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Product of Range Symmetric Block Matrices in Minkowski Space

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Abstract. Necessary and sufficient conditions for the product of range symmetric matrices of rank r to be range symmetric in Minkowski space $\mathcal{M}$ is derived. Also equivalent conditions for the product of two range symmetric block matrices to be range symmetric are established. As an application we have shown that a block matrix in Minkowski space can be expressed as a product of range symmetric matrices in $\mathcal{M}$.

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1. Introduction

Throughout we shall deal with $C^{n\times n}$, the space of $n\times n$ complex matrices. Let C^n be the space of complex n-tuples, we shall index the components of a complex vector in C^n from 0 to $n-1$, that is $u=(u_0,u_1,u_2,\dots u_{n-1})$. Let G be the Minkowski metric tensor defined by $Gu=(u_0,-u_1,-u_2,\dots,-u_{n-1})$. Clearly the Minkowski metric matrix

$$G=\begin{bmatrix}1 & 0\\ 0 & -I_{n-1}\end{bmatrix}, G=G^* \text{ and } G^2=I_n. \tag{1.1}$$

In [8], Minkowski inner product on C^n is defined by $(u,v)=[u,Gv]$, where [.,.] denotes the conventional Hilbert(unitary) space inner product. A space with Minkowski inner product is called a Minkowski space and denoted as $\mathcal{M}$. For $A\in C^{n\times n}$, $x,y\in C^n$, by using (1.1),

$$\begin{aligned}(Ax,y)&=[Ax,Gy]=[x,A^*Gy]\\ &=[x,G(GA^*G)y]=[x,GA^{\sim}y]=(x,A^{\sim}y)\end{aligned} \tag{1.2}$$

where $A^{\sim}=GA^*G$. The matrix $A^{\sim}$ is called the Minkowski adjoint of A in $\mathcal{M}$ (A^* is usual hermitian adjoint of A). Naturally, we call a matrix $A\in C^{n\times n}$

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$\mathcal{M}$-symmetric in $\mathcal{M}$ if $A = A^{\sim}$. From the definition $A^{\sim} = GA^*G$ we have the following equivalence: A is $\mathcal{M}$-symmetric $\Leftrightarrow AG$ is hermitian $\Leftrightarrow GA$ is hermitian. For $A \in C^{n\times n}$, $\text{rk}(A)$, $N(A)$, and $R(A)$ are respectively the rank of A, null space of A and range space of A. By a generalized inverse of A we mean a solution of the equation $AXA = A$ and is denoted as $A^{(1)}$. $A\{1\}$ is the set of all generalized inverses of A . Throughout I refers to identity matrix of appropriate order unless otherwise specified.

Definition 1.1. [2, Definition 1, p.7] *For $A \in C^{m\times n}$, A^+ is the Moore-Penrose inverse of A if $AA^+A = A$, $A^+AA^+ = A^+$, AA^+ and A^+A are Hermitian. The Minkowski inverse of A, analogous to Moore-Penrose inverse of A is introduced and its existence is discussed in [5].*

Definition 1.2. [5, Definition 4, p.2] *For $A \in C^{m\times n}$, $A^{\textcircled{m}}$ is the Minkowski inverse of A if $AA^{\textcircled{m}}A = A$, $A^{\textcircled{m}}AA^{\textcircled{m}} = A$, $AA^{\textcircled{m}}$ and $A^{\textcircled{m}}A$ are $\mathcal{M}$-symmetric.*

Theorem 1.1. [5, Theorem 1, p.4] *For $A \in C^{m\times n}$, $A^{\textcircled{m}}$ exists in $\mathcal{M} \Leftrightarrow \text{rk}(A) = \text{rk}(AA^{\sim}) = \text{rk}(A^{\sim}A)$.*

Theorem 1.2. [7, Lemma 3.3, p.143] *Let A and B be matrices in $\mathcal{M}$. Then $N(A^*) \subseteq N(B^*) \Leftrightarrow N(A^{\sim}) \subseteq N(B^{\sim})$.*

Theorem 1.3. [4, Lemma 1, p.193] *For $A, B, C \in C^{m\times n}$, the following are equivalent:*

