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Semi θ-Continuity in Intuitionistic Fuzzy Topological Spaces

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Abstract. This paper is devoted to the study of intuitionistic fuzzy topological spaces with specific attention to the strong forms of intuitionistic fuzzy continuity. Here we introduce intuitionistic fuzzy semi θ-continuity and intuitionistic fuzzy semi θ-open(closed) functions. Besides many properties and basic results, results related to the product of functions and the graph of functions are obtained.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [9], and later Atanassov [1] generalized this idea to intuitionistic fuzzy sets. On the other hand, Coker [2] introduced the notions of intuitionistic fuzzy topological spaces, fuzzy continuity and some other related concepts. In this paper, we introduce the concept of semi θ-continuity in intuitionistic fuzzy topological spaces. This type of functions have been characterized and investigated in light of notions of quasi-coincident [5] and θ-neighbourhood [8]. For definitions and results not explained in this paper, we refer to the papers [1, 2, 5, 7], assuming them to be well known. The words "neighbourhood", "continuous", "quasi-coincident", "not quasi-coincident" and "irresolute" will be abbreviated as respectively "*nbd*", "*cont.*", "q", "$\widetilde{q}$" and "i".

2. Preliminaries

First, we present the fundamental definitions (see [2]).

Definition 2.1. [1] *Let X be a nonempty fixed set. An intuitionistic fuzzy set (IFS, for short) U is an object having the form $U = \{\langle x, \mu_U(x), \gamma_U(x)\rangle : x \in X\}$ where the functions $\mu_U : X \to I$ and $\gamma_U : X \to I$ denote respectively the degree of membership (namely $\mu_U(x)$) and the degree of nonmembership (namely $\gamma_U(x)$) of each element $x \in X$ to the set U, and $0 \leq \mu_U(x) + \gamma_U(x) \leq 1$ for each $x \in X$.*

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The reader may consult [2, 3, 5] to see several types of relations and operations on IFS's, intuitionistic fuzzy points (IFP's, for short) and some properties of images and preimages of IFS's.

Definition 2.2. [2] *An intuitionistic fuzzy topology (IFT, for short) on a nonempty set X is a family Ψ of IFS's in X containing $\underset{\sim}{0}, \underset{\sim}{1}$ and closed under finite infima and arbitrary suprema. In this case the pair (X, Ψ) is called an intuitionistic fuzzy topological space (IFTS, for short) and each IFS in Ψ is known as an intuitionistic fuzzy open set (IFOS, for short) in X. The complement $\overline{U}$ of an $IFOS$ U in an $IFTS$ (X, Ψ) is called an intuitionistic fuzzy closed set (IFCS, for short) in X.*

Definition 2.3. [7] *Let X, Y be nonempty sets and $U = \langle x, \mu_U(x), \gamma_U(x)\rangle$, $V = \langle y, \mu_V(y), \gamma_V(y)\rangle$ IFS's of X and Y, respectively. Then $U \times V$ is an IFS of $X \times Y$ defined by:*

$$(U \times V)(x, y) = \langle (x, y), \min(\mu_U(x), \mu_V(y)), \max(\gamma_U(x), \gamma_V(y))\rangle.$$

Definition 2.4. [7] *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$. The product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is defined by:*

$$(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2)) \ \forall (x_1, x_2) \in X_1 \times X_2.$$

Definition 2.5. [7] *Let $f : X \to Y$ be a function. The graph $g : X \to X \times Y$ of f is defined by: $g(x) = (x, f(x))$ $\forall x \in X$.*

Lemma 2.1. [7] *Let $f_i : X_i \to Y_i (i = 1, 2)$ be functions and U, V IFS's of Y_1, Y_2, respectively, then*

$$(f_1 \times f_2)^{-1}(U \times V) = f_1^{-1}(U) \times f_2^{-1}(V).$$

Definition 2.6. [7] *Let $(X, \Psi), (Y, \Phi)$ be IFTS's and $A \in \Psi$, $B \in \Phi$. We say that (X, Ψ) is product related to (Y, Φ) if for any IFS's U of X and V of Y, whenever $(\overline{A} \ngeq U$ and $\overline{B} \ngeq V$) $\Rightarrow (\overline{A} \times \underset{\sim}{1} \cup \underset{\sim}{1} \times \overline{B} \geq U \times V$), there exist $A_1 \in \Psi, B_1 \in \Phi$ such that $\overline{A}_1 \geq U$ or $\overline{B_1} \geq V$ and $\overline{A}_1 \times \underset{\sim}{1} \cup \underset{\sim}{1} \times \overline{B}_1 = \overline{A} \times \underset{\sim}{1} \cup \underset{\sim}{1} \times \overline{B}$.*

Theorem 2.1. [7] *Let (X, Ψ) and (Y, Φ) be IFTS's such that X is product related to Y. Then for IFS's U of X and V of Y we have:*

(i) $\mathrm{cl}(U \times V) = \mathrm{cl}(U) \times \mathrm{cl}(V)$;
(ii) $\mathrm{int}(U \times V) = \mathrm{int}(U) \times \mathrm{int}(V)$.

