BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **29***(1) (2006), 51–57*

A Note on ξ-Conformally Flat Contact Manifolds

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Abstract. We prove that a contact manifold with the structure vector field ξ belonging to the k-nullity distribution is ξ -conformally flat if and only if it is an η-Einstein manifold and we give some applications.

2000 Mathematics Subject Classification: 53C15, 53C25

Key words and phrases: Contact manifolds, ξ-conformally flat, η-Einstein manifold.

1. Introduction

Let M be a $(2n+1)$–dimensional Riemannian manifold with metric g and let $T(M)$ be the Lie algebra of differentiable vector fields in M. The Ricci operator Q of (M, g) is defined by $g(QX, Y) = S(X, Y)$, where S denotes the Ricci tensor of type (0, 2) on M and $X, Y \in T(M)$. Weyl ([19], [20]) constructed a generalized curvature tensor on a Riemannian manifold which vanishes, whenever the metric is (locally) conformally equivalent to a flat metric. The Weyl conformal curvature tensor is defined by

$$
\begin{aligned}
C(X,Y)Z = R(X,Y)Z &- \frac{1}{2n-1}[g(QY,Z)X + g(Y,Z)QX \\
&- g(QX,Z)Y - g(X,Z)QY] \\
&+ \frac{\mathbf{r}}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y],
\end{aligned}
\tag{1.1}
$$

for $X, Y, Z \in T(M)$, where R and r denote the Riemannian curvature tensor and the scalar curvature of M respectively.

Now let M be a $(2n+1)$–dimensional contact metric manifold with a contact structure (φ, η, ξ, g) [1]. Then φ is a (1, 1)–tensor field, ξ is the associated vector field, η is the contact 1-form and g is the associated Riemannian metric.

For a contact metric manifold, the following relations hold [1]:

$$\varphi^2 = -I + \eta \otimes \xi, \varphi(\xi) = 0, \eta(\xi) = 1,$$

Received: August 12, 2004; *Accepted:* May 2, 2005.

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$g(X, \varphi Y) = d\eta(X, Y), for X, Y \in T(M).$$

At each point $p \in M$, we have

$$T_p(M) = \varphi(T_p(M)) \oplus \{\xi_p\},$$

where $\{\xi_p\}$ is 1-dimensional linear subspace of $T_p(M)$ generated by ξ_p. Then the Weyl conformal curvature tensor C is a map:

$$C : T_p(M) \times T_p(M) \times T_p(M) \to \varphi(T_p(M)) \oplus \{\xi\}.$$

Three particular cases can be considered as follows:

(1) $C : T_p(M) \times T_p(M) \times T_p(M) \to \{\xi\}$, that is, the projection of the image of C in $\varphi(T_p(M))$ is zero.
(2) $C : T_p(M) \times T_p(M) \times T_p(M) \to \varphi(T_p(M))$, that is, the projection of the image of C in $\{\xi\}$ is zero.
(3) $C : \varphi(T_p(M)) \times \varphi(T_p(M)) \times \varphi(T_p(M)) \to \{\xi\}$, that is, when C is restricted to $\varphi(T_p(M)) \times \varphi(T_p(M))$, the projection of the image of C in $\varphi(T_p(M))$ is zero, which is equivalent to $\varphi^2 C(\varphi X, \varphi Y)\varphi Z = 0$.

Okumura [15] proved that a conformally flat Sasakian manifold is locally isometric to the unit sphere. Later, Miyazawa and Yamaguchi [14] proved that a conformally symmetric Sasakian manifold is also locally isometric to the unit sphere. Chaki and Tarafdar [6] obtained the same result for a Sasakian manifold satisfying the condition $R(X, Y).C = 0$, for $X, Y \in T(M)$. Among the above three cases (1), (2) and (3), case (1) for K-contact manifold is studied in [11] and M is locally isometric to the unit sphere; case (2) for K-contact manifold is studied in [12] and M is an η-Einstein Sasakian manifold. In particular, if the manifold M is of dimension 3, a K-contact metric structure is ξ-conformally flat [12] and Sasakian and therefore, it is η-Einstein [2]. In this paper, we will study the case (2) for a contact manifold with ξ belonging to the k-nullity distribution, where the k-nullity distribution of a Riemannian manifold (M, g), for a real number k, is given by

$$N(k) : p \to N_p(k) = \{Z \in T_p(M)/R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\}$$

for $X, Y \in T_p(M)$. Thus

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y]. \tag{1.2}$$

Contact Riemannian manifolds whose characteristic vector field ξ belonging to the k-nullity distribution have been studied by various authors ([3], [4], [5], [7], [8], [9], [10], [13], [17], [18]). Observe that a Sasakian manifold is characterized by

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

Consequently, if $k = 1$, a contact manifold with $\xi \in N(k)$ is a Sasakian manifold.

