \ldots, n-1$. Further, (2.1) is seen to be equivalent to

$$p(z) - \frac{1}{n} z p'(z) = 1 + \lambda w(z).$$

An algebraic computation implies that

$$p(z) = 1 - n a_{n+1} z^n - n\lambda \int_0^1 \frac{w(tz)}{t^{n+1}}\, dt. \tag{2.2}$$

As $w(z) \in \mathcal{B}_n$, Schwarz's lemma gives that $|w(z)| \le |z|^{n+1}$ for $z \in \Delta$ and therefore,

$$|p(z) - 1| \le n|z|^n \left(|a_{n+1}| + \lambda |z|\right), \quad z \in \Delta,$$

which is

$$\left|\left(\frac{z}{f(z)}\right)^n - 1\right| \le n|z|^n \left(|a_{n+1}| + \lambda |z|\right), \quad z \in \Delta, \tag{2.3}$$

so that

$$1 - n|z|^n(|a_{n+1}| + \lambda|z|) \le \operatorname{Re}\left(\frac{z}{f(z)}\right)^n \le 1 + n|z|^n(|a_{n+1}| + \lambda|z|). \tag{2.4}$$

Equality holds in each of the last two inequalities (2.3) and (2.4) for functions of the form

$$f(z) = \frac{z}{\left(1 \pm n|a_{n+1}|z^n + \lambda n z^{n+1}\right)^{1/n}}.$$

## 3. Strongly starlikeness and convexity for functions in $\mathcal{U}_n(\lambda)$

We are now in a position to state our main results and their consequences. The proof of these results will be given in Section 4.

**Theorem 3.1.** *Let* $\gamma \in (0,1]$, $n \geq 1$ *and*

$$\lambda_*(\gamma,n) = \frac{-n(n+\cos(\gamma\pi/2))|a_{n+1}| + \sin(\gamma\pi/2)\sqrt{1+n^2(1-|a_{n+1}|^2)+2n\cos(\gamma\pi/2)}}{1+2n\cos(\gamma\pi/2)+n^2}.$$

*If* $f \in \mathcal{U}_n(\lambda)$, *then* $f \in \mathcal{S}_\gamma$ *for* $0 < \lambda \leq \lambda_*(\gamma,n)$.

Theorem 3.1 for $n=1$ is due to Obradović *et al* [7]. In the case $\gamma = 1$, Theorem 3.1 yields criteria for starlike functions.

**Corollary 3.1.** *If* $f \in \mathcal{U}_n(\lambda)$ *and* $0 < \lambda \leq \dfrac{-n^2|a_{n+1}| + \sqrt{1+n^2(1-|a_{n+1}|^2)}}{1+n^2}$, *then* $f \in \mathcal{S}^*$.

For $n=1$, Corollary 3.1 yields

**Corollary 3.2.** *If* $f \in \mathcal{U}(\lambda)$, *then* $f \in \mathcal{S}^*$ *for* $0 < \lambda \leq \dfrac{-|a_2| + \sqrt{2-|a_2|^2}}{2}$.

This corollary was stated as a conjecture in [6] but was settled later in [7]. The same reasoning indicated in the proof of Theorem 3.1 helps to obtain the following result.

**Theorem 3.2.** *Let* $f \in \mathcal{U}_n(\lambda)$ *and* $\lambda_*(\gamma,n)$ *be as in Theorem* 3.1. *Then, for* $\lambda_*(\gamma,n) \leq \lambda$, $f$ *is strongly starlike in* $|z| < r(\lambda,\gamma,n)$, *where* $r = r(\lambda,\gamma,n)$ *is the smallest positive root of the equation* $E_\lambda(n,r) = 0$, *where*

$$\begin{aligned} E_\lambda(n,r) &= \lambda^2 r^{2(n+1)}\left(1+n^2+2n\cos(\gamma\pi/2)\right) + 2\lambda n\left(n+\cos(\gamma\pi/2)\right)|a_{n+1}|r^{2n+1} \\ &\quad + n^2|a_{n+1}|^2 r^{2n} - \sin^2(\gamma\pi/2). \end{aligned}$$

In the case $\gamma = 1$, Theorem 3.2 yields

**Corollary 3.3.** *If* $f \in \mathcal{U}_n(\lambda)$ *and*

$$\frac{-n^2|a_{n+1}| + \sqrt{1+n^2(1-|a_{n+1}|^2)}}{1+n^2} < \lambda \leq 1,$$

*then* $f \in \mathcal{S}^*$ *in* $|z| < r = r(\lambda,n)$, *where* $r$ *is the smallest root of*

$$\lambda^2(1+n^2)r^{2(n+1)} + 2\lambda n^2|a_{n+1}|r^{2n+1} + n^2|a_{n+1}|^2 r^{2n} - 1 = 0.$$