(1) *$CA^{(1)}B$ is invariant for every $A^{(1)} \in C^{nxm}$.*
(2) *$N(A) \subseteq N(C)$ and $N(A^*) \subseteq N(B^*)$ 3. $C = CA^{(1)}A$ and $B = AA^{(1)}B$ for every $A^{(1)} \in A\{1\}$.*

Theorem 1.4. [7, Lemma 2.3, p.139] *For A_1, $A_2 \in C^{n\times n}(A_1A_2)^{\sim} = A_2^{\sim}A_1^{\sim}$ and $(A_1^{\sim})^{\sim} = A_1$.*

A matrix $A \in C^{n\times n}$ is said to be range symmetric in unitary space (or) equivalently A is said to be EP if $N(A) = N(A^*)$ (or $AA^+ = A^+A$) [2, p.163]]. For further properties of EP matrices one may refer [1,2, 4 and 9]. In [6] the concept of range symmetric matrix in $\mathcal{M}$ is introduced and developed analogous to that of EP matrices. A matrix $A \in C^{n\times n}$ said to be range symmetric in $\mathcal{M} \Leftrightarrow N(A) = N(A^{\sim})$. In the sequel we shall make use of the following results.

Theorem 1.5. [6, Theorem 2.2, p.47] *For $A \in C^{n\times n}$, the following are equivalent:*

(i) *A is range symmetric in $\mathcal{M}$*
(ii) *GA is EP*
(iii) *AG is EP*
(iv) *$N(A^*) = N(AG)$*
(v) *$R(A) = R(A^{\sim})$*
(vi) *$A^{\sim} = HA = AK$ for some non-singular matrices H and K.*
(vii) *$R(A^*) = R(GA)$*

2. Product of range symmetric matrices in $\mathcal{M}$

In this section we have obtained necessary and sufficient conditions for the product of two range symmetric matrices of rank r to be range symmetric in $\mathcal{M}$. Later we have extended the result to block matrices in $\mathcal{M}$.

Theorem 2.1. *Let A and B be range symmetric matrices of rank r in $\mathcal{M}$ and AB be of rank r. Then AB is range symmetric in $\mathcal{M}$ if and only if $R(A) = R(B)$.*

Proof. Let A and B be range symmetric matrices of rank r in $\mathcal{M}$. Let AB be of rank r and $R(A) = R(B)$. We prove that AB is range symmetric in $\mathcal{M}$. Clearly $R(AB) \subseteq R(A)$. Since $\text{rk}(AB) = \text{rk}(A) = r$, it follows that $R(AB) = R(A)$. Also $R(AB)^{\sim} \subseteq R(B^{\sim}) = R(B)$ and $\text{rk}(AB)^{\sim} = \text{rk}(AB) = \text{rk}(A) = \text{rk}(B) = r$. This implies $R(AB)^{\sim} = R(B)$. Since $R(A) = R(B)$, if follows that $R(AB) = R(AB)^{\sim}$. Hence AB is range symmetric in $\mathcal{M}$.

Conversely, AB is range symmetric in $\mathcal{M}$ implies $R(AB)^{\sim} = R(AB)$. $R(AB) \subseteq R(A)$ and $rk(AB) = \text{rk}(A) = r$ implies $R(AB) = R(A)$. $R(AB)^{\sim} \subseteq R(B^{\sim}) = R(B)$. Thus $R(A) = R(B)$. □

Hence forth we are concerned with $n \times n$ matrices M partitioned in the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ with } \text{rk}(M) = \text{rk}(A) = r \tag{2.1}$$

It is well known that in [3] M of the form (2.1) satisfies $N(A) \subseteq N(C), N(A^*) \subseteq N(B^*)$ and $D = CA^+B$.

Definition 2.1. [4, Lemma 1.2, p.193] *Let*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

be an $n \times n$ matrix. The Schur complement of A in M, denoted by M/A is defined as $D - CA^{(1)}B$, where $A^{(1)}$ is a generalized inverse of A.