Definition 2.7. *Let U be an IFS of an IFTS X. Then*

(i) U is said to be an intuitionistic fuzzy semi open(semi closed) set [6] *($IFSOS$ ($IFSCS$), for short) if $U \leq \mathrm{cl}(\mathrm{int}(U))$ ($\mathrm{int}(\mathrm{cl}(U)) \leq U$).*
(ii) The semi-closure of U is denoted and defined by :

$$\mathrm{cl}_s(U) = \wedge\{K : K \text{ is } IFSCS \text{ in } X \text{ and } U \leq K\}.$$

(iii) The semi-interior of U is denoted and defined by :

$$\mathrm{int}_s(U) = \vee\{G : G \text{ is } IFSOS \text{ in } X \text{ and } G \leq U\}.$$

Definition 2.8. *An IFS U of an IFTS X is called*

(i) $\varepsilon - nbd$ [3]($\varepsilon\theta - nbd$) *of an IFP* $c(a,b)$, *if there exists an* $IFOS$ $(IF\theta OS)$ G *in* X *such that* $c(a,b) \in G \leq U$.

(ii) $\varepsilon q - nbd$ [8]($\varepsilon sq - nbd$) *of an IFP* $c(a,b)$, *if there exists an* $IFOS$ $(IFSOS)$ G *in* X *such that* $c(a,b) q G \leq U$.

(iii) $\varepsilon\theta q - nbd$ [8] *of an IFP* $c(a,b)$, *if there exists an* $\varepsilon q - nbd$ G *of* $c(a,b)$ *such that* $\mathrm{cl}(G)\ \widetilde{q}\ \overline{U}$.

The family of all $\varepsilon - nbd$ *(resp.* $\varepsilon\theta - nbd$, $\varepsilon q - nbd$, $\varepsilon sq - nbd$, $\varepsilon\theta q - nbd$ *) of an IFP* $c(a,b)$ *will be denoted by* N_ε*(resp.* N_ε^θ, N_ε^q, N_ε^{sq}, $N_\varepsilon^{\theta q}$*)*$(c(a,b))$.

Definition 2.9. [8] *An IFP* $c(a,b)$ *is said to be intuitionistic fuzzy* θ*-cluster point (IF*θ*-cluster point, for short) of an* IFS U *iff for each* $A \in N_\varepsilon^q(c(a,b))$, $\mathrm{cl}(A)\ q\ U$.

The set of all $IF\theta$*-cluster points of* U *is called the intuitionistic fuzzy* θ*-closure of* U *and denoted by* $\mathrm{cl}_\theta(U)$. *An IFS* U *will be called* $IF\theta$*-closed (IF*θ*CS, for short) iff* $U = \mathrm{cl}_\theta(U)$. *The complement of an* $IF\theta$*-closed set is* $IF\theta$*-open (IF*θ*OS, for short). The* θ*-interior of* U *is denoted and defined by* $\mathrm{int}_\theta(U) = \underset{\sim}{1} - \mathrm{cl}_\theta(\underset{\sim}{1} - U)$.

Definition 2.10. *Let* (X,Ψ) *and* (Y,Φ) *be two IFTS's. A function* $f : X \to Y$ *is said to be:*

(i) $IFcont.$ [2]*(*$IFsemi$*-cont.(IFS-cont., for short)* [6]*) if the preimage of each* $IFOS$ *in* Y *is* $IFOS(IFSOS)$ *in* X.

(ii) IFi *(*$IFsuper$ i*) function if the preimage of each* $IFSOS$ *in* Y *is* $IFSOS$ *(*$IFOS$*) in* X.

(iii) IF*-*$q\theta$*(resp.* $IFstr$*-*θ [8]*,* $IFfaintly$*,* $IF\lambda\theta$ [8]*) cont. if the preimage of each* $IF\theta OS$ *(resp.* $IFOS, IF\theta OS$*,* $IF\lambda OS$ *) of* Y *is* $IF\theta OS$ *(resp.* $IF\theta OS$*,*
$IFOS, IF\theta OS$*) in* X.

(iv) $IFopen$ *(resp.* $IFclosed$*,* $IFsemiclosed$*,* $IFstr$*-*$\theta open$*,* $IFsemiopen$*,* $IFfaintly$*-*$open$*,* $IFfaintly$ $semiopen$*,* $IF\lambda\theta$*-*$open$*) if the image of each* $IFOS$ *(resp.* $IFCS, IFCS, IFOS, IFOS, IF\theta OS, IF\theta OS, IF\lambda OS$*) of* X *is* $IFOS$ *(resp.* $IFCS, IFSCS, IF\theta OS, IFSOS, IFOS, IFSOS, IF\theta OS$*) in* Y.

Definition 2.11. *An IFTS* (X,Ψ) *is called :*

(i) $IFcompact$ [2] *(resp.* $IFalmost$ $compact$ [4]*) if for every IFopen cover* $\{U_j : j \in J\}$ *of* X*, there exists a finite subfamily* $J_\circ \subset J$ *such that* $X = \vee\{U_j : j \in J_\circ\}$ *(resp.* $X = \vee\{\mathrm{cl}(U_j) : j \in J_\circ\}$*).*

(ii) $IFsemi$*-*$compact$ *if for every IFsemiopen cover* $\{U_j : j \in J\}$ *of* X*, there exists a finite subfamily* $J_\circ \subset J$ *such that* $X = \vee\{U_j : j \in J_\circ\}$.