For $n=1$, Corollary 3.3 yields

**Example 3.1.** If $f \in \mathcal{U}(\lambda)$, then $\frac{1}{r}f(rz) \in \mathcal{S}^*$ for $\dfrac{-|a_2| + \sqrt{2-|a_2|^2}}{2} \leq \lambda \leq 1$, where $r$ is the smallest positive root of

$$2\lambda^2 r^4 + 2\lambda|a_2|r^3 + |a_2|^2 r^2 - 1 = 0.$$

**Example 3.2.** Suppose that $f \in \mathcal{U}_n(\lambda)$ with $a_{n+1} = 0$, and

$$\lambda_0(\gamma, n) = \frac{\sin(\pi\gamma/2)}{\sqrt{1 + 2n\cos(\pi\gamma/2) + n^2}}.$$

Then, by Theorems 3.1 and 3.2, we have the following:

(i) $f \in \mathcal{S}_\gamma$ whenever $0 < \lambda \le \lambda_0 = \lambda_0(\gamma, n)$

(ii) $f \in \mathcal{S}_\gamma$ for $|z| < r = \left(\frac{\lambda_0(\gamma, n)}{\lambda}\right)^{1/(n+1)}$ whenever $\lambda_0(\gamma, n) < \lambda \le 1$.

In the following theorem, we consider similar results for certain subsets of the set of all starlike functions. To do this, we define

$$\mathcal{S}_b^*(\beta) = \left\{ f \in \mathcal{S}^* : \left| \frac{zf'(z)}{f(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta},\ z \in \Delta \right\},$$

where $0 < \beta < 1$.

**Theorem 3.3.** *Let $n \in \mathbb{N}$ and $\lambda \in (0, 1]$. If $f \in \mathcal{U}_n(\lambda)$, then for $0 < \beta < 1$ we have*

$$\left| \frac{zf'(z)}{f(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta} \quad \textit{for} \ \ |z| < r_0 = r_0(\lambda, n, \beta),$$

*where $r_0$ is the positive root of the equation*

$$2\lambda(\beta + n)r^{n+1} + 2n|a_{n+1}|r^n + |2\beta - 1| - 1 = 0.$$

For $n = 1$, Theorem 3.3 has been obtained by Obradović *et al* [7].

**Theorem 3.4.** *Let $n \in \mathbb{N}$ and $\lambda \in (0, 1]$. If $f \in \mathcal{U}_n(\lambda)$, then for $0 < \beta \le 1$ we have*

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta} \quad \textit{for} \ \ |z| < r_{\lambda,n}(\beta),$$

*where $r = r_{\lambda,n}(\beta)$ is the smallest positive root of the equation*

$$\begin{aligned} &2\beta\lambda^2 nr^{2n+3} + 2\beta\lambda n|a_{n+1}|r^{2n+2} + 2\lambda\left(\beta n^2 + (\beta+1)n + \beta\right) r^{n+3} \\ &+2\left(\beta\lambda(\lambda n - 1) + (\beta n + 1)n|a_{n+1}|\right) r^{n+2} - 2\lambda\left(\beta n^2 + (1+\beta)n + \beta - \beta n|a_{n+1}|\right) r^{n+1} \\ &-2\left(\beta n + 1\right) n|a_{n+1}|r^n - \left(1 - |2\beta - 1|\right) r^2 - 2\beta\lambda r - |2\beta - 1| + 1 = 0. \end{aligned} \tag{3.1}$$

*In particular, $r^{-1}f(rz) \in \mathcal{K}$, where $\mathcal{K}$ denotes the class of all convex functions $g$, i.e. $zg'(z)$ belongs $\mathcal{S}^*$.*

If we choose $\beta = 1/2$, we obtain

**Corollary 3.4.** *Let $f \in \mathcal{U}_n(\lambda)$. Then*

$$\left| \frac{zf''(z)}{f'(z)} \right| < 1 \quad \textit{for} \ \ |z| < r_{\lambda,n}(1/2),$$

*where $r_{\lambda,n}(1/2)$ is the smallest positive root of the equation*

$$\begin{aligned} &\lambda^2 nr^{2n+3} + \lambda n|a_{n+1}|r^{2n+2} + \lambda(n^2 + 3n + 1)r^{n+3} + \left[(n+2)n|a_{n+1}| + \lambda(\lambda n - 1)\right] r^{n+2} \\ &-\lambda\left(n^2 + 3n - n|a_{n+1}| + 1\right) r^{n+1} - (n+2)n|a_{n+1}|r^n - r^2 - \lambda r + 1 = 0. \end{aligned}$$

**Example 3.3.** In particular, the last corollary gives the following:
$f \in \mathcal{U}_n(1)$ with $a_{n+1} = 0$ implies that

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 \quad \text{for} \quad |z| < r_{1,n}(1/2)$$

where $r_{1,n}(1/2)$ is the smallest positive root of the equation

$$nr^{2n+3} + (n^2 + 3n + 1)r^{n+3} + (n-1)r^{n+2} - (n^2 + 3n + 1)r^{n+1} - r^2 - r + 1 = 0.$$