Theorem 2.2. *Let M be of the form (2.1), then M is range symmetric in $\mathcal{M} \Leftrightarrow A$ is range symmetric in $\mathcal{M}$ and $CA^+ = -G_1(A^+B)^{\sim}$, where G_1 is the Minkowski metric tensor of order as that of A.*

Proof. Since A is range symmetric in $\mathcal{M}$ and $CA^+ = -G_1(A^+B)^{\sim}$, by Theorem 1.5 (ii), G_1A is EP and $(G_1A)^+ = A^+G_1$. Hence $G_1A(G_1A)^+ = (G_1A)^+G_1A$. Since $G_1^+ = G_1$, $G_1AA^+G_1 = A^+G_1G_1A$. By (1.1) for G_1, we have

$$G_1AA^+G_1 = A^+A. \tag{2.2}$$

Since $\text{rk}(M) = \text{rk}(A) = r$, we have $N(A) \subseteq N(C)$, $N(A^{\sim}) \subseteq N(B^{\sim})$ and Schur complement of A in M is zero. By Theorem 1.3, we have $C = CA^+A$, $B = AA^+B$ and $D = CA^+B$.

Let us consider the matrices

$$P = \begin{bmatrix} I & 0 \\ CA^+ & I \end{bmatrix}, \quad Q = \begin{bmatrix} I & A^+B \\ 0 & I \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

where P, Q are non-singular. Now

$$\begin{aligned} PLQ &= \begin{bmatrix} I & 0 \\ CA^{+} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & A^{+}B \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} A & AA^{+}B \\ CA^{+}A & CA^{+}B \end{bmatrix} \\ &= \begin{bmatrix} A & B \\ C & D \end{bmatrix} \\ &= M \end{aligned}$$

and by using $CA^{+} = -G_1(A^{+}B)^{\sim}$, we have

$$\begin{aligned} Q^{\sim} &= GQ^{*}G \\ &= \begin{bmatrix} G_1 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ (A^{+}B)^{*} & I \end{bmatrix} \begin{bmatrix} G_1 & 0 \\ 0 & -I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ -(A^{+}B)^{*}G_1 & I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ -G_1(A^{+}B)^{\sim} & I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ CA^{+} & I \end{bmatrix} \\ &= P \end{aligned}$$

Since A is range symmetric in $\mathcal{M}$, L is range symmetric in $\mathcal{M}$. We claim M is range symmetric in $\mathcal{M}$, that is $N(M) = N(M^{\sim})$. Since $M = PLP^{\sim}$, $x \in N(M) \Leftrightarrow Mx = 0 \Leftrightarrow PLP^{\sim}x = 0 \Leftrightarrow Ly = 0$ [where $y = P^{\sim}x$] $\Leftrightarrow L^{\sim}y = 0$ [by $N(L) = N(L^{\sim})$] $\Leftrightarrow PL^{\sim}P^{\sim}x = 0 \Leftrightarrow M^{\sim}x = 0 \Leftrightarrow x \in N(M^{\sim})$. Thus $N(M) = N(M^{\sim})$ and hence M is range symmetric in $\mathcal{M}$.

Conversely, let us assume that M is range symmetric in $\mathcal{M}$. Since $M = PLQ$, one choice of $M^{(1)}$ is

$$M^{(1)} = Q^{-1} \begin{bmatrix} A^{+} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}.$$

Since M is range symmetric in $\mathcal{M}$, we have $N(M) = N(M^{\sim})$. By using Theorem 1.2, and Theorem 1.3 we get $M^{\sim} = M^{\sim}M^{(1)}M$. By using (1.2), we have $GM^{*}G =$

$GM^*GM^{(1)}M$ and by using (1.1),

$$\begin{aligned}
M^* &= M^*GM^{(1)}MG \\
&= M^*GQ^{-1}\begin{bmatrix} A^+ & 0 \\ 0 & 0 \end{bmatrix} P^{-1}P \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} QG \\
&= M^*GQ^{-1}\begin{bmatrix} A^+A & 0 \\ 0 & 0 \end{bmatrix} QG \\
&= M^*G\begin{bmatrix} I & -(A^+B) \\ 0 & I \end{bmatrix}\begin{bmatrix} A^+A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & (A^+B) \\ 0 & I \end{bmatrix} G \\
&= M^*G\begin{bmatrix} A^+A & A^+B \\ 0 & 0 \end{bmatrix} G \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \\
&= \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}\begin{bmatrix} G_1 & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} A^+A & A^+B \\ 0 & 0 \end{bmatrix}\begin{bmatrix} G_1 & 0 \\ 0 & -I \end{bmatrix} \\
&= \begin{bmatrix} A^*G_1A^+AG_1 & -A^*G_1A^+B \\ B^*G_1A^+AG_1 & -B^*G_1A^+B \end{bmatrix}
\end{aligned}$$

