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Rare α-Continuity

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Abstract. Popa [16] introduced the notion of rare continuity. In this paper, we introduce a new class of functions called rare α-continuous functions and investigate some of its fundamental properties. These functions are generalizations of both rare continuous and weak α-continuous functions [14].

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1. Introduction

Popa [16] introduced the notion of rare continuity as a generalization of weak continuity [6] which has been further investigated by Long and Herrington [8] and the present author [2, 3] Noiri [14] introduced and investigated weakly α-continuity as a generalization of weak continuity. He also introduced and investigated almost α-continuity [15] and showed that this type of continuity is a generalization of both α-continuity [12] and almost continuity [1]. Noiri showed that almost α-continuity is equivalent with almost feeble continuity [9].

The purpose of the present paper is to introduce the concept of rare α-continuity in topological spaces as a generalization of rare continuity and weak α-continuity. We investigate several properties of rarely α-continuous functions. It turns out that rare α-continuity implies both rare quasi continuity and rare precontinuity. The notion of $I.\alpha$-continuity is also introduced which is weaker than α-continuity and stronger than rare α-continuity. It is shown that when the codomain of a function is regular, then the notions of rare α-continuity and $I.\alpha$-continuity are equivalent.

2. Main results

Throughout this paper, X and Y are topological spaces. Recall that a rare set is a set R such that $\mathrm{Int}(R) = \emptyset$. A nowhere dense set is a set R with $\mathrm{Int}(\mathrm{Cl}(R)) = \emptyset$

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if $\mathrm{Cl}(R)$ is codense. Mashhour et al. [11] introduced the notion of preopen sets: A set A in X is called preopen if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$. The complement of a preopen set is called preclosed [11]. A set A in X is called semi-open [7] if there exists an open set such that $U \subset A \subset \mathrm{Cl}(U)$. Levine [7, Theorem 1] proved that A is semi-open if and only if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a semi-open set is called semi-closed. A set A in X is called α-open [13] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$. The complement of an α-open set is called α-closed. The intersection of all α-closed sets containing A is called the α-closure of A and is denoted by $\alpha\,\mathrm{Cl}(A)$. The subset A is α-closed if and only if $A = \alpha\,\mathrm{Cl}(A)$. The α-interior of A, denoted by $\alpha\,\mathrm{Int}(A)$, is defined by the union of all α-open sets of X contained in A. The family of all preopen (resp. semi-open and α-open) sets will be denoted by $PO(X)$ (resp. $SO(X)$ and $\alpha(X)$). We set $PO(X,x) = \{U \mid x \in U \in PO(X)\}$, $SO(X,x) = \{U \mid x \in U \in SO(X)\}$ and $\alpha(X,x) = \{U \mid x \in U \in \alpha(X)\}$.

Definition 2.1. *A function $f : X \to Y$ is called α-continuous* [12] *(resp. almost α-continuous* [15]*, weakly α-continuous* [14]*) if for each $x \in X$ and each open set G containing $f(x)$, there exists $U \in \alpha(X,x)$ such that $f(U) \subset G$ (resp. $f(U) \subset \mathrm{Int}(\mathrm{Cl}(G))$, $f(U) \subset \mathrm{Cl}(G)$).*

Definition 2.2. *A function $f : X \to Y$ is called rarely α-continuous (resp. rarely continuous* [16]*, rarely precontinuous* [4] *and rarely quasi continuous* [17]*) if for each $x \in X$ and each open set $G \subset Y$ containing $f(x)$, there exist a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ and $U \in \alpha(X,x)$ (resp. an open set $U \subset X$ containing x, $U \in PO(X,x)$ and $U \in SO(X,x)$) such that $f(U) \subset G \cup R_G$.*

Example 2.1. Let $X = \{a,b,c\}$, $\tau = \{X, \emptyset, \{c\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}$. Let $f : (X,\tau) \to (X,\sigma)$ be the identity function. Then f is rarely α-continuous.

