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Relatively Absolutely Countably Compact Spaces

YAN-KUI SONG
Department of Mathematics, Nanjing Normal University, Nanjing, 210097 P.R. China
songyankui@pine.njnu.edu.cn

Abstract. A subspace Y of a space X is *absolutely countably compact*(=*acc*) (*str-ongly absolutely countably compact*(=*strongly acc*)) in X if for every open cover $\mathcal{U}$ of X and every dense subspace $D \subseteq X$ (respectively, $D \subseteq Y$) there exists a finite subset $F \subseteq D$ such that $Y \subseteq St(F,\mathcal{U})$, where $St(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$. In this paper, we investigate the relationships between these spaces and other starcompact spaces by giving some examples, and also study topological properties of relatively absolutely countably compact spaces.

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1. Introduction

By a space, we mean a topological space. Let us recall that a space X is *countably compact* if every countable open cover of X has a finite subcover. Fleischman [1] defined a space X to be *starcompact* if for every open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $St(F,\mathcal{U}) = X$, where $St(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$. He proved that every countably compact space is starcompact. A subspace Y of a space X is called *countably compact* in X (see, for example[3]) if for every countable open cover $\mathcal{U}$ of X, there exists a countable subfamily covering Y. Recall from [6] that a space X is *absolutely countably compact*(=*acc*) if for every open cover $\mathcal{U}$ of X and every dense subspace D of X, there exists a finite subset $F \subseteq D$ such that $St(F,\mathcal{U}) = X$. This motivates the following definitions:

Definition 1.1. [3] *A subspace Y of a space X is starcompact (strongly starcompact) in X if for every open cover $\mathcal{U}$ of X, there exists a finite subset $F \subseteq X$ (respectively, $F \subseteq Y$) such that $Y \subseteq St(F,\mathcal{U})$.*

Definition 1.2. [5] *A subspace Y of a space X is absolutely countably compact- (=acc) (strongly absolutely countably compact (=strongly acc)) in X if for every open cover $\mathcal{U}$ of X and every dense subspace $D \subseteq X$ (respectively, $D \subseteq Y$) there exists a finite subset $F \subseteq D$ such that $Y \subseteq St(F,\mathcal{U})$.*

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Definition 1.3. *A subspace Y of a space X is (strongly absolutely countably compact (=strongly acc)) in X if for every open cover $\mathcal{U}$ of X and every dense subspace $D \subseteq Y$ there exists a finite subset $F \subseteq D$ such that $Y \subseteq St(F,\mathcal{U})$.*

Let X be a space and Y a subspace of X. From the above definitions, it is clear that if Y is strongly starcompact in X, then Y is starcompact in X; if Y is strongly acc in X, then Y is strongly starcompact in X; if Y is acc in X, then Y is starcompact in X. But, the converses do not hold. The purpose of this paper is to show the difference between these properties in the class of Tychonoff spaces by giving various examples, and also to study topological properties of relatively absolutely countably compact spaces.

Let ω denote the first infinite cardinal and ω_1 the first uncountable cardinal. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. Other terms and symbols that we do not define will be used as in [2].

2. Some examples on relatively acc spaces

In this section, we distinguish the properties defined the previous section. Kočin-ac [3] showed that relative countable compactness and relative starcompactness are equivalent for Hausdorff spaces. More, Matveev [4] showed that they are equivalent with relative strong starcompactness for Hausdorff spaces, but this not not true for T_1 spaces (see [4, Example 1]). Vaughan [7] proved that every countably compact GO-space is absolutely countably compact. Thus, every cardinal with uncountable cofinality is absolutely countably compact.

Example 2.1. There exist a Tychonoff space X and a subspace Y of X such that Y is strongly starcompact in X, but Y is not strongly acc in X.