## 4. Proofs of the main theorems

**4.1. Proof of Theorem 3.1 .** Suppose that $f \in \mathcal{U}_n(\lambda)$ for some $\lambda \in (0, 1]$ and $n \in \mathbb{N}$. Then, by the definition of $\mathcal{U}_n(\lambda)$, we have

$$\left|\left(\frac{z}{f(z)}\right)^{n+1} f'(z) - 1\right| \leq \lambda |z|^{n+1} < \lambda$$

and, by (2.3), we get

$$\left|\left(\frac{z}{f(z)}\right)^{n} - 1\right| \leq n|z|^n (|a_{n+1}| + \lambda |z|) < n(|a_{n+1}| + \lambda).$$

Therefore, it follows that

$$\left|\arg\left(\frac{z}{f(z)}\right)^{n+1} f'(z)\right| < \arcsin(\lambda) \tag{4.2}$$

and

$$\left|\arg\left(\frac{z}{f(z)}\right)^{n}\right| < \arcsin(n(|a_{n+1}| + \lambda)). \tag{4.3}$$

Using (4.2), (4.3) and the addition formula for the inverse of sine function, namely,

$$\arcsin(x) + \arcsin(y) = \arcsin\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right],$$

we find that

$$\begin{aligned} \left|\arg\frac{zf'(z)}{f(z)}\right| &\leq \left|\arg\left(\frac{z}{f(z)}\right)^{n+1} f'(z)\right| + \left|\arg\left(\frac{z}{f(z)}\right)^{n}\right| \\ &< \arcsin(\lambda) + \arcsin\left(n(|a_{n+1}| + \lambda)\right) \\ &= \arcsin\left[\lambda\sqrt{1 - n^2(|a_{n+1}| + \lambda)^2} + n(|a_{n+1}| + \lambda)\sqrt{1-\lambda^2}\right]. \end{aligned}$$

Thus, $f \in \mathcal{S}_\gamma$ whenever $\lambda \in (0, \lambda_*(\gamma, n)]$. Here $\lambda_*(\gamma, n)$ is the solution of the equation

$$\phi(\lambda) = \lambda\sqrt{1 - n^2(|a_{n+1}| + \lambda)^2} + n(|a_{n+1}| + \lambda)\sqrt{1-\lambda^2} - \sin\left(\frac{\pi\gamma}{2}\right) = 0$$

which proves the Theorem. □

**4.4. Proof of Theorem 3.2 .** Let $f \in \mathcal{U}_n(\lambda)$. Following the proof of Theorem 3.1, we obtain that

$$\left|\arg\left(\frac{z}{f(z)}\right)^{n+1} f'(z)\right| \leq \arcsin(\lambda r^{n+1})$$

and

$$\left|\arg\left(\frac{z}{f(z)}\right)^{n}\right| \leq \arcsin\left(nr^n(|a_{n+1}|+\lambda r)\right).$$

Combining the last two inequalities, we get

$$\left|\arg\frac{zf'(z)}{f(z)}\right| \leq \arcsin\left[\lambda r^{n+1}\sqrt{1-n^2r^{2n}(|a_{n+1}|+\lambda r)^2}\right.$$
$$\left. + nr^n(|a_{n+1}|+\lambda r)\sqrt{1-\lambda^2 r^{2(n+1)}}\right].$$

By a simple calculation, we see that the right hand side of the last inequality is less than or equal to $\pi\gamma/2$ provided that $E_\lambda(n,r) \leq 0$, where $E_\lambda(n,r)$ is as in Theorem 3.2. □

**4.5. Proof of Theorem 3.3 .** Let $f \in \mathcal{U}_n(\lambda)$. Then, by the representations (2.1) and (2.2), it follows that

$$\frac{zf'(z)}{f(z)} = \frac{1+\lambda w(z)}{1-na_{n+1}z^n - \lambda n\displaystyle\int_0^1 \frac{w(tz)}{t^{n+1}}\,dt}, \tag{4.6}$$

where $w \in \mathcal{B}_n$. We proceed with the method of proof of Theorem 1.9 in [7]. According to this,