2. Preliminaries

Let M be a $(2n+1)$–dimensional contact metric manifold with contact structure (φ, η, ξ, g) [1], [21]. $A(1,1)$ type tensor field h on M is defined by $h = \frac{1}{2}L_\xi\varphi$, where L denotes Lie differentiation. It is easy to show that h is symmetric and it satisfies

$$h\xi = 0, \ Trh = 0, \ Tr\varphi h = 0 \quad \text{and } h\varphi = -\varphi h. \tag{2.1}$$

Let ∇ be the Riemannian connection of g. Then we know

$$\nabla_X \xi = -\varphi X - \varphi hX. \tag{2.2}$$

We assume that ξ belongs to the k-nullity distribution. Then

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y]. \tag{2.3}$$

Contracting X in (2.3), it follows that

$$S(Y,\xi) = 2nk\eta(Y). \tag{2.4}$$

Therefore $g(Q\xi, Y) = 2nkg(\xi, Y)$, where Q is the Ricci operator defined by

$$g(QX,Y) = S(X,Y).$$

Hence

$$Q\xi = 2nk\xi. \tag{2.5}$$

3. ξ-conformally flat contact manifolds with $\xi \in N(k)$

Let M^{2n+1} be a contact metric manifold with the contact structure (φ, η, ξ, g). Then

$$\eta(\varphi T(M)) = d\eta(\xi, T(M)) = 0.$$

Conversely, if $\eta(X) = 0$, then $X = -\varphi^2 X \in \varphi(T(M))$. The Weyl conformal curvature tensor with respect to the metric g is the tensor field of type (1, 3) defined by (1.1). On the other hand, the Lie algebra $T(M)$ can be decomposed in a direct sum $T(M) = \varphi(T(M)) \oplus \{\xi\}$, where $\{\xi\}$ is the 1-dimensional distribution on M generated by the characteristic field ξ.

Definition 3.1. [12] *A contact metric manifold $(M^{n+1}, \varphi, \eta, \xi, g)$ is said to be ξ-conformally flat if the linear operator $C(X,Y)$ is an endomorphism of $\varphi(T(M))$, that is, if*

$$C(X,Y)\varphi(T(M)) \subset \varphi(T(M)).$$

Then it follows immediately that [12]

Proposition 3.1. *On a contact metric manifold M^{2n+1}, the following conditions are equivalent.*

(i) M^{2n+1} *is ξ-conformally flat,*
(ii) $\eta(C(X,Y)Z) = 0,$
(iii) $\varphi^2 C(X,Y)Z = -C(X,Y)Z,$
(iv) $C(X,Y)\xi = 0,$

where $X, Y, Z \in T(M)$.

From (iv), in Proposition 3.1, it follows that a contact metric manifold is ξ-conformally flat if and only if

$$R(X,Y)\xi = \frac{1}{2n-1}[g(QY,\xi)X + \eta(Y)QX - g(QX,\xi)Y - \eta(X)QY] + \frac{r}{2n(2n-1)}[\eta(X)Y - \eta(Y)X]. \tag{3.1}$$

Now we prove the main theorem of the paper.

Theorem 3.1. *A contact metric manifold M^{2n+1} with $\xi \in N(k)$ is ξ-conformally flat if and only if it is an η-Einstein manifold.*

Proof. First suppose that the manifold M^{2n+1} is η-Einstein. Then there are functions α and β such that

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y). \tag{3.2}$$

Hence, we get

$$QX = \alpha X + \beta\eta(X)\xi. \tag{3.3}$$

From (2.5) and (3.2), we obtain

$$\alpha + \beta = 2nk. \tag{3.4}$$

Also from (3.3), it follows that

$$r = Tr(Q) = (2n+1)\alpha + \beta. \tag{3.5}$$

Using (3.4) in (3.5) yields

$$r = 2n(\alpha + k). \tag{3.6}$$

From (1.1), we get

$$\begin{aligned} C(X,Y)\xi &= R(X,Y)\xi - \frac{1}{2n-1}[g(QY,\xi)X + g(Y,\xi)QX \\ &\quad - g(QX,\xi)Y - g(X,\xi)QY] \\ &\quad + \frac{r}{2n(2n-1)}[g(Y,\xi)X - g(X,\xi)Y] \\ &= R(X,Y)\xi - \frac{1}{2n-1}[S(Y,\xi)X + \eta(Y)QX \\ &\quad - S(X,\xi)Y - \eta(X)QY] \\ &\quad + \frac{r}{2n(2n-1)}[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{3.7}$$

With the help of (2.4), (3.3) and (3.6), the relation (3.7) reduces to

$$C(X,Y)\xi = R(X,Y)\xi - k[\eta(Y)X - \eta(X)Y] = 0,$$

since $\xi \in N(k)$. Hence, the manifold is ξ-conformally flat.

Next, we will prove that a ξ-conformally flat contact metric manifold M^{2n+1} with $\xi \in N(k)$ is an η-Einstein manifold.