(iii) IFT_2 [2] *iff for every IFP's* $c(a,b)$*,* $d(m,n)$ *in* X *and* $c \neq d$*, there exist* $G = \langle x, \mu_G(x), \gamma_G(x)\rangle$*,* $H = \langle x, \mu_H(x), \gamma_H(x)\rangle \in \Psi$ *with* $\mu_G(c) = 1$, $\gamma_G(c) = 0$, $\mu_H(d) = 1$, $\gamma_H(d) = 0$ *and* $G \wedge H = \underset{\sim}{0}$.

3. Intuitionistic fuzzy semi $\theta-$continuity

Definition 3.1. *A function $f:(X,\Psi)\to(Y,\Phi)$ is said to be intuitionistic fuzzy semi θ-cont.($IFS\theta$-cont., for short), if for each IFP $c(a,b)$ in X and $V\in N_\varepsilon^{sq}(f(c(a,b)))$, there exists $U\in N_\varepsilon^{\theta q}(c(a,b))$ such that $f(U)\leq V$.*

Theorem 3.1. *Let $f:(X,\Psi)\to(Y,\Phi)$ be a function. Then the following are equivalent:*

(i) f *is an* $IFS\theta$*-cont..*
(ii) $f^{-1}(V)$ *is an* $IF\theta OS$ *in* X*, for each* $IFSOS$ V *in* Y.
(iii) $f^{-1}(H)$ *is an* $IF\theta CS$ *in* X*, for each* $IFSCS$ H *in* Y.
(iv) $\mathrm{cl}_\theta(f^{-1}(V))\leq f^{-1}(\mathrm{cl}_s(V))$*, for each* IFS V *in* Y.
(v) $f^{-1}(\mathrm{int}_s(G))\leq \mathrm{int}_\theta(f^{-1}(G))$*, for each* IFS G *in* Y.

Proof. $(i)\Rightarrow(ii)$: Let V be an $IFSOS$ in Y and $c(a,b)$ be IFP in X such that $c(a,b)qf^{-1}(V)$. Since f is $IFS\theta$ cont., there exists $U\in N_\varepsilon^{\theta q}(c(a,b))$ such that $f(U)\leq V$. Then $c(a,b)qU\leq f^{-1}f(U)\leq f^{-1}(V)$ which shows that $f^{-1}(V)\in N_\varepsilon^{\theta q}(c(a,b))$ and then is an $IF\theta OS$ of X.

$(ii)\Rightarrow(iii)$: By taking the complement.

$(iii)\Rightarrow(iv)$: Let V be an IFS in Y Since $V\leq \mathrm{cl}_s(V)$,then $f^{-1}(V)\leq f^{-1}(\mathrm{cl}_s(V))$. Using (iii), $f^{-1}(\mathrm{cl}_s(V))$ is an $IF\theta OS$ in X. Thus $\mathrm{cl}_\theta(f^{-1}(V))\leq \mathrm{cl}_\theta(f^{-1}(\mathrm{cl}_s(V)))=f^{-1}(\mathrm{cl}_s(V))$.

$(iv)\Rightarrow(v)$: Using (iv), $\mathrm{cl}_\theta(f^{-1}(V))\leq f^{-1}(\mathrm{cl}_s(V))$, then $\overline{\mathrm{cl}_\theta(f^{-1}(V))}\geq\overline{f^{-1}(\mathrm{cl}_s(V))}$. Hence $\mathrm{int}_\theta(\overline{f^{-1}(V)})\geq f^{-1}(\overline{\mathrm{cl}_s(V)})$. Thus $f^{-1}(\mathrm{int}_s(\overline{V}))\leq \mathrm{int}_\theta(f^{-1}(\overline{V}))$. Put $G=\overline{V}$, then $f^{-1}(\mathrm{int}_s(G))\leq\mathrm{int}_\theta(f^{-1}(G))$.

$(v)\Rightarrow(i)$: Let V be an $IFSOS$ in Y. Then $\mathrm{int}_s(V)=V$. Using (v), $f^{-1}(V)\leq \mathrm{int}_\theta(f^{-1}(V))$. Hence $f^{-1}(V)=\mathrm{int}_\theta(f^{-1}(V))$ i.e. $f^{-1}(V)$ is an $IF\theta OS$ in X. Let $c(a,b)$ be any IFP in $f^{-1}(V)$. Then $c(a,b)qf^{-1}(V)$ implies $f(c(a,b))qff^{-1}(V)\leq V$. Thus for any IFP $c(a,b)$ and each $V\in N_\varepsilon^{sq}(f(c(a,b)))$, there exists $U=f^{-1}(V)\in N_\varepsilon^{\theta q}(c(a,b))$ such that $f(U)\leq V$. Thus f is $IFS\theta$*-cont.*. □

Theorem 3.2. *Let f be a bijective function from an $IFTS(X,\Psi)$ into an $IFTS(Y,\Phi)$. Then f is an $IFS\theta$-cont. iff $\mathrm{int}_s(f(U))\leq f(\mathrm{int}_\theta(U))$, for each IFS U of X.*

Proof. ($\Rightarrow$:): Let f be an $IFS\theta$ cont. function and U be an IFS in X. Hence $f^{-1}(\mathrm{int}_s(f(U)))$ is an $IF\theta OS$ in X. Since f is injective function and using Theorem 3.1(v), we have: $f^{-1}(\mathrm{int}_s(f(U)))\leq\mathrm{int}_\theta(f^{-1}(f(U)))=\mathrm{int}_\theta(U)$. Since f is surjective, $ff^{-1}(\mathrm{int}_s(f(U)))\leq f(\mathrm{int}_\theta(U))$ i.e. $\mathrm{int}_s(f(U))\leq f(\mathrm{int}_\theta(U))$.