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)} - \frac{1}{2\beta}\right| &= \frac{1}{2\beta}\left[\frac{\left|2\beta-1+na_{n+1}z^n+2\beta\lambda w(z)+\lambda n\displaystyle\int_0^1\frac{w(tz)}{t^{n+1}}\,dt\right|}{\left|1-na_{n+1}z^n-\lambda n\displaystyle\int_0^1\frac{w(tz)}{t^{n+1}}\,dt\right|}\right]\\ &\leq \frac{1}{2\beta}\left[\frac{|2\beta-1|+n|a_{n+1}|\,|z|^n+(2\beta+n)\lambda|z|^{n+1}}{1-n|a_{n+1}|\,|z|^n-\lambda n|z|^{n+1}}\right],\end{aligned}$$

since $|w(z)| \leq |z|^{n+1}$. It is a simple exercise to see that the square bracketed term in the last step is less than 1 provided

$$2\lambda(\beta+n)|z|^{n+1} + 2n|a_{n+1}|\,|z|^n + |2\beta-1| - 1 < 0.$$

Thus, it follows that

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{2\beta}\right| < \frac{1}{2\beta} \quad \text{for } |z| < r_0,$$

where $r_0$ is the positive root of the equation

$$2\lambda(\beta+n)r^{n+1}+2n|a_{n+1}|r^n+|2\beta-1|-1=0.$$

We complete the proof. □

**4.7. Proof of Theorem 3.4 .** Let $f\in\mathcal{U}_n(\lambda)$. Then the logarithmic derivative of the representation given by (2.1) yields that

$$1+\frac{zf''(z)}{f'(z)}=(n+1)\frac{zf'(z)}{f(z)}-n+\frac{\lambda zw'(z)}{1+\lambda w(z)},\quad w\in\mathcal{B}_n.$$

In view of this equation and the representation (4.6), we see that

$$1+\frac{zf''(z)}{f'(z)}-\frac{1}{2\beta}=(n+1)\left(\frac{1+\lambda w(z)}{1-na_{n+1}z^n-\lambda n\displaystyle\int_0^1\frac{w(tz)}{t^{n+1}}\,dt}\right)-n+\frac{\lambda zw'(z)}{1+\lambda w(z)}-\frac{1}{2\beta}.$$

Since $w\in\mathcal{B}_n$, by the definition of $\mathcal{B}_n$, we have $|w(z)|\leq|z|^{n+1}$. By the well-known Schwarz-Pick lemma, we obtain that

$$|w'(z)|\leq\frac{1-|w(z)|^2}{1-|z|^2}.$$

It follows that (as $\lambda\leq 1$)

$$\left|\frac{zw'(z)}{1+\lambda w(z)}\right|\leq\frac{|z|}{1-\lambda|w(z)|}\left(\frac{1-|w(z)|^2}{1-|z|^2}\right)\leq\frac{|z|(1+|z|^{n+1})}{1-|z|^2}.$$

With the help of this inequality and the fact that $|w(z)|\leq|z|^{n+1}$, after some computation, we get that

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{1}{2\beta}\right|<\frac{1}{2\beta}R_n(\lambda,\beta,|z|),$$

where

$$R_n(\lambda,\beta,|z|)=\frac{|2\beta-1|+(2\beta n+1)n|a_{n+1}|\,|z|^n+\lambda[2\beta(n^2+n+1)+n]|z|^{n+1}}{1-n|a_{n+1}|\,|z|^n-\lambda n|z|^{n+1}}+\frac{2\beta\lambda|z|(1+|z|^{n+1})}{1-|z|^2}.$$

It can be easily seen that the inequality $R_n(\lambda,\beta,|z|)<1$ is equivalent to (3.1). The desired conclusion follows. □

## 5. Order of starlikeness for functions in $\mathcal{U}_n(\lambda)$

**Theorem 5.1.** *If $f \in \mathcal{U}_n(\lambda)$ and $b = |a_{n+1}| \leq 1/n$, then $f \in \mathcal{S}^*(\alpha)$ whenever $0 < \lambda \leq \lambda_0(\alpha)$, where*

$$\lambda_0(\alpha) = \begin{cases} \dfrac{\sqrt{(1-2\alpha)(1+n^2(1-2\alpha-b^2))} - n^2 b(1-2\alpha)}{1+n^2(1-2\alpha)} & \text{if } 0 \leq \alpha < \alpha_0(n,b), \\ \dfrac{1-\alpha(1+nb)}{1+n\alpha} & \text{if } \alpha_0(n,b) \leq \alpha < \dfrac{1}{1+nb} \end{cases}$$

*with* $\alpha_0(n,b) = \frac{n(b+1)}{n(b+2)+1}$.

We observe that if we choose $\alpha = 0$ in Theorem 5.1, then Corollary 3.1 follows. Further, we believe that the order of starlikeness given above for functions in $\mathcal{U}_n(\lambda)$ is sharp although at present we do not have a concrete proof for our claim. However, from Theorem 5.1, one can obtain a number of new results.

**Corollary 5.1.** *If $f \in \mathcal{U}_n(\lambda)$ with $f^{(n+1)}(0) = 0$, then $f \in \mathcal{S}^*(\alpha)$ whenever $0 < \lambda \leq \lambda_0(\alpha)$, where*

$$\lambda_0(\alpha) = \begin{cases} \sqrt{\dfrac{1-2\alpha}{1+n^2(1-2\alpha)}} & \text{if } 0 \leq \alpha < \dfrac{n}{2n+1} \\ \dfrac{1-\alpha}{1+n\alpha} & \text{if } \dfrac{n}{2n+1} \leq \alpha < 1. \end{cases}$$

The following corollary is an equivalent form of Corollary 5.1 which is some what handy and is of independent interest in some special situations.