For a ξ-conformally flat contact metric manifold with $\xi \in N(k)$ and by (3.1) and (1.2), we have

$$\begin{aligned}k[\eta(Y)X - \eta(X)Y] \\ &= \frac{1}{2n-1}[g(QY,\xi)X + \eta(Y)QX - g(QX,\xi)Y - \eta(X)QY] \\ &+ \frac{r}{2n(2n-1)}[\eta(X)Y - \eta(Y)X].\end{aligned} \tag{3.8}$$

Putting $Y = \xi$, (3.8) gives

$$\begin{aligned}QX &= [2nk - k - g(Q\xi,\xi) + \frac{r}{2n}]X \\ &+ [g(Q\xi,X) - (2nk - k + \frac{r}{2n})\eta(X)]\xi + \eta(X)Q\xi\end{aligned} \tag{3.9}$$

for $X \in T(M)$. From (2.5) and (3.9), we get

$$QX = (-k + \frac{r}{2n})X + \{(2n+1)k - \frac{r}{2n}\}\eta(X)\xi = \alpha X + \beta\eta(X)\xi \tag{3.10}$$

where

$$\alpha = -k + \frac{r}{2n} \quad \text{and} \quad \beta = (2n+1)k - \frac{r}{2n}.$$

Hence, the manifold is an η-Einstein manifold. This completes the proof. □

Corollary 3.1. *Let M^{2n+1} be a ξ-conformally flat contact metric manifold with $\xi \in N(k)$. If there exists functions L and M on M^{2n+1} such that*

$$(\nabla_X Q)Y - (\nabla_Y Q)X = LX + MY \tag{3.11}$$

for $X, Y \in T(M)$, then

$$QX = 2nkX \tag{3.12}$$

Proof. From Theorem 3.1, we have $QX = \alpha X + \beta\eta(X)\xi$, where

$$\alpha = -k + \frac{r}{2n} \quad \text{and} \quad \beta = (2n+1)k - \frac{r}{2n}$$

and thus using (2.2), we have

$$\begin{aligned}&(\nabla_X Q)Y - (\nabla_Y Q)X \\ &= (X\alpha)Y - (Y\alpha)X + (X\beta)\eta(Y)\xi - (Y\beta)\eta(X)\xi \\ &- \beta\eta(Y)\varphi X + \beta\eta(X)\varphi Y - 2\beta\{g(\varphi X, Y) + g(\varphi hX, Y)\}\xi.\end{aligned} \tag{3.13}$$

Replacing X by φX and Y by φY in (3.13), we get

$$\begin{aligned}&(\nabla_{\varphi X} Q)\varphi Y - (\nabla_{\varphi Y} Q)\varphi X \\ &= (\varphi X\alpha)\varphi Y - (\varphi Y\alpha)\varphi X - 2\beta\{g(\varphi^2 X, \varphi Y) + g(\varphi h\varphi X, \varphi Y)\}\xi \\ &= (\varphi X\alpha)\varphi Y - (\varphi Y\alpha)\varphi X - 2\beta\{g(\varphi^2 X, \varphi Y) - g(h\varphi^2 X, \varphi Y)\}\xi.\end{aligned} \tag{3.14}$$

From (3.11) and (3.14), we obtain

$$(L + (\varphi Y\alpha))\varphi X + (M - (\varphi X\alpha))\varphi Y = -2\beta\{g(\varphi^2 X, \varphi Y) - g(h\varphi^2 X, \varphi Y)\}\xi,$$

which shows that

$$-2\beta\{g(\varphi^2 X, \varphi Y) - g(h\varphi^2 X, \varphi Y)\} = 0. \tag{3.15}$$

Replacing X by φY in (3.15), we obtain

$$\beta\{g(\varphi Y, \varphi Y) - g(h\varphi Y, \varphi Y)\} = 0$$

and this implies

$$\beta g(\varphi Y, \varphi Y) = 0, \tag{3.16}$$

since $g(hZ, Z) = -g(Z, hZ)$ and thus $g(hZ, Z) = 0$, for $Z \in T(M)$.

Hence, from (3.16), it follows that $\beta = 0$. Thus from the definition of η-Einstein manifold, we get $QX = \alpha X$. Hence, form (2.5) and $QX = \alpha X$, it follows that $QX = 2nkX$, which proves the corollary. □

Now as an application of the Corollary 3.1, we have:

Corollary 3.2. *Any conformally flat contact metric manifold with $\xi \in N(k)$ is a space form.*

Proof. On a conformally flat Riemannian manifold for $n > 1$ [20], we have

$$(\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{4n}\{(Xr)Y - (Yr)X\}.$$

Then by Corollary 3.1, we get $QX = 2nkX$ and therefore, $r = Tr(Q) = 2nk$. Now for $X, Y, Z \in T(M)$, $C(X,Y)Z = 0$ gives

$$R(X,Y)Z = \frac{1}{2n-1}[g(QY,Z)X + g(Y,Z)QX -$$

$$g(QX,Z)Y - g(X,Z)QY] + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y].$$

Using $QX = 2nkX$ and $r = 2nk$ in the above expression we get

$$R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y].$$

This completes the proof. □

Acknowledgement. The authors are thankful to the referees for their valuable suggestions in the improvement of the paper.

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