($\Leftarrow$:): Let V be an $IFSOS$ in Y. Then $V=\mathrm{int}_s(V)$. Using the hypothesis, we have: $V=\mathrm{int}_s(V)=\mathrm{int}_s(ff^{-1}(V))\leq f(\mathrm{int}_\theta(f^{-1}(V)))$, which implies that $f^{-1}(V)\leq f^{-1}f(\mathrm{int}_\theta(f^{-1}(V)))$. From the fact that f is injective, we have: $f^{-1}(V)\leq\mathrm{int}_\theta(f^{-1}(V))$. Hence $f^{-1}(V)=\mathrm{int}_\theta(f^{-1}(V))$ i.e. $f^{-1}(V)$ is an $IF\theta OS$ in X. Thus f is an $IFS\theta$*-cont.*. □

Theorem 3.3. *Let $f:(X,\Psi)\to(Y,\Phi)$ be a bijective function. Then f is an $IFS\theta$-cont. iff $f(\mathrm{cl}_\theta(U))\leq\mathrm{cl}_s(f(U))$, for each IFS U of X.*

Proof. Similar to the proof of Theorem 3.2. □

Lemma 3.1. *Every $IFS\theta$-cont. function is $IF\lambda\theta$-cont.*

Proof. From the fact that every $IF\lambda OS$ is $IFSOS$. $\square$

Remark 3.1. From the above discussion, one can illustrate the following implications: $IFS\theta$-*cont.* $\Longrightarrow$ $IF\lambda\theta$-*cont.* $\Longrightarrow$ $IFstr\theta$-*cont.* $\Longrightarrow$ IF-*cont.*

The converse of the above implications need not be true in general, as shown in the following example and remark.

Example 3.1. Let $X = [0,1]$ and consider the IFS's $U = \langle x, \mu_U(x), \gamma_U(x)\rangle$ and $V = \langle x, \mu_V(x), \gamma_V(x)\rangle$ as follows: $\mu_U(x) = \frac{3}{4}$, for all $x \in I$; $\gamma_U(x) = \frac{1}{3}\ \forall\ x \in I$ and $\mu_V(x) = \frac{2}{3}$, for all $x \in I$; $\gamma_V(x) = \frac{1}{4}\ \forall\ x \in I$.

Now, the families $\Psi = \{\underset{\sim}{0}, \underset{\sim}{1}, U, \overline{U}\}$ and $\Phi = \{\underset{\sim}{0}, \underset{\sim}{1}, U,\}$ are IFTS's on X. If we define the identity function $f : (X,\Psi) \to (Y,\Phi)$, then f is an $IFstr\theta$-*cont.* function since $f^{-1}(U) = U$ is an $IF\theta OS$ in (X,Ψ). But f is not $IF\lambda\theta$-*cont.* (Indeed, in (X,Ψ), $V \leq \text{int}(\text{cl}(\text{int}(V))) = \underset{\sim}{1}$, then V is an $IF\lambda OS$ in (X,Ψ). We notice that $f^{-1}(V) = V$ is not $IF\theta OS$ in (X,Ψ) because there exists an IFP $c(\frac{1}{2},\frac{1}{4})$ in (X,Ψ), $c(\frac{1}{2},\frac{1}{4})qV$ and only $\underset{\sim}{1}, U \in N^q_{\varepsilon}(c(\frac{1}{2},\frac{1}{4}))$ but $c(\frac{1}{2},\frac{1}{4})qcl(U) = U \nleq V$).

Remark 3.2. From the above Example it is obvious that:

(i) $IFstr\theta$-cont. $\not\Longrightarrow$ $IFS\theta$-cont.
(ii) IF-cont.$\not\Longrightarrow$ $IF\lambda\theta$-*cont.*
(iii) IF-cont.$\not\Longrightarrow$ $IFS\theta$-*cont.*

Definition 3.2. *Let X,Y be non empty sets and $c(a,b)$, $d(m,n)$ IFP's of X,Y, respectively.*

(i) $c(a,b) \times d(m,n)$ is an IFP of $X \times Y$ defined by :

$$(c(a,b) \times d(m,n))(x,y) = \langle (x,y), \min(a,m), \max(b,n)\rangle$$

(ii) Let $U = \langle x, \mu_U, \gamma_U\rangle$ and $V = \langle y, \mu_V, \gamma_V\rangle$ be IFS's of X and Y, respectively. Then, $(c(a,b), d(m,n))(x,y)q(U \times V)(x,y)$ iff $a > \gamma_U(c)$ and $m > \gamma_V(d)$ or $b < \mu_U(c)$ and $n < \mu_V(d)$.