**Corollary 5.2.** *If $f \in \mathcal{U}_n(\lambda)$ with $f^{(n+1)}(0) = 0$ and $0 < \lambda \leq 1/\sqrt{n^2+1}$, then $f \in \mathcal{S}^*(\alpha)$, where*

$$\alpha := \alpha(\lambda) = \begin{cases} \dfrac{1-\lambda}{1+n\lambda} & \text{if } 0 < \lambda \leq 1/(n+1) \\ \dfrac{1-(1+n^2)\lambda^2}{2(1-n^2\lambda^2)} & \text{if } 1/(n+1) < \lambda \leq 1/\sqrt{n^2+1}. \end{cases} \tag{5.1}$$

For $n = 1$, Theorem 5.1 is due to [13].

### 5.2. Proof of Theorem 5.1.

Suppose that $f \in \mathcal{U}_n(\lambda)$. Then, we can write

$$-\frac{1}{n} z \left\{ \left( \frac{z}{f(z)} \right)^n \right\}' + \left( \frac{z}{f(z)} \right)^n = \left( \frac{z}{f(z)} \right)^{n+1} f'(z) = 1 + \lambda w(z), \tag{5.3}$$

where $w \in \mathcal{B}_n$. It follows that (see Section 2)

$$\left( \frac{z}{f(z)} \right)^n = 1 - n a_{n+1} z^n - n\lambda \int_0^1 \frac{w(tz)}{t^{n+1}} \, dt,$$

and therefore, by (5.3), we see that

$$\frac{zf'(z)}{f(z)} = \frac{1+\lambda w(z)}{1-na_{n+1}z^n - n\lambda\displaystyle\int_0^1 \frac{w(tz)}{t^{n+1}}dt}.$$

Thus,

$$\frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)}-\alpha\right) = \frac{1+\dfrac{\lambda}{1-\alpha}w(z)+\dfrac{n\alpha}{1-\alpha}\left[\lambda\displaystyle\int_0^1 \frac{w(tz)}{t^{n+1}}\,dt + a_{n+1}z^n\right]}{1-na_{n+1}z^n - n\lambda\displaystyle\int_0^1 \frac{w(tz)}{t^{n+1}}\,dt}.$$

Now, $\mathrm{Re}\left(\dfrac{zf'(z)}{f(z)}\right) > \alpha$ is equivalent to the condition

$$\frac{1+\dfrac{\lambda}{1-\alpha}w(z)+\dfrac{n\alpha}{1-\alpha}\left[\lambda\displaystyle\int_0^1 \frac{w(tz)}{t^{n+1}}\,dt + a_{n+1}z^n\right]}{1-na_{n+1}z^n - n\lambda\displaystyle\int_0^1 \frac{w(tz)}{t^{n+1}}\,dt} \neq -iT, \quad \text{for all } T\in\mathbb{R} \text{ and } z\in\Delta,$$

which can be rewritten as

$$\lambda\left[\frac{w(z)+n(\alpha-i(1-\alpha)T)\displaystyle\int_0^1 \frac{w(tz)}{t^{n+1}}\,dt}{(1-\alpha)(1+iT)+n(\alpha-iT(1-\alpha))a_{n+1}z^n}\right] \neq -1, \quad \text{for all } T\in\mathbb{R} \text{ and } z\in\Delta.$$

If we let

$$M = \sup_{z\in\Delta,\, w\in\mathcal{B}_n,\, T\in\mathbb{R}}\left|\frac{w(z)+n(\alpha-i(1-\alpha)T)\displaystyle\int_0^1 \frac{w(tz)}{t^{n+1}}dt}{(1-\alpha)(1+iT)+n(\alpha-iT(1-\alpha))a_{n+1}z^n}\right|$$

then, in view of the rotation invariance property of the space $\mathcal{B}_n$, we obtain that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \text{if } \lambda M \leq 1.$$

This observation shows that it suffices to find $M$. Since $|w(z)| \leq |z|^{n+1}$ for $z\in\Delta$, we first we notice that

$$M \leq \sup_{T\in\mathbb{R}}\left\{\frac{1+n\sqrt{\alpha^2+(1-\alpha)^2T^2}}{|(1-\alpha)\sqrt{1+T^2}-nb\sqrt{\alpha^2+(1-\alpha)^2T^2}|}\right\},$$

where, for convenience, we use the notation $b=|a_{n+1}|$. Define $\phi:\ [0,\infty)\to\mathbb{R}$ by

(5.4) $$\phi(x) = \frac{1+n\sqrt{\alpha^2+(1-\alpha)^2x}}{(1-\alpha)\sqrt{1+x}-nb\sqrt{\alpha^2+(1-\alpha)^2x}}.$$

First we observe that the denominator in the expression of $\phi(x)$ is positive for all $x \in [0,\infty)$ provided $0 \le \alpha < 1/(1+nb)$ and $0 \le b \le 1/n$.

