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Intersection Preserving and Global Expansions of Subalgebras and Filters in Lattice Implication Algebras

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Abstract. The notion of (intersection preserving, global) expansions of subalgebras and filters in lattice implication algebras is introduced. Also the notion of σ-primary filters in lattice implication algebras is discussed. The concept of residual division is defined, and related properties are investigated. We show that the homomorphic and inverse image of σ-primary filter are also σ-primary.

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1. Introduction

In the field of many-valued logic, lattice-valued logic plays an important role for two aspects: One is that it extends the chain-type truth-value field of some well-known presented logic [1] to some relatively general lattices. The other is that the incompletely comparable property of truth value characterized by general lattice can more efficiently reflect the uncertainty of people's thinking, judging and decision. Hence, lattice-valued logic is becoming a research field which strongly influences the development of Algebraic Logic, Computer Science and Artificial Intelligence Technology. Therefore Goguen [2], Pavelka [12] and Novak [11] researched on this lattice-valued logic formal systems. Moreover, in order to establish a logic system with truth value in a relatively general lattice, in 1990, during the study of the project "The Study of Abstract Fuzzy Logic" granted by National Natural Science Foundation in China, Xu established the lattice implication algebra by combining lattice and implication algebra, and investigated many useful structures [9, 10, 14, 15, 16]. We note that lattice implication algebras are isomorphic to MV-algebras. Lattice implication algebra provided the foundation to establish the

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corresponding logic system from the algebraic viewpoint. For the general development of lattice implication algebras, the filter theory plays an important role (see [3, 4, 5, 6, 7, 8, 16].) In this paper, we introduce the notion of (intersection preserving, global) expansions of subalgebras and filters in lattice implication algebras, and the notion of σ-primary filters in lattice implication algebras. We also define the notion of residual division, and investigates related properties. We show that the homomorphic image and inverse image of σ-primary filter are also σ-primary.

2. Preliminaries

We give herein the basic notions on lattice implication algebras. For further information, we refer the readers to [14, 15, 16, 17].

By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution "$'$" and a binary operation "$\to$" satisfying the following axioms:

(I1) $x \to (y \to z) = y \to (x \to z)$,
(I2) $x \to x = 1$,
(I3) $x \to y = y' \to x'$,
(I4) $x \to y = y \to x = 1 \Rightarrow x = y$,
(I5) $(x \to y) \to y = (y \to x) \to x$,
(L1) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(L2) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,

for all $x, y, z \in L$. In a lattice implication algebra, we can define a partial ordering $\leq$ by $x \leq y$ if and only if $x \to y = 1$. A subset S of a lattice implication algebra L is called a *subalgebra* of L if it satisfies

- $0 \in S$,
- $(\forall x, y \in S)\ (x \to y \in S)$.

A subset F of a lattice implication algebra L is called a *filter* of L if it satisfies

- $1 \in F$,
- $(\forall x \in F)\ (\forall y \in L)\ (x \to y \in F \ \Rightarrow\ y \in F)$.

A proper filter F of a lattice implication algebra L is said to be *prime* if it satisfies

- $(\forall a, b \in L)\ (a \vee b \in F \ \Rightarrow\ a \in F \text{ or } b \in F)$.

3. Expansions of subalgebras and filters

In what follows let L denote a lattice implication algebra unless otherwise specified.

Definition 3.1. *Let $\mathfrak{O}(L)$ be a set of objects in L, that is, a family of subsets of L. An* expansion of objects *in L is defined to be a function $\sigma : \mathfrak{O}(L) \to \mathfrak{O}(L)$ such that*

(o1) $(\forall G \in \mathfrak{O}(L))\,(G \subseteq \sigma(G))$.

(o2) $(\forall G, H \in \mathfrak{O}(L))\,(G \subseteq H \ \Rightarrow\ \sigma(G) \subseteq \sigma(H))$.

Let $\mathfrak{S}(L)$ (resp., $\mathfrak{F}(L)$) denote the set of all subalgebras (resp., filters) of L. If $\mathfrak{O}(L) = \mathfrak{S}(L)$ (resp., $\mathfrak{O}(L) = \mathfrak{F}(L)$), we say that σ is an expansion of subalgebras *(resp.,* filters*).*

Example 3.1. (1) The function $\sigma_0 : \mathfrak{S}(L) \to \mathfrak{S}(L)$ (resp., $\sigma_0 : \mathfrak{F}(L) \to \mathfrak{F}(L)$) defined by $\sigma_0(G) = G$ for all $G \in \mathfrak{S}(L)$ (resp., $\mathfrak{F}(L)$) is an expansion of subalgebras (resp., filters) in L.

(2) The function ν that assigns the largest subalgebra (resp., filter) L to each subalgebra (resp., filter) of L is an expansion of subalgebras (resp., filters) in L.