Lemma 3.2. *Let $c(a,b)$, $d(m,n)$ be IFP's in X and $U = \langle x, \mu_U, \gamma_U\rangle$ and $V = \langle y, \mu_V, \gamma_V\rangle$ be IFS's in X then the following implication hold:*

$$c(a,b)qU \text{ and } d(m,n)qV \Rightarrow (c(a,b), d(m,n))q(U \times V)$$

Proof. Since $c(a,b)qU$ and $d(m,n)qV$. Using [5], we have $a > \gamma_U(c)$ or $b < \mu_U(c)$ and $m > \gamma_V(d)$ or $n < \mu_V(d)$. Hence using Definition 3.2, we have:

$$(c(a,b) \times d(m,n))(x,y) = \langle (x,y), \min(a,m), \max(b,n)\rangle$$

and

$$(U \times V)(x,y) = \langle (x,y), \min(\mu_U(x), \mu_V(y)), \max(\gamma_U(x), \gamma_V(y))\rangle.$$

Since $a > \gamma_U(c)$ and $m > \gamma_V(d)$ or $b < \mu_U(c)$ and $n < \mu_V(d)$. Hence using Definition 3.2, $(c(a,b), d(m,n))q(U \times V)$. $\square$

Lemma 3.3. *Let X,Y be $IFTS's$ such that X is product related to Y. Then the product $U \times V$ of $IF\theta OS$ U of X and $IF\theta OS$ V of Y is an $IF\theta OS$ of $X \times Y$.*

Proof. Let $c(a,b)$ and $d(m,n)$ are $IFP's$ in X and Y respectively, such that $c(a,b)qU$ and $d(m,n)qV$. Since U and V are $IF\theta OS's$, there exists $IFOS's$ G and H in X and Y respectively, such that $c(a,b)qcl(G) \leq U$ and $d(m,n)qcl(H) \leq V$. Using Lemma 3.2 and Theorem 2.1, we have:

$$(c(a,b), d(m,n))q(\mathrm{cl}(G) \times \mathrm{cl}(H)) = \mathrm{cl}(G \times H) \leq U \times V$$

Hence $U \times V$ is an $IF\theta OS$. □

Theorem 3.4. *Let X_1, X_2, Y_1 and Y_2 are $IFTS's$ such that X_1 is product related to X_2 and $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$. Then the product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ of $IFS\theta$-cont. functions f_1 and f_2 is an $IFS\theta$-cont..*

Proof. Let $G = \vee(U_i \times V_j)$ be an $IFSOS$ of $Y_1 \times Y_2$, where U_i's and V_j's are $IFSOS's$ of Y_1 and Y_2 respectively. Using [3, Corollary 2.10(e)] and Lemma 2.1, we have:

$$\begin{aligned}(f_1 \times f_2)^{-1}(G) &= (f_1 \times f_2)^{-1}(\vee(U_i \times V_j)) \\ &= \vee(f_1 \times f_2)^{-1}(U_i \times V_j) \\ &= \vee(f_1^{-1}(U_i) \times f_2^{-1}(V_j)).\end{aligned}$$

Since $f_1^{-1}(U_i)$ and $f_2^{-1}(V_j)$ are $IF\theta OS's$ of X and Y respectively. Hence by Lemma 3.3, $f_1^{-1}(U_i) \times f_2^{-1}(V_j)$ is an $IF\theta OS$. So, $\vee(f_1^{-1}(U_i) \times f_2^{-1}(V_j)$ is an $IF\theta OS$. Hence $f_1 \times f_2$ is an $IFS\theta$-*cont.*. □

Theorem 3.5. *A function $f : (X, \Psi) \to (Y, \Phi)$ is an $IFS\theta$-cont. if the graph function $g : X \to X \times Y$ is an $IFS\theta$-cont..*

Proof. Let g be an $IFS\theta$-*cont.* function and $c(a,b)$ be any IFP in X. If $V \in N_\varepsilon^{sq}(f(c(a,b)))$, then $X \times V \in N_\varepsilon^{sq}(g(c(a,b)))$ in $X \times Y$. Since g is $IFS\theta$-*cont.*, there exists $U \in N_\varepsilon^{\theta}(c(a,b))$ such that $g(U) \leq X \times V$. This implies that $f(U) \leq V$. Thus f is $IFS\theta$-*cont.*.

□

4. Compositions and some preservation results

Theorem 4.1. *If $f : X \to Y$ is an $IFS\theta$-cont. and $g : Y \to Z$ is an IFi function, then $g \circ f : X \to Z$ is an $IFS\theta$-cont. function.*

Proof. Straightforward. □

Corollary 4.1. *The composition of two $IFS\theta$-cont. functions is an $IFS\theta$-cont. function.*

Theorem 4.2. *The following hold for functions $f : X \to Y$ and $g : Y \to Z$:*

(i) If f is an $IFS\theta$-cont. and g is an IFS-cont., then $g \circ f : X \to Z$ is an $IFstr$-θ cont. function.
(ii) If f is an $IFq\theta$-cont. and g is an $IFS\theta$-cont., then $g \circ f : X \to Z$ is an $IFS\theta$-cont. function.
(iii) If f is an $IF faintly$ cont. and g is an $IFS\theta$-cont., then $g \circ f : X \to Z$ is an $IF super\ i$ function.