Further, it is a simple exercise to see that

$$\phi'(x) = \frac{(1-\alpha)N(x)}{2\left[(1-\alpha)\sqrt{1+x} - nb\sqrt{\alpha^2+(1-\alpha)^2x}\right]^2 \sqrt{1+x}\ \sqrt{\alpha^2+(1-\alpha)^2x}}$$

where

$$N(x) = n(1-2\alpha) - \sqrt{\alpha^2+(1-\alpha)^2x} + nb(1-\alpha)\sqrt{1+x}.$$

**Case (I):** Let $b=0$. Then, we have

$$\phi'(x) = \frac{n(1-2\alpha) - \sqrt{\alpha^2+(1-\alpha)^2x}}{2(1-\alpha)\sqrt{(1+x)^3}\ \sqrt{\alpha^2+(1-\alpha)^2x}}.$$

For $\alpha \ge \dfrac{n}{2n+1}$, we note that $\phi'(x) \le 0$ for all $x \ge 0$ and therefore,

$$\phi(x) \le \phi(0) = \frac{1+n\alpha}{1-\alpha}.$$

If $0 \le \alpha < n/(2n+1)$, then

$$x_0 = \frac{n^2(1-2\alpha)^2-\alpha^2}{(1-\alpha)^2}$$

is the only critical point and that $\phi''(x_0) < 0$. This observation shows that, for $0 < \alpha < n/(2n+1)$, $\phi$ attains its maximum value at $x_0$ so that

$$\phi(x_0) = \sqrt{\frac{1+n^2(1-2\alpha)}{1-2\alpha}}.$$

This gives essentially a direct proof for Corollary 5.1.

**Case (II):** Now we consider the case $b \neq 0$. In this case, the proofs run into several subcases. Firstly, we consider $1/2 \le \alpha < 1/(1+nb)$. It follows that

$$N(x) \le n(1-2\alpha) \le 0,$$

because

$$nb(1-\alpha)\sqrt{1+x} \le \sqrt{\alpha^2+(1-\alpha)^2x}.$$

Indeed, the last inequality follows from the fact that $nb \le 1$,

$$0 \ge (1-\alpha)^2 - \alpha^2 = 1-2\alpha \ge n^2b^2(1-\alpha)^2 - \alpha^2,$$

and

$$x(1-\alpha)^2(1-n^2b^2) \ge n^2b^2(1-\alpha)^2 - \alpha^2.$$

Thus, $\phi'(x) \le 0$ for all $x \ge 0$ whenever $1/2 \le \alpha < 1/(1+nb)$. Next, we consider the case

$$\frac{n(b+1)}{n(b+2)+1} \le \alpha < 1/2.$$

In this case, it suffices to compute

$$N'(x) = -\frac{(1-\alpha)^2}{2\sqrt{\alpha^2+(1-\alpha)^2x}} + \frac{nb(1-\alpha)}{2\sqrt{1+x}}$$

and note that $N'(x) \le 0$ holds for $x \ge 0$ if and only if

$$x(1-\alpha)^2(1-n^2b^2) \ge n^2b^2\alpha^2 - (1-\alpha)^2.$$

Since $\alpha < 1/2$ implies that

$$0 > 2\alpha - 1 = \alpha^2 - (1-\alpha)^2 \ge n^2b^2\alpha^2 - (1-\alpha)^2,$$

the function $N(x)$ is decreasing for $x \ge 0$. Therefore, for $\dfrac{n(b+1)}{n(b+2)+1} \le \alpha < 1/2$, we have

$$N(x) \le N(0) = n(b+1) - \alpha(2n+nb+1) \le 0 \quad \text{for } x \ge 0.$$

The above observation shows that $\phi(x)$ defined by (5.4) is a decreasing function on $[0,\infty)$ whenever $\dfrac{n(b+1)}{n(b+2)+1} \le \alpha < \dfrac{1}{1+nb}$. In particular,

$$\phi(x) \le \phi(0) = \frac{1+n\alpha}{1-(1+nb)\alpha} \quad \text{for} \quad \frac{n(b+1)}{n(b+2)+1} \le \alpha < \frac{1}{1+nb}.$$