Theorem 2.1. *The following statements are equivalent for a function $f : X \to Y$:*

(1) *The function f is rarely α-continuous at $x \in X$.*
(2) *For each open set G containing $f(x)$, there exists a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ such that $x \in \alpha\,\mathrm{Int}(f^{-1}(G \cup R_G))$.*
(3) *For each open set G containing $f(x)$, there exists a rare set R_G with $\mathrm{Cl}(G) \cap R_G = \emptyset$ such that $x \in \alpha\,\mathrm{Int}(f^{-1}(\mathrm{Cl}(G) \cup R_G))$.*
(4) *For each regular open set G containing $f(x)$, there exists a rare set R_G with $G \cap R_G = \emptyset$ such that $x \in \alpha\,\mathrm{Int}(f^{-1}(G \cup R_G))$.*
(5) *For each open set G containing $f(x)$, there exists $U \in \alpha(X,x)$ such that $\mathrm{Int}[f(U) \cap (Y \setminus G)] = \emptyset$.*
(6) *For each open set G containing $f(x)$, there exists $U \in \alpha(X,x)$ such that $\mathrm{Int}[f(U)] \subset \mathrm{Cl}(G)$.*

Proof. Similar to [17, Theorem 3.1]. □

Theorem 2.2. *A function $f : X \to Y$ is rarely α-continuous if and only if $f^{-1}(G) \subset \alpha\,\mathrm{Int}[f^{-1}(G \cup R_G)]$ for every open set G in Y, where R_G is a rare set with $G \cap \mathrm{Cl}(R_G) = \emptyset$.*

Proof. It is an immediate consequence of the above theorem. □

Definition 2.3. *A function $f : X \to Y$ is $I.\alpha$-continuous at $x \in X$ if for each open set $G \in Y$ containing $f(x)$, there exists an α-open set $U \subset X$ containing x such that* $\text{Int}[f(U)] \subset G$.

If f has this property at each point $x \in X$, then we say that f is $I.\alpha$-continuous on X.

Example 2.2. Let $X = Y = \{a, b, c\}$ and $\tau = \sigma = \{X, \emptyset, \{a\}\}$. Then a function $f : X \to Y$ defined by $f(a) = f(b) = a$ and $f(c) = c$ is $I.\alpha$-continuous.

Remark 2.1. It should be noted that $I.\alpha$-continuity is weaker than α-continuity and stronger than rare α-continuity.

Theorem 2.3. *Let Y be a regular space. Then the function $f : X \to Y$ is $I.\alpha$-continuous on X if and only if f is rarely α-continuous on X.*

Proof. "Necessity". It is evident.
"Sufficiency". Let f be rarely α-continuous on X. Suppose that $f(x) \in G$, where G is an open set in Y and $x \in X$. By the regularity of Y, there exists an open set G_1 in Y such that $f(x) \in G_1$ and $\text{Cl}(G_1) \subset G$. Since f is rarely α-continuous, then there exists $U \in \alpha(X, x)$ such that $\text{Int}[f(U)] \subset \text{Cl}(G_1)$. This implies that $\text{Int}[f(U)] \subset G$, which means that f is $I.\alpha$-continuous on X. □

Theorem 2.4. *If $f : X \to Y$ is rarely precontinuous and rarely quasi continuous, then f is rarely α-continuous.*

Proof. It follows from the well-known fact that $\alpha(X) = SO(X) \cap PO(X)$. □

Remark 2.2. It should be noted that rare α-continuity is weaker than both rare continuity and weak α-continuity and stronger than both rare precontinuity and rare quasi continuity.

Definition 2.4. *A function $f : X \to Y$ is called pre-α-open if for every $U \in \alpha(X)$, $f(U) \in \alpha(Y)$.*

Theorem 2.5. *If a function $f : X \to Y$ is a pre-α-open rarely α-continuous, then f is almost α-continuous.*

Proof. Suppose that $x \in X$ and G is any open set of Y containing $f(x)$. Since f is rarely α-continuous at x, then there exists $U \in \alpha(X, x)$ such that $\text{Int}[f(U)] \subset \text{Cl}(G)$. Since f is pre-α-open, we have $f(U) \in \alpha(Y)$. This implies that $f(U) \subset \text{Int}(\text{Cl}(\text{Int}(f(U)))) \subset \text{Int}(\text{Cl}(G))$. Hence f is almost α-continuous. □

We say that a function $f : X \to Y$ is $r.\alpha$-open if the image of an α-open set is open.