Proof. Let

$$Y = \omega_1 \times (\omega_1 + 1)$$

be the product of ω_1 and $\omega_1 + 1$. Then, Y is a Tychonoff starcompact space, since Y is countably compact and Y is not acc (see [6, Example 2.2]). Let

$$X = Y \cup \{\infty\} \text{ such that } \infty \notin Y.$$

We topologize X as follows: Y has the usual product topology and is a clopen subspace of X, ∞ is isolated. Then, Y is strongly starcompact in X, since Y is countably compact. By the definition of topology of X, it is not difficult to show that Y is not strongly acc in X, which completes the proof. □

Example 2.2. There exist a Tychonoff space X and a subspace Y of X such that Y is strongly acc in X, but Y is not acc in X.

Proof. Let $X = \omega_1 \times (\omega_1 + 1)$ be the product of ω_1 and $\omega_1 + 1$ and $Y = \omega_1 \times \{\omega_1\}$. Then, Y is strongly acc in X, since Y is homeomorphic with ω_1 and ω_1 is acc. Similar to the proof of [6, Example 1.5], it is not difficult to prove that Y is not acc in X. □

Recall that the Alexandroff duplicate of a space X, denoted by $A(X)$, is constructed in the following way: the underlying set of $A(X)$ is $X \times \{0,1\}$ and each point of $X \times \{1\}$ is isolated; a basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is a set of the from $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X. It is well-known that $A(X)$ is Hausdorff (regular, Tychonoff, normal) iff so is X and $A(X)$ is compact iff so is X. In the next example, we use the following lemma from [8].

Lemma 2.1. *If X is countably compact, then $A(X)$ is acc.*

Example 2.3. There exist a Tychonoff space X and a subspace Y in X such that Y is acc in X, but Y is not strongly acc in X.

Proof. Let $X = A(X')$, where X' is the space X of Example 2.2. Let $Y = X' \times \{0\}$. Then, Y is not strongly acc in X, since Y is homeomorphic to $\omega_1 \times (\omega_1 + 1)$ and $\omega_1 \times (\omega_1 + 1)$ is not acc. Since $\omega_1 \times (\omega_1 + 1)$ is countably compact, then X is acc by the Lemma 2.1. hence Y is acc in X, which completes the proof.

□

3. Topological properties of relatively acc spaces

From the definition of relatively acc space, it is not difficult to see that every subset A of an acc space X is acc in X. But, A need not be strongly acc in X even if it is a regular-closed subset as the following example shows.

Example 3.1. There exist a Tychonoff acc space X and a regular-closed subset Y of X such that Y is not strongly acc in X.

Proof. Let X be the same X as Example 2.3 and $Y = X' \times \{0\}$. Clearly, X is acc, Y is a regular-closed subset of X and Y is not strongly acc in X, since Y is homeomorphic to $\omega_1 \times (\omega_1 + 1)$, which completes that proof. □

In the following, we can prove a positive result.

Proposition 3.1. *If X is acc and Y is a closed and open subset of X, then Y is strongly acc in X.*

The continuous images of relative acc spaces (relative strongly acc spaces) need not be relative acc spaces (respectively, relative strongly acc spaces).

Example 3.2. There exist a Tychonoff space X, a subspace A of X and a closed 2-to-1 continuous mapping $f : X \to Y$ such that A is strongly acc in X, but $f(A)$ is not strongly acc in Y.

Proof. Let X' be the same space X as Example 2.1. Let $X = A(X')$ and $A = A(\omega_1 \times (\omega_1 + 1))$. It follows from Lemma 2.1 that X and A are acc, hence A is strongly acc in X. Let $Y = X' \times \{0\}$ and $f : X \to Y$ be the projection. Then, f is a closed 2-to-1 continuous mapping. $f(A) = (\omega_1 \times (\omega_1 + 1)) \times \{0\}$. It follows from Example 2.1 that $f(A)$ is not strongly acc in Y. □

Proposition 3.2. *If there exist a continuous one to one open mapping $f : X \to Y$ from a space X onto a space Y and A is strongly acc in X, then $f(A)$ is strongly acc in Y.*

Proof. Let $\mathcal{U}$ be an open cover of Y and a dense subspace D of $f(A)$. Then, $\mathcal{V} = \{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of X and $D' = f^{-1}(D) \cap A$ is dense in A, since f is one to one and open. Thus, there exists a finite subset $F' \subseteq D'$ such that $A \subseteq St(F', \mathcal{V})$, since A is strongly acc in X. Hence, $F = f(D')$ is a finite subset of D and $f(A) \subseteq St(F, \mathcal{U})$, and this implies that $f(A)$ is strongly acc in Y, which completes the proof. □

Example 3.3. There exist a Tychonoff space X, a subspace A of X and $f : X \to Y$ a continuous mapping such that A is acc in X, but $f(A)$ is not acc in Y.