Equating the corresponding blocks, we get

$$A^* = A^*G_1A^+AG_1 \Rightarrow G_1A^*G_1 = G_1A^*G_1A^+A \Rightarrow A^{\sim} = A^{\sim}A^+A.$$

By Theorem 1.3, it follows that $N(A) \subset N(A^{\sim})$ and $\mathrm{rk}(A) = \mathrm{rk}(A^{\sim})$. Hence $N(A) = N(A^{\sim})$ and therefore A is range symmetric in $\mathcal{M}$. Also

$$C^* = -A^*G_1A^+B \Rightarrow G_1C^*G_1 = -G_1A^*G_1A^+BG_1 \Rightarrow C^{\sim} = -A^{\sim}A^+BG_1.$$

Taking Minkowski adjoint, and by using $G_1^{\sim} = G_1$ and Theorem 1.4, we get $C = -G_1(A^+B)^{\sim}A$. Now

$$\begin{aligned}
CA^+ &= -G_1(A^+B)^{\sim}AA^+ \\
&= -G_1(A^+B)^{\sim}G_1A^+AG_1 \\
&= -G_1(A^+B)^{\sim}G_1(A^+A)^*G_1 \quad \text{[By using (2.2)]} \\
&= -G_1(A^+B)^{\sim}(A^+A)^{\sim} \\
&= -G_1(A^+AA^+B)^{\sim} \quad \text{[By using Theorem 1.4]} \\
&= -G_1(A^+B)^{\sim}.
\end{aligned}$$

This completes the proof. □

Lemma 2.1. *Let M be of the form (2.1) be range symmetric in $\mathcal{M}$, then A is range symmetric in $\mathcal{M}$ and there exists an $r \times (n-r)$ matrix X such that*

$$M = \begin{bmatrix} A & AX \\ -G_1X^{\sim}A & -G_1X^{\sim}AX \end{bmatrix} \tag{2.3}$$

where G_1 is the Minkowski metric tensor of order as that of A.

Proof. Since M is of the form (2.1) by using Theorem 1.2 and Theorem 1.3, M satisfy $N(A) \subset N(C)$, $N(A^{\sim}) \subset N(B^{\sim})$ and $D = CA^+B$. Hence there exist $(n-r) \times r$ matrix Y and $n \times (n-r)$ matrix X such that $C = YA$ and $B = AX$.

Since A is range symmetric in $\mathcal{M}$ by using Theorem 1.5(ii). G_1A is EPr, again by using (2.2) we have $AA^+ = G_1A^+AG_1$, $CA^+ = -G_1(A^+B)^{\sim}$, and

$$\begin{aligned} YAA^+ &= -G_1(A^+AX)^{\sim} \\ &= -G_1X^{\sim}(A^+A)^{\sim} \quad \text{[By using Theorem 1.4]} \\ &= -G_1X^{\sim}G_1(A^+A)^*G_1 \quad \text{[By using (1.2)]} \\ &= -G_1X^{\sim}G_1A^+AG_1 \\ &= -G_1X^{\sim}AA^+ \quad \text{[By using (2.2)]} \end{aligned}$$

Therefore $YA = -G_1X^{\sim}A = C$ and $D = CA^+B = -G_1X^{\sim}AA^+AX = -G_1X^{\sim}AX$. Thus

$$M = \begin{bmatrix} A & AX \\ -G_1X^{\sim}A & -G_1X^{\sim}AX \end{bmatrix}.$$

□

Theorem 2.3. *Let*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ and } L = \begin{bmatrix} F & U \\ H & K \end{bmatrix}$$

be range symmetric matrices in $\mathcal{M}$ both of the form (2.1) and ML of rank r, then the following are equivalent:

(i) *ML is range symmetric in $\mathcal{M}$.*
(ii) *AF is range symmetric in $\mathcal{M}$ and $CA^+ = HF^+$*
(iii) *AF is range symmetric in $\mathcal{M}$ and $A^+B = F^+U$*

Proof. Since M and L are of the form (2.1) by Lemma 2.1 there exist $r \times (n-r)$ matrices X and Y such that

$$M = \begin{bmatrix} A & AX \\ -G_1X^{\sim}A & -G_1X^{\sim}AX \end{bmatrix} \text{ and } L = \begin{bmatrix} F & FY \\ -G_1Y^{\sim}F & -G_1Y^{\sim}FY \end{bmatrix}.$$

Now

$$\begin{aligned} ML &= \begin{bmatrix} A & AX \\ -G_1X^{\sim}A & -G_1X^{\sim}AX \end{bmatrix} \begin{bmatrix} F & FY \\ -G_1Y^{\sim}F & -G_1Y^{\sim}FY \end{bmatrix} \\ &= \begin{bmatrix} A(I-XG_1Y^{\sim})F & A(I-XG_1Y^{\sim})FY \\ -G_1X^{\sim}A(I-XG_1Y^{\sim})F & -G_1X^{\sim}AX(I-XG_1Y^{\sim})FY \end{bmatrix} \\ &= \begin{bmatrix} AZF & AZFY \\ -G_1X^{\sim}AZF & -G_1X^{\sim}AZFY \end{bmatrix} \end{aligned}$$

where $Z = I - XG_1Y^{\sim}$. Clearly

$$N(AZF) \subset N(-G_1X^{\sim}AZF) = N(G_1X^{\sim}AZF),\ N(AZF)^{\sim} \subset N(AZFY)^{\sim}$$

and the Schur complement of AZF in ML is zero. For

$$\begin{aligned} ML/AZF &= -G_1X^{\sim}AZFY + G_1X^{\sim}(AZF)(AZF)^+AZFY \\ &= -G_1X^{\sim}AZFY + G_1X^{\sim}AZFY = 0. \end{aligned}$$

Hence $\text{rk}(AZF) = \text{rk}(ML) = r$. Thus ML is also of the form (2.1). Since M and L are range symmetric in $\mathcal{M}$ by Theorem 2.2 and Lemma 2.1, A and F are range symmetric in $\mathcal{M}$. Now

$$\begin{aligned} -G_1(A^+AX)^\sim &= -G_1X^\sim(A^+A)^\sim \quad \text{[By Theorem 1.4]} \\ &= -G_1X^\sim G_1(A^+A)^*G_1 \quad \text{[By using 1.2]} \\ &= -G_1X^\sim G_1A^+AG_1 \\ &= -G_1X^\sim AA^+ \quad \text{[By using 2.2]} \end{aligned}$$

Similarly it can be proved that $-G_1Y^\sim FF^+ = -G_1(F^+FY)^\sim$. We now claim AZF is range symmetric in $\mathcal{M}$. $N(F) \subset N(AZF)$, and $\text{rk}(AZF) = rk(F) = r$, hence it follows $N(F) = N(AZF)$. Also $N(A^\sim) \subset N(AZF)^\sim$ and $\text{rk}(AZF)^\sim = \text{rk}(AZF) = \text{rk}(A) = r = rk(F), N(A) = N(A^\sim) = N(AZF)^\sim = N(F)$. Thus $N(AZF) = N(AZF)^\sim$ and hence AZF is range symmetric in $\mathcal{M}$. By Theorem 1.5 (ii), G_1AZF in EPr and by using (2.2) we have

$$G_1AZF(AZF)^+G_1 = (AZF)^+AZF \tag{2.4}$$

By using (2.4) for

$$N(AZF) = N(F),\ N(AZF)^\sim = N(A^\sim) = N(A)$$

we get

$$\begin{aligned} &AZF(AZF)^+ = FF^+ = G_1(AZF)^+AZFG_1 \text{ and} \\ &(AZF)^+AZF = A^+A = G_1AA^+G_1. \end{aligned} \tag{2.5}$$