(iv) *If f is an $IFS\theta$-cont. and g is an $IFcont.$, then $g \circ f : X \to Z$ is an $IFS\theta$-cont. function.*

Proof. Straightforward. □

Theorem 4.3. *Let X, Y and Z are $IFTS's$. If $f : X \to Y$ is an $IFfaintly$ semiopen and $IFS\theta$-cont. surjection function and $g : Y \to Z$ is a function such that $g \circ f$ is an $IFS\theta$-cont., hence g is IFS-cont..*

Proof. Let V be an $IFOS$ in Z, hence V is an $IFSOS$ [since every $IFOS$ is $IFSOS$]. Since $g \circ f$ is an $IFS\theta$-*cont.*, then $(g \circ f)^{-1}(V)$ is an $IF\theta OS$ in X. Since f is an $IFfaintly\ open$ surjection, hence $f((g \circ f)^{-1}(V)) = f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is an $IFSOS$ in Y. Thus g is IFS-*cont.*. □

Theorem 4.4. *Let $f : X \to \prod_{j \in J} X_j$ be $IFS\theta$-cont. function and let $f_j : X \to X_j$, for each $j \in J$, defined by $f_j(c(a, b)) = c_j(a_j, b_j)$ if $f(c(a, b)) = c_j(a_j, b_j)$. Then the function f_j is an $IFS\theta$-cont., for each $j \in J$.*

Proof. Let P_j denote the projection of $\prod_{j \in J} X_j$ onto X_j. Then obviously $f_j = P_j \circ f$ for each $j \in J$. Since f is an $IFS\theta$-*cont.*, then each f_j is so by Theorem 4.2(iv). □

Definition 4.1. *An $IFTS$ (X, Ψ) is said to be an $IF\theta T_2$ iff for every IFP's $c(a, b)$, $d(m, n)$ in X and $c \neq d$, there exist $IF\theta OS\prime s$ $G = \langle x, \mu_G(x), \gamma_G(x)\rangle$, $H = \langle x, \mu_H(x)$,*
$\gamma_H(x)\rangle \in \Psi$ with

$$\mu_G(c) = 1\ , \quad \gamma_G(c) = 0\ ,$$
$$\mu_H(d) = 1\ , \quad \gamma_H(d) = 0 \quad and \quad G \wedge H = \underset{\sim}{0}.$$

Remark 4.1. From Definition 4.6 and Definition 2.11(iii), it is clear that:

$$IF\theta T_2S \Rightarrow IFT_2S.$$

Theorem 4.5. *Let $f : (X, \Psi) \to (Y, \Phi)$ be an injective and $IFS\theta$-cont. function. If (Y, Φ) is an IFT_2S, then (X, Ψ) is an $IF\theta T_2S$.*

Proof. Let $c(a, b)$, $d(m, n)$ be IFP's in X and $c \neq d$. By injective f, $f(c) \neq f(d)$ and by the IFT_2 property of Y, there exist $IFOS\prime s$ $G = \langle y, \mu_G, \gamma_G\rangle$, $H = \langle y, \mu_H, \gamma_H\rangle$ of Φ with $\mu_G(f(c)) = 1$, $\gamma_G(f(c)) = 0$, $\mu_H(f(d)) = 1$, $\gamma_H(f(d)) = 0$ and $G \wedge H = \underset{\sim}{0}$. Since f is an $IFS\theta$-*cont.*, then

$$f^{-1}(G) = \langle x, f^{-1}(\mu_G), f^{-1}(\gamma_G)\rangle,\ f^{-1}(H) = \langle x, f^{-1}(\mu_H), f^{-1}(\gamma_H)\rangle$$

are $N^{\theta}_{\varepsilon}(c(a, b))$ and $N^{\theta}_{\varepsilon}(d(m, n))$, respectively such that

$$f^{-1}(\mu_G)(c(a, b)) = \mu_G(f(c)) = 1,\ f^{-1}(\gamma_G)\ c(a, b) = \gamma_G(f(c)) = 0,$$

$$f^{-1}(\mu_H)(d(m, n)) = \mu_H(f(d)) = 1,\ \ f^{-1}(\gamma_H)(d(m, n)) = \gamma_H(f(d)) = 0$$

and

$$f^{-1}(G) \wedge f^{-1}(H) = f^{-1}(G \wedge H) = f^{-1}(\underset{\sim}{0}) = \underset{\sim}{0}.$$

Hence (X, Ψ) is an $IF\theta T_2S$. □

Corollary 4.2. *Let $f:(X,\Psi)\to(Y,\Phi)$ be an injective and $IFS\theta$-cont. function. If (Y,Φ) is an IFT_2S, then (X,Ψ) is so.*

Definition 4.2. *An $IFTS$ (X,Ψ) is said to be $IF\theta S$ iff the collection of all $IF\theta OS\prime s$ of X forms a base for the IFT Ψ of X.*

Lemma 4.1. *If an $IFTS$ (X,Ψ) is an $IF\theta S$, then for each IFP $c(a,b)$ in X and each $U\in N_\varepsilon^{\theta q}(c(a,b))$, there is $V\in N_\varepsilon^{\theta q}$ $(c(a,b))$ such that $V\leq U$.*