(3) For each filter F of L, let

$$\mathfrak{M}(F) = \cap\{M \mid F \subseteq M,\ M \text{ is a maximal filter of } L\}.$$

Then $\mathfrak{M}$ is an expansion of filters in L.

(4) Let $F \in \mathfrak{F}(L)$. For each $a \in L$, the set

$$a^{-1}F := \{x \in L \mid a \vee x \in F\}$$

is a filter of L containing F, and if F and G are filters of L such that $F \subseteq G$ then $a^{-1}F \subseteq a^{-1}G$ (see [6]). Hence the function $\sigma_a : \mathfrak{F}(L) \to \mathfrak{F}(L)$ given by $\sigma_a(F) = a^{-1}F$ for all $F \in \mathfrak{F}(L)$ is an expansion of filters in L.

Definition 3.2. *Let σ be an expansion of filters in L. Then a filter G of L is said to be* σ-primary *if*

$$(\forall a, b \in L)\,(a \vee b \in G,\ a \notin G\ \Rightarrow\ b \in \sigma(G)).$$

Note that a filter G of L is σ_0-primary if and only if it is a prime filter of L, where σ_0 is the function in Example 3.1(1).

Theorem 3.1. *If σ and δ are expansions of filters in L such that $\sigma(G) \subseteq \delta(G)$ for every $G \in \mathfrak{F}(L)$, then every σ-primary filter is also δ-primary.*

Proof. Let F be a σ-primary filter of L and let $a, b \in L$ be such that $a \vee b \in F$ and $a \notin F$. Then $b \in \sigma(F) \subseteq \delta(F)$ by assumption. Hence F is a δ-primary filter of L. □

Corollary 3.1. *Let σ be an expansion of filters in L. Then every prime filter of L is σ-primary.*

Proof. Let G be a prime filter of L. Then G is σ_0-primary, and $\sigma_0(G) = G \subseteq \sigma(G)$. It follows from Theorem 3.1 that G is a σ-primary filter of L. □

Theorem 3.2. *Let α and β be expansions of subalgebras* (resp., *filters*) *in L. Let $\sigma : \mathfrak{S}(L) \to \mathfrak{S}(L)$* (resp., *$\sigma : \mathfrak{F}(L) \to \mathfrak{F}(L)$*) *be a function defined by $\sigma(G) = \alpha(G) \cap \beta(G)$ for all $G \in \mathfrak{S}(L)$* (resp., *$\mathfrak{F}(L)$*). *Then σ is an expansion of subalgebras* (resp., *filters*) *in L.*

Proof. For every $G \in \mathfrak{S}(L)$ (resp., $\mathfrak{F}(L)$), we have $G \subseteq \alpha(G)$ and $G \subseteq \beta(G)$ by (o1), and so $G \subseteq \alpha(G) \cap \beta(G) = \sigma(G)$. Let $G, H \in \mathfrak{S}(L)$ (resp., $\mathfrak{F}(L)$) be such that $G \subseteq H$. Then $\alpha(G) \subseteq \alpha(H)$ and $\beta(G) \subseteq \beta(H)$ by (o2), which imply that

$$\sigma(G) = \alpha(G) \cap \beta(G) \subseteq \alpha(H) \cap \beta(H) = \sigma(H).$$

Therefore σ is an expansion of subalgebras (resp., filters) in L. □

Generally, the intersection of expansions of subalgebras (resp., filters) is an expansion of subalgebras (resp., filters).

Theorem 3.3. *Let σ be an expansion of filters in L. If $\{G_i \mid i \in D\}$ is a directed collection of σ-primary filters of L where D is an index set, then the filter $G := \bigcup_{i\in D} G_i$ is σ-primary.*

Proof. Let $a, b \in L$ be such that $a \vee b \in G$ and $a \notin G$. Then there exists G_i such that $a \vee b \in G_i$ and $a \notin G_i$. Since G_i is σ-primary and $G_i \subseteq G$, it follows that $b \in \sigma(G_i) \subseteq \sigma(G)$ so that G is σ-primary. □

Theorem 3.4. *Let σ be an expansion of filters in L. If P is a σ-primary filter of L, then*

$$(\forall F, G \in \mathfrak{F}(L))\,(F \vee G \subseteq P,\ F \nsubseteq P \ \Rightarrow\ G \subseteq \sigma(P)),$$

where $F \vee G = \{x \vee y \mid x \in F,\ y \in G\}$.