Theorem 2.6. *Let $f : X \to Y$ be an $r.\alpha$-open rarely α-continuous function. Then f is weakly α-continuous.*

Proof. It is an immediate consequence of Theorem 2.5. □

Theorem 2.7. *If $f : X \to Y$ is rarely α-continuous function, then the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for every x in X is rarely α-continuous.*

Proof. Suppose that $x \in X$ and W is any open set containing $g(x)$. It follows that there exists open sets U and V in X and Y, respectively, such that $(x, f(x)) \in U \times V \subset W$. Since f is rarely α-continuous, there exists $G \in \alpha(X, x)$ such that $\mathrm{Int}[f(G)] \subset \mathrm{Cl}(V)$. Let $E = U \cap G$. It follows that $E \in \alpha(X, x)$ and we have $\mathrm{Int}[g(E)] \subset \mathrm{Int}(U \times f(G)) \subset U \times \mathrm{Cl}(V) \subset \mathrm{Cl}(W)$. Therefore, g is rarely α-continuous. □

Recall that a topological space X is called α-compact [10] if every cover of X by α-open sets has a finite subcover.

Definition 2.5. *Let $A = \{G_i\}$ be a class of subsets of X. By rarely union sets* [2] *of A we mean $\{G_i \cup R_{G_i}\}$, where each R_{G_i} is a rare set such that each of $\{G_i \cap \mathrm{Cl}(R_{G_i})\}$ is empty.*

Definition 2.6. *A space X is called rarely almost compact* [2] *if each open cover has a finite subfamily whose rarely union sets cover the space.*

Theorem 2.8. *Let $f : X \to Y$ be rarely α-continuous and K be an α-compact set in X. Then $f(K)$ is a rarely almost compact subset of Y.*

Proof. Its proof is similar to the proof of [2, Theorem 4.1]. □

Lemma 2.1. [8] *If $g : Y \to Z$ is continuous and one-to-one, then g preserves rare sets.*

Theorem 2.9. *If $f : X \to Y$ is rarely α-continuous surjection and $g : Y \to Z$ is continuous and one-to-one, the $g \circ f : X \to Z$ is rarely α-continuous.*

Proof. Suppose that $x \in X$ and $(gf)(x) \in V$, where V is an open set in Z. By hypothesis, g is continuous, therefore there exists an open set $G \subset Y$ containing $f(x)$ such that $g(G) \subset V$. Since f is rarely α-continuous, there exists a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ and an α-open set U containing x such that $f(U) \subset G \cup R_G$. It follows from Lemma 2.1 that $g(R_G)$ is a rare set in Z. Since R_G is a subset of $Y \setminus G$ and g is injective, we have $\mathrm{Cl}(g(R_G)) \cap V = \emptyset$. This implies that $gf(U) \subset V \cup g(R_G)$. Hence the result. □

Theorem 2.10. *Let $f : X \to Y$ be pre-α-open and $g : Y \to Z$ a function such that $g \circ f : X \to Z$ is rarely α-continuous. Then g is rarely α-continuous.*

Proof. Let $y \in Y$ and $x \in X$ such that $f(x) = y$. Let G be an open set containing $(gf)(x)$. Then there exists a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ and an α-open set U containing x such that $(f)(U) \subset G \cup R_G$. But $f(U)$ is an α-open set containing $f(x) = y$ such that $g(f(U)) = (gf)(U) \subset G \cup R_G$. This shows that g is rarely α-continuous at y. □

Theorem 2.11. *If $f : X \to Y$ is rarely α-continuous and for each point $x \in X$ and each open set $G \subset Y$ containing $f(x)$, $f^{-1}(\mathrm{Cl}(R_G))$ is a closed set in X, then f is α-continuous.*

Proof. Similar to [17, Theorem 3.2]. □

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