Proof. Let $c(a,b)$ be an IFP in X and $U\in N_\varepsilon^{\theta q}(c(a,b))$. Since X is $IF\theta S$, $U=\underset{j\in J}{\vee}A_j$, where for each $j\in J$, $A_j=\langle x,\mu_{A_j},\gamma_{A_j}\rangle$ are some $IF\theta OS\prime s$ in X. We claim that for some j, $A_j\in N_\varepsilon^{\theta q}(c(a,b))$. If not, i.e. $c(a,b)\ \widetilde{q}A_j$ for all $j\in J$. Then $a<\gamma_{A_j}$ or $b>\mu_{A_j}$ for all $j\in J$. Then $a<\wedge\gamma_{A_j}$ or $b>\vee\mu_{A_j}$, so that $c(a,b)\widetilde{q}\underset{j\in J}{\vee}A_j=U$ which is a contraduction. Hence for some $j_\circ\in J$, $A_{j_\circ}=\langle x,\mu_{A_{j_\circ}},\gamma_{A_{j_\circ}}\rangle\in N_\varepsilon^{\theta q}(c(a,b))$. Also $\mu_{A_{j_\circ}}<\vee\mu_{A_j}$, $\gamma_{A_{j_\circ}}>\wedge\gamma_{A_j}$. Hence $A_{j_\circ}\leq U$. Putting $V=A_{j_\circ}$, we have $V\leq U$. □

Theorem 4.6. *If $f:(X,\Psi)\to(Y,\Phi)$ is an $IFsuper\ i$ function and X is an $IF\theta S$, then f is an $IFS\theta$-cont..*

Proof. Let $c(a,b)$ be an IFP in X and $V\in N_\varepsilon^{Sq}(f(c(a,b)))$. Then $f^{-1}(V)\in N_\varepsilon^{q}(c(a,b))$ since f is an $IFsuper\ i$ function. Also, since X is $IF\theta S$ and by Lemma 4.1., there is $U\in N_\varepsilon^{\theta q}(c(a,b))$ such that $U\leq f^{-1}(V)$ and so $f(U)\leq V$. Hence f is an $IFS\theta$-*cont.*. □

Theorem 4.7. *Every $IFS\theta$-cont. image of an IFcompact space is an IFsemi-compact.*

Proof. Let $f:X\to Y$ be an $IFS\theta$-*cont.* of an $IFcompact$ space X onto an $IFTS$ Y. Let $\{U_j:j\in J\}$ be any $IFsemi$ open cover of Y. Then $\{f^{-1}(U_j):j\in J\}$ is an $IF\theta$ open cover of X. Since X is an $IFcompact$, then there exists a finite subcover $\{f^{-1}(U_j):j=1,...,n\}$ of $\{f^{-1}(U_j):j\in J\}$. It implies that $\{U_j:j=1,...,n\}$ is a finite subcover of $\{U_j:j\in J\}$. Hence Y is an $IFsemi$-*compact*. □

Theorem 4.8. *Every IFSθ-cont. image of an IFalmost compact space is an IFalmost compact.*

Proof. Similar to the proof of Theorem 4.7.

□

5. Intuitionistic fuzzy semi θ-open(closed) functions

Definition 5.1. *A function $f:(X,\Psi)\to(Y,\Phi)$ is said to be IFsemiθ-open (IFsemi θ-closed) (IFSθ-open(IFSθ-closed), for short) if $f(U)$ is an $IF\theta OS$ ($IF\theta CS$) of Y for each $IFSOS$ ($IFSCS$) U of X.*

Theorem 5.1. *For a function $f:(X,\Psi)\to(Y,\Phi)$, the following are equivalent:*

(i) f *is an* $IFS\theta$*-open.*

(ii) *For each IFS V of Y and each $IFSCS$ U of X, when $f^{-1}(V)\leq U$, there is an $IF\theta CS$ H of Y with $V\leq H$ such that $f^{-1}(H)\leq U$.*

(iii) $f^{-1}(\mathrm{cl}_\theta(V))\leq\mathrm{cl}_s(f^{-1}(V))$ *for each IFS V of Y.*

(iv) $f(\mathrm{int}_s(U)) \leq \mathrm{int}_\theta(f(U))$ *for each* IFS U *of* X.

Proof. $(i) \Rightarrow (ii)$: Suppose f is $IFS\theta$*-open* and V be any IFS in Y. Let U is an $IFSCS$ in X such that $f^{-1}(V) \leq U$. Let $H = \overline{f(\overline{U})}$. Then H is an $IF\theta CS$ H in Y and, $V \leq H$, we have:

$$f^{-1}(H) = f^{-1}(\overline{f(\overline{U})}) = \overline{f^{-1}f(\overline{U})} \leq U.$$

(ii) $\Rightarrow$ (i): Let G be an $IFSOS$ in X, $V = \overline{f(G)}$ and $U = \overline{G}$. We obtain $f^{-1}(V) = f^{-1}(\overline{f(G)}) = \overline{f^{-1}f(G)} \leq \overline{G}$. By hypothesis, there exists an $IF\theta CS$ H in Y with $V \leq H$ such that $f^{-1}(H) \leq U = \overline{G}$. Then $G \leq \overline{f^{-1}(H)} = f^{-1}(\overline{H})$. Hence, $f(G) \leq ff^{-1}(\overline{H}) \leq \overline{H}$. Also, since $V \leq H, f(G) = \overline{V} \geq \overline{H}$. Hence $f(G) = \overline{H}$ is an $IF\theta OS$ in Y and hence f is $IFS\theta$*-open*.