**Case (III):** Assume $b \neq 0$ and $0 \le \alpha < \dfrac{n(b+1)}{n(b+2)+1}$. We make the substitution

$$t = \frac{1}{\sqrt{\alpha^2+(1-\alpha)^2x}}$$

and note that

$$\sup_{x\in[0,\infty)} \phi(x) = \sup_{t\in(0,1/\alpha]} \psi(t),$$

where $\phi(x)$ becomes

$$\psi(t) = \frac{n+t}{\sqrt{1+(1-2\alpha)t^2} - nb},$$

with the above substitution. Now we compute

$$\psi'(t) = \frac{R(t)}{\left[\sqrt{1+(1-2\alpha)t^2} - nb\right]^2 \sqrt{1+(1-2\alpha)t^2}},$$

where

$$R(t) = 1 - n(1-2\alpha)t - nb\sqrt{1+(1-2\alpha)t^2}.$$

Since $R(t)$ decreases,

$$R(0) = 1 - nb \ge 0 > R(1/\alpha) = \frac{n(2+b)+1}{\alpha}\left[\alpha - \frac{n(1+b)}{n(b+2)+1}\right],$$

$R(t) \neq 0$ for $t > 1/(n(1-2\alpha))$, we get the estimate

$$M \leq \{\psi(t) : 0 \leq t \leq 1/(n(1-2\alpha)),\ R(t) = 0\} = \psi(s),$$

where

$$s = \frac{-(1-2\alpha) + b\sqrt{(1-2\alpha)(n^2+1-2\alpha n^2 - n^2 b^2)}}{n(1-2\alpha)(b^2-1+2\alpha)}.$$

A simple calculation shows that $f \in \mathcal{S}^*(\alpha)$ whenever

$$\lambda \leq \frac{1}{\psi(s)} = \frac{b\sqrt{(1-2\alpha)(1+n^2(1-2\alpha-b^2))} - (1-2\alpha)}{b - \sqrt{(1-2\alpha)(1+n^2(1-2\alpha-b^2))}}$$

which, by multiplying both the numerator and the denominator by the quantity

$$b + \sqrt{(1-2\alpha)(1+n^2(1-2\alpha-b^2))},$$

is seen to be equivalent to

$$\lambda \leq \frac{1}{\psi(s)} = \frac{\sqrt{(1-2\alpha)(1+n^2(1-2\alpha-b^2))} - n^2 b(1-2\alpha)}{1+n^2(1-2\alpha)}$$

for $0 \leq \alpha < \dfrac{n(b+1)}{n(b+2)+1}$. This completes the proof. □

## 6. Integral transforms

In this section we consider the following integral transform $I(f)$ of $f \in \mathcal{A}$ defined by

$$(6.1)\quad [I(f)](z) = F(z) = z\left[\frac{c+1-n}{z^{c+1-n}}\int_0^z \left(\frac{t}{f(t)}\right)^n t^{c-n}\,dt\right]^{1/n}, \quad c+1-n > 0.$$

When $c = n = 1$, (6.1) becomes

$$\int_0^z \frac{t}{f(t)}\,dt$$

which is similar to Alexander transform. Also, $I(f)$ is similar to Bernadi transformation when $n = 1$ and $c > 0$.

**Theorem 6.1.** *Let $f \in \mathcal{U}_n(\lambda)$ for some $\lambda > 0$ and $n \geq 1$. For $c+1-n > 0$, $F = I(f)$ be defined by (6.1). Then $F \in \mathcal{S}^*_\alpha$ whenever $|a_{n+1}|$, $c$, $\lambda$ are related by*

$$(6.2)\quad 0 < \lambda \leq \frac{c+2}{(c+1-n)(c+1)}\left[\frac{(1-\alpha)(c+1) - n(2-\alpha)(c+1-n)|a_{n+1}|}{1+(2-\alpha)n}\right].$$

*Proof.* From (6.2) we observe that

$$|a_{n+1}| < \left(\frac{c+1}{n(c+1-n)}\right)\left(\frac{1-\alpha}{2-\alpha}\right).$$

By (6.1), we see that

$$(c+1-n)\left(\frac{F(z)}{z}\right)^n + z\frac{d}{dz}\left(\frac{F(z)}{z}\right)^n = (c+1-n)\left(\frac{z}{f(z)}\right)^n.$$

It is a simple exercise to show that

$$\frac{1}{n(c+1-n)}\left[(c-n)(n+1)\left(\frac{F(z)}{z}\right)^n - (c-2n)\frac{d}{dz}\left(z\left(\frac{F(z)}{z}\right)^n\right)\right.$$
$$\left. -z\frac{d^2}{dz^2}\left(z\left(\frac{F(z)}{z}\right)^n\right)\right] = \left(\frac{z}{f(z)}\right)^{n+1} f'(z).$$