Proof. Let $X = (\omega_1 \times \omega_1) \oplus \omega_1$ be the discrete sum of $\omega_1 \times \omega_1$ and ω_1. It follows from [6, Lemma 1.7] that $\omega_1 \times \omega_1$ is acc and from [6, Theorem 1.8] that ω_1 is acc. Thus, X is acc. Let $A = \omega_1$. Then, A is acc in X. Let $Y = \omega_1 \times (\omega_1 + 1)$. Let $f : X \to Y$ be a mapping defined by $f(\langle \alpha, \beta \rangle) = \langle \alpha, \beta \rangle$ for $\alpha < \omega_1$, $\beta < \omega_1$ and $f(\alpha) = \langle \alpha, \omega_1 \rangle$ for $\alpha < \omega_1$. Clearly, f is a continuous mapping and $f(A) = \omega_1 \times \{\omega_1\}$. It follows from Example 2.2 that $f(A)$ is not acc in Y, which completes the proof. □

Similar to the proof of Proposition 3.2, it is not difficult to show the following proposition:

Proposition 3.3. *If there exist a continuous open mapping $f : X \to Y$ from a space X onto a space Y and A is acc in X, then $f(A)$ is acc in Y.*

Example 3.4. There exist a Tychonoff space X, a subspace A of X and a compact space Y such that A is acc (strongly acc) in X, but $A \times Y$ is not acc (respectively, strongly acc) in $X \times Y$.

Proof. Let $X = \omega_1 \cup \{a\}$ where $a \notin \omega_1$ and $A = \omega_1$. We topologize X as follows: A has the usual topology and is a clopen subspace of X, $\{a\}$ is an isolated point. Then, A is strongly acc and acc in X, since X is acc and A is also acc. Let $Y = \omega_1 + 1$. Then, Y is compact. Since $A \times Y = \omega_1 \times (\omega_1 + 1)$, then $A\times$ is not acc. hence $A \times Y$ is not strongly acc and acc in $X \times Y$. □

For Hausdorff space X and a subset A of X, if A is strongly acc and acc in X, then A is countably compact in X (see [3,4]). Similar to the proof of [6, Theorem 2.3], it is not difficult to prove the following proposition:

Proposition 3.4. *If A is acc (strongly acc) in X and Y is a first countable, compact space, then $A \times Y$ is acc (respectively, strongly acc) in $X \times Y$.*

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References

[1] W. M. Fleischman, A new extension of countable compactness, *Fund. Math.* **67** (1970), 1–9.

[2] R. Engelking, *General Topology*, Translated from the Polish by the author, Second edition, Heldermann, Berlin, 1989.

[3] Lj. D. Kočinac, Some relative topological properties, *Mat. Vesnik* **44**(1–2) (1992), 33–44.

[4] M. V. Matveev, A survey on star covering properties, *Topology Atlas*, preprint No. 330, 1998.

[5] M. V. Matveev, O. I. Pavlov and J. K. Tartir, On relatively normal spaces, relatively regular spaces, and on relative property (a), *Topology Appl.* **93**(2) (1999), 121–129.

[6] M. V. Matveev, Absolutely countably compact spaces, *Topology Appl.* **58**(1) (1994), 81–92.

[7] J. E. Vaughan, On the product of a compact space with an absolutely countably compact space, in *Papers on general topology and applications (Amsterdam, 1994)*, 203–208, *Ann. New York Acad. Sci.,* 788, *New York Acad. Sci.,* New York.

[8] J. E. Vaughan, Absolute countable compactness and property (a), Talk at 1996 *Praha Symposium on General Topology.*