Proof. Assume that P is a σ-primary filter of L and let $F, G \in \mathfrak{F}(L)$ be such that $F \vee G \subseteq P$ and $F \nsubseteq P$. Suppose that $G \nsubseteq \sigma(P)$. Then there exist $a \in F \setminus P$ and $b \in G \setminus \sigma(P)$, which imply that $a \vee b \in F \vee G \subseteq P$. But $a \notin P$ and $b \notin \sigma(P)$. This contradicts the assumption that P is σ-primary. Consequently, the result is valid. □

Theorem 3.5. *If σ is an expansion of filters in L, then the function $E_\sigma : \mathfrak{F}(L) \to \mathfrak{F}(L)$ defined by*

$$E_\sigma(G) := \cap\{H \in \mathfrak{F}(L) \mid G \subseteq H, \text{ and } H \text{ is } \sigma\text{-primary}\}$$

for all $G \in \mathfrak{F}(L)$ is an expansion of filters in L.

Proof. Clearly, $G \subseteq E_\sigma(G)$ for all $G \in \mathfrak{F}(L)$. Let $F, G \in \mathfrak{F}(L)$ be such that $F \subseteq G$. Then

$$\begin{aligned} E_\sigma(F) &= \cap\{H \in \mathfrak{F}(L) \mid F \subseteq H \text{ and } H \text{ is } \sigma\text{-primary}\} \\ &\subseteq \cap\{H \in \mathfrak{F}(L) \mid G \subseteq H \text{ and } H \text{ is } \sigma\text{-primary}\} \\ &= E_\sigma(G). \end{aligned}$$

Hence E_σ is an expansion of filters in L. □

For any filters P and Q of L, the *residual division* of P and Q is defined to be the filter

$$P : Q = \bigcap_{x\in Q} x^{-1}P = \{y \in L \mid x \vee y \in P \text{ for all } x \in Q\}.$$

Theorem 3.6. *Let σ be an expansion of filters in L and let P be a σ-primary filter of L. Then*

(i) *if F is a filter of L which is not contained in $\sigma(P)$, then $P : F = P$.*
(ii) *if G is any filter of L, then $P : G$ is σ-primary.*

Proof. (i) Obviously, $P \subseteq P : F$. Also we have $F \vee (P : F) \subseteq P$ by the definition of $P : F$. Since $F \nsubseteq \sigma(P)$, it follows from Theorem 3.4 that $P : F \subseteq P$. Therefore $P : F = P$.

(ii) Let $a, b \in L$ be such that $a \vee b \in P : G$ and $a \notin P : G$. Then $a \vee x \notin P$ for some $x \in G$. But $(a \vee x) \vee b = (a \vee b) \vee x \in P$, and so $b \in \sigma(P) \subseteq \sigma(P : G)$. Thus $P : G$ is σ-primary. This completes the proof. □

A mapping $f : Y \to L$ of lattice implication algebras is called an *implication homomorphism* if it satisfies

- $(\forall x, y \in Y)\ f(x \to y) = f(x) \to f(y)$.

If an implication homomorphism $f : Y \to L$ of lattice implication algebras satisfies the further conditions:

- $(\forall x, y \in Y)\ f(x \vee y) = f(x) \vee f(y)$,
- $(\forall x, y \in Y)\ f(x \wedge y) = f(x) \wedge f(y)$,
- $(\forall x \in Y)\ f(x') = f(x)'$,

then we say that f is a *homomorphism* of lattice implication algebras. Note that f is a homomorphism of lattice implication algebras if and only if f is an implication homomorphism and $f(x') = f(x)'$ for all $x \in Y$ (see [13]).

Definition 3.3. *Let σ be an expansion of filters. Then*

(i) *σ is said to be* intersection preserving *if it satisfies:*

$$(\forall F, G \in \mathfrak{F}(L))\,(\sigma(F \cap G) = \sigma(F) \cap \sigma(G)),$$

(ii) *σ is said to be* global *if for each homomorphism $f : Y \to L$ of lattice implication algebras, the following holds:*

$$(\forall F \in \mathfrak{F}(L))\,(\sigma(f^{-1}(F)) = f^{-1}(\sigma(F))).$$

Example 3.2. (1) The expansion of filters $\sigma_0 : \mathfrak{F}(L) \to \mathfrak{F}(L)$ defined by $\sigma_0(G) = G$ for all $G \in \mathfrak{F}(L)$ is both intersection preserving and global.