If we set

$$P(z) = z\left(\frac{F(z)}{z}\right)^n, \tag{6.3}$$

then, from the last equation and the assumption $f \in \mathcal{U}_n(\lambda)$, it follows that $P(z)$ satisfies the second order differential equation

$$\left(\frac{(c-n)(n+1)}{n(c+1-n)}\right)\frac{P(z)}{z} - \frac{(c-2n)P'(z)}{n(c+1-n)} - \frac{zP''(z)}{n(c+1-n)} = 1 + \lambda w(z) \tag{6.4}$$

where $w \in \mathcal{B}_n$. If we let $P(z) = z + \sum_{k=n+1}^{\infty} c_k z^k$ and $w(z) = \sum_{k=n+1}^{\infty} w_k z^k$ in (6.4), then, by equating the coefficients of $z^n$, we get the representations

$$\frac{P(z)}{z} = 1 + c_{n+1}\, z^n - \frac{\lambda n(c+1-n)}{c+1}\int_0^1 \frac{w(tz)}{t^{n+1}}(1-t^{c+1})\, dt \tag{6.5}$$

and

$$P'(z) = 1 + (n+1)c_{n+1}z^n - \frac{\lambda n(c+1-n)}{c+1}\int_0^1 \frac{w(tz)}{t^{n+1}}(n+1+(c-n)t^{c+1})\, dt \tag{6.6}$$

where

$$c_{n+1} = -\frac{n(c+1-n)}{c+1}a_{n+1}. \tag{6.7}$$

In view of the representation

$$\left(\frac{z}{f(z)}\right)^{n+1} f'(z) = \left(\frac{z}{f(z)}\right)^n - \frac{1}{n}z\left\{\left(\frac{z}{f(z)}\right)^n\right\}' = 1 + \lambda w(z) \quad (w \in \mathcal{B}_n),$$

it follows that (see Section 2)

$$\left(\frac{z}{f(z)}\right)^n = 1 - n\, a_{n+1}\, z^n - \lambda n \int_0^1 \frac{w(tz)}{t^{n+1}}\, dt\,.$$

From (6.3), we have

$$\frac{zF'(z)}{F(z)} - 1 = \frac{1}{n}\left(\frac{zP'(z)}{P(z)} - 1\right). \tag{6.8}$$

Using (6.5), (6.6) and (6.8), we find that

$$\begin{aligned}
&\frac{zF'(z)}{F(z)} - 1 \\
&= \frac{1}{n}\left[-1 + \frac{1 + (n+1)c_{n+1}z^n - \dfrac{\lambda n(c+1-n)}{c+1}\displaystyle\int_0^1 \frac{w(tz)}{t^{n+1}}(n+1+(c-n)t^{c+1})\,dt}{1 + c_{n+1}z^n - \dfrac{\lambda n(c+1-n)}{c+1}\displaystyle\int_0^1 \frac{w(tz)}{t^{n+1}}(1-t^{c+1})\,dt}\right] \\
&= \frac{1}{n}\left[\frac{nc_{n+1}z^n - \dfrac{\lambda n(c+1-n)}{c+1}\displaystyle\int_0^1 \frac{w(tz)}{t^{n+1}}(n+(c+1-n)t^{c+1})\,dt}{1 + c_{n+1}z^n - \dfrac{\lambda n(c+1-n)}{c+1}\displaystyle\int_0^1 \frac{w(tz)}{t^{n+1}}(1-t^{c+1})\,dt}\right]
\end{aligned}$$

so that

$$\begin{aligned}
\left|\frac{zF'(z)}{F(z)} - 1\right| &< \frac{|c_{n+1}| + \dfrac{\lambda(c+1-n)}{c+1}\displaystyle\int_0^1 (n+(c+1-n)t^{c+1})\,dt}{1 - |c_{n+1}| - \dfrac{\lambda n(c+1-n)}{c+1}\displaystyle\int_0^1 (1-t^{c+1})\,dt} \\
&= \frac{(c+1-n)\left[\dfrac{n|a_{n+1}|}{c+1} + \dfrac{\lambda(n+1)}{c+2}\right]}{1 - n(c+1-n)\left[\dfrac{|a_{n+1}|}{c+1} + \dfrac{\lambda}{c+2}\right]} \le 1-\alpha, \quad \text{by (6.2) and (6.7).}
\end{aligned}$$

This completes the proof. □

If we let $n = 1$ in Theorem 6.1, then we have the following

**Corollary 6.1.** *Let $f(z) = z + a_2z^2 + \cdots \in \mathcal{U}(\lambda)$ for some $\lambda > 0$. If $c > 0$ and $\alpha < 1$, then $F(z)$ defined in (6.1) is in $S^*_\alpha$ whenever $c, \lambda$ are related by*

$$0 < \lambda \le \frac{c+2}{c(c+1)}\left[\frac{(1-\alpha)(c+1) - (2-\alpha)c|a_2|}{3-\alpha}\right]. \tag{6.9}$$

From (6.9), we note that

$$|a_2| < \left(\frac{c+1}{c}\right)\left(\frac{1-\alpha}{2-\alpha}\right).$$

Corollary 6.1 was obtained recently by Ponnusamy *et al.* in [12]. We end the paper with the following conjecture which we are unable to handle at present.

**Conjecture.** *The results of Theorems 3.1, 3.2 and 6.1 are all sharp.*

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