(ii) $\Rightarrow$ (iii): Let V be an IFS in Y. Since $\mathrm{cl}_s(f^{-1}(V))$ is an $IFSCS$ in X, with $f^{-1}(V) \leq \mathrm{cl}_s(f^{-1}(V))$. Then by (ii), there exists an IFθCS H of Y with $V \leq H$ such that $f^{-1}(H) \leq \mathrm{cl}_s(f^{-1}(V))$. Since $V \leq H$, we have $f^{-1}(\mathrm{cl}_\theta(V)) \leq f^{-1}(\mathrm{cl}_\theta(H)) \leq f^{-1}(H) \leq \mathrm{cl}_s(f^{-1}(V))$.

(iii) $\Rightarrow$ (iv): Easy by putting $V = \overline{f(U)}$ in (iii).

(iv) $\Rightarrow$ (i): Obvious. □

Theorem 5.2. *A function* $f : (X, \sigma) \to (Y, \Phi)$ *is said to be* $IFS\theta$*-closed iff for each* IFS V *of* Y *and each* $IFSOS$ U *of* X*, when* $f^{-1}(V) \leq U$*, there is an* $IF\theta OS$ G *of* Y *such that* $V \leq G$ *and* $f^{-1}(G) \leq U$*.*

Proof. Analogous to the proof of Theorem 5.1. □

Remark 5.1. For a function $f : (X, \Psi) \to (Y, \Phi)$, the following implications hold:
$IFS\theta$*-open* $\Longrightarrow$ $IF\lambda\theta$*-open* $\Longrightarrow$ $IFstr\theta$*-open* $\Longrightarrow$ IF*-open*

The converse of the above implications need not be true in general, as shown in the following example and remark.

Example 5.1. Let $X = [0, 1]$ and consider the IFS's $U = \langle x, \mu_U(x), \gamma_U(x)\rangle$ and $V = \langle x, \mu_V(x), \gamma_V(x)\rangle$ as follows: $\mu_U(x) = \frac{5}{6}$, $\forall\ x \in I$; $\gamma_U(x) = \frac{1}{5}$ $\forall\ x \in I$. and $\mu_V(x) = \frac{4}{5}$, $forall\ x \in I$; $\gamma_V(x) = \frac{1}{6}$ for all $x \in I$.

Now, the family $\Psi = \{\underset{\sim}{0}, \underset{\sim}{1}, U, \overline{U}\}$ and $\Phi = \{\underset{\sim}{0}, \underset{\sim}{1}, U, \overline{U}\}$ is IFTS on X. If we define the identity function $f : (X, \Psi) \to (X, \Psi)$, then f is an $IFstr\theta$*-open* function since U and $\overline{U}$ are $IF\theta OS$ in (X, Ψ). But f is not $IF\lambda\theta$*-open* (Indeed, V is an $IF\lambda OS$ in (X, Ψ) and $f(V) = V$ is not $IF\theta OS$ in (X, Ψ).

Remark 5.2. From the above Example it is obvious that:

(i) $IFstr\theta$*-open* function $\not\Longrightarrow$ $IFS\theta$*-open* function.

(ii) IF*-open* function $\not\Longrightarrow$ $IF\lambda\theta$*-open* function.

(iii) IF*-open* function $\not\Longrightarrow$ $IFS\theta$*-open* function.

Theorem 5.3. *The following hold for functions* $f : X \to Y$ *and* $g : Y \to Z$ *:*

(i) The composition of two $IFS\theta$*-open functions is an* $IFS\theta$*-open function.*

(ii) g *is an* $IFS\theta$*-open, if* f *is a bijective* IFi *and* $g \circ f$ *is* $IFS\theta$*-open.*

(iii) *If f is an $IFsemiopen$ and g is an $IFstr\theta$-open, then $g \circ f : X \to Z$ is an $IFS\theta$-open .*

(iv) *$g \circ f$ is $IFopen$, If f is $IFS\theta$-open and g is an $IFfaintly$ $open$.*

Proof. Straightforward. □

Theorem 5.4. *Let $f : (X, \Psi) \to (Y, \Phi)$ be a function. Then the following are equivalent:*

(i) *f is an $IFS\theta$-open.*

(ii) *f is an $IFS\theta$-closed.*

(iii) *f^{-1} is an $IFS\theta$-cont.*

Proof. $(i) \Rightarrow (ii)$: Let H be an $IFSCS$ in X, then $\overline{H}$ is an $IFSOS$ in X. Since f is bijective and $IFS\theta$-*open*, then $f(\overline{H}) = \overline{f(H)}$ is an $IF\theta OS$ in Y and hence $f(H)$ is an $IF\theta CS$. Therefore f is an $IFS\theta$-*closed*.

$(ii) \Rightarrow (iii)$: Let U be an $IFSCS$ in X, by (ii), $f(U)$ is an $IF\theta CS$. Now $(f^{-1})^{-1}(U) = f(U)$ is an $IF\theta CS$ in Y, hence f^{-1} is an $IFS\theta$-*cont.*.

$(iii) \Rightarrow (i)$: Let U be an $IFSOS$ in X. Since f^{-1} is bijective and $IFS\theta$-*cont.*, then $f(U) = (f^{-1})^{-1}(U)$ is an $IF\theta OS$ in Y and hence f is an $IFS\theta$-*open*. □

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