(2) The expansion of filters

$$\sigma_a : \mathfrak{F}(L) \to \mathfrak{F}(L),\ F \mapsto a^{-1}F$$

in Example 3.1(4) is intersection preserving. Because, for every $F, G \in \mathfrak{F}(L)$ we have

$$\sigma_a(F \cap G) = a^{-1}(F \cap G) = a^{-1}F \cap a^{-1}G = \sigma_a(F) \cap \sigma_a(G).$$

Theorem 3.7. *Let σ be an expansion of filters which is intersection preserving. If $Q_1, Q_2, \cdots, Q_n$ are σ-primary filters of L and $P = \sigma(Q_i)$ for all i, then $Q = \bigcap_{i=1}^{n} Q_i$ is σ-primary.*

Proof. Let $a, b \in L$ be such that $a \vee b \in Q$ and $a \notin Q$. Then $a \notin Q_k$ for some $k \in \{1, 2, \cdots, n\}$. But $a \vee b \in Q \subseteq Q_k$ and Q_k is σ-primary, which imply that $b \in \sigma(Q_k)$. Now

$$\sigma(Q) = \sigma(\bigcap_{i=1}^{n} Q_i) = \bigcap_{i=1}^{n} \sigma(Q_i) = P = \sigma(Q_k),$$

and so $b \in \sigma(Q)$. Therefore Q is σ-primary. □

Let $f : Y \to L$ be a homomorphism of lattice implication algebras. Note that if F is a filter of L, then $f^{-1}(F)$ is a filter of Y, and that if f is surjective and G is a filter of Y then $f(G)$ is a filter of L.

Theorem 3.8. *Let σ be a global expansion of filters and let $f : Y \to L$ be a homomorphism of lattice implication algebras. If F is a σ-primary filter of L, then $f^{-1}(F)$ is a σ-primary filter of Y.*

Proof. Let $a, b \in Y$ be such that $a \vee b \in f^{-1}(F)$ and $a \notin f^{-1}(F)$. Then

$$f(a) \vee f(b) = f(a \vee b) \in f(f^{-1}(F)) \subseteq F$$

and $f(a) \notin F$. Since F is σ-primary, it follows that $f(b) \in \sigma(F)$ so that $b \in f^{-1}(\sigma(F)) = \sigma(f^{-1}(F))$. Hence $f^{-1}(F)$ is σ-primary. □

Lemma 3.1. *Let $f : Y \to L$ be a homomorphism of lattice implication algebras. If G is a filter of Y that contains the dual kernel of f, denoted by* Dker$f$, *then $f^{-1}(f(G)) = G$.*

Proof. Clearly $G \subseteq f^{-1}(f(G))$. Now let $y \in f^{-1}(f(G))$. Then $f(y) \in f(G)$, and so there exists $x \in G$ such that $f(y) = f(x)$. Hence $f(y \to x) = f(y) \to f(x) = 1$, which implies that $y \to x \in \text{Dker}f \subseteq G$. Since G is a filter containing x, it follows that $y \in G$ so that $f^{-1}(f(G)) \subseteq G$. Therefore $f^{-1}(f(G)) = G$. □

Theorem 3.9. *Let $f : Y \to L$ be a surjective homomorphism of lattice implication algebras and let G be a filter of Y that contains* Dker$f$. *Then G is σ-primary if and only if $f(G)$ is a σ-primary filter of L, where σ is a global expansion of filters.*

Proof. Sufficiency follows from Theorem 3.8 and Lemma 3.1. Suppose that G is σ-primary. Let $a, b \in L$ be such that $a \vee b \in f(G)$ and $a \notin f(G)$. Since f is surjective, we have $f(x) = a$ and $f(y) = b$ for some $x, y \in Y$. Then

$$f(x \vee y) = f(x) \vee f(y) = a \vee b \in f(G),$$

which implies $x \vee y \in f^{-1}(f(G)) = G$. Now $f(x) = a \notin f(G)$ implies $x \notin G$. Since G is σ-primary, it follows that $y \in \sigma(G)$ so that $b = f(y) \in f(\sigma(G))$. Using Lemma 3.1 and the fact that σ is global, we get

$$\sigma(G) = \sigma(f^{-1}(f(G))) = f^{-1}(\sigma(f(G))),$$

and so $f(\sigma(G)) = f(f^{-1}(\sigma(f(G)))) = \sigma(f(G))$ by the surjectivity of f. Therefore $f(G)$ is σ-primary. This completes the proof.

□

4. Conclusions

Combining lattice and implication algebra, Xu established the concept of lattice implication algebra which is a new logical algebraic system. The theory of MV-algebras, developed by Chang, first appeared in 1958. MV-algebras were developed to provide an algebraic proof of the completeness theorem for Łukasiewicz infinite-valued propositional logic. We note that lattice implication algebras and MV-algebras are isomorphic. The aim of this paper is to introduce the notion of (intersection preserving, global) expansions of subalgebras and filters in lattice implication algebras, and the notion of σ-primary filters in lattice implication algebras. We also define the notion of residual division, and investigates related properties. We show that the homomorphic image and inverse image of σ-primary filter are also σ-primary. Based on this idea and results, our future work will focus on applying such notions to MV-algebras, BL-algebras, R_0-algebras, and arbitrary residuated lattice etc.

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