Since $H = -G_1Y^\sim F$ and $C = -G_1X^\sim A$, we have

$$\begin{aligned} HF^+ &= -G_1Y^\sim FF^+ \quad \text{[by (2.5)]} \\ &= -G_1Y^\sim G_1AZF(AZF)^+ \\ &= -G_1Y^\sim G_1(AZF)^+AZFG_1 \\ &= -G_1Y^\sim[(AZF)^+AZF]^\sim \\ &= -G_1[(AZF)^+AZFY]^\sim. \end{aligned}$$

Similarly by using (2.5), we have

$$CA^+ = -G_1X^\sim AZF(AZF)^+.$$

Therefore

$$CA^+ = HF^+ \Leftrightarrow -G_1X^\sim AZF(AZF)^+ = -G_1[(AZF)^+AZFY]^\sim. \tag{2.6}$$

Now the proof runs as follows: ML is range symmetric in $\mathcal{M} \Leftrightarrow AZF$ is range symmetric in $\mathcal{M}$ and $G_1X^\sim AZF(AZF)^+ = -G_1[(AZF)^+AZFY]^\sim \Leftrightarrow AZF$ is range symmetric in $\mathcal{M}$ and $CA^+ = HF^+$ [By using (2.6)] $\Leftrightarrow N(AZF) = N(AZF)^\sim$ and $CA^+ = HF^+ \Leftrightarrow N(F) = N(A)^\sim = N(A)$ and $CA^+ = HF^+ \Leftrightarrow AF$ is range symmetric in $\mathcal{M}$ and $CA^+ = HF^+$ [By using Theorem 2.1] $\Leftrightarrow AF$ is range symmetric in $\mathcal{M}$ and $A^+B = F^+U$ [By Theorem 2.2]. $\square$

3. Factorization

In this section a set of conditions under which a matrix can be expressed as product of range symmetric matrices in m are derived.

Definition 3.1. *A matrix $A \in C^{n\times n}$ is said to be unitary in unitary space if and only if $AA^* = A^*A = I$.*

Lemma 3.1. [9, Theorem 1] *Let $A \in Cr^{n\times n}$ be EPr matrix, then there exist a unitary matrix U such that*

$$A = U^*DU = U^* \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} U,$$

where D_1 is $r \times r$ non-singular.

Lemma 3.2. *Let $A \in C^{n\times n}$, A is range symmetric in $\mathcal{M}$ if and only if*

$$A = P \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} P^{\sim},$$

where D is $r \times r$ matrix and $\mathrm{rk}(D) = r$ and P is unitary.

Proof. By Theorem 1.5 (ii), the matrix A is range symmetric in $\mathcal{M} \Leftrightarrow AG$ is EP. By Lemma 3.1, this holds if and only if

$$AG = U \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} U^*$$

or

$$A = U \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} U^*G$$

or

$$A = UG \begin{bmatrix} G_1 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} U^*G$$

where G_1 is $r \times r$ or

$$A = UG \begin{bmatrix} G_1D_1 & 0 \\ 0 & 0 \end{bmatrix} U^*G.$$

Thus the matrix A is range symmetric in $\mathcal{M}$ if and only if

$$A = P \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} P^{\sim},$$

where $P = UG$, $P^{\sim} = GU^{\sim} = U^*G$ and $D = G_1D_1$ is $r \times r$ non-singular. □

Lemma 3.3. *Let A and B be range symmetric matrices in $\mathcal{M}$ of rank r. Then $N(A) = N(B)$ if and only if $N(PAP^{\sim}) = N(PBP^{\sim})$ where P is unitary.*

Proof. Let A and B be range symmetric matrices in $\mathcal{M}$. Assume $N(A) = N(B)$. Then $x \in N(PAP^{\sim})$ $PAP^{\sim}x = 0$ $AP^{\sim}x = 0$ $Ay = 0$ where $y = P^{\sim}x$ $y \in N(A) = N(B) By = 0$ $BP^{\sim}x = 0$ $PBP^{\sim}x = 0$ $x \in N(PBP^{\sim})$. Thus $N(A) = N(B)$ $N(PAP^{\sim}) = N(PBP^{\sim})$.

The proof the converse is similar and hence omitted. □

Theorem 3.1. *Let M be of the form (2.1) be range symmetric in $\mathcal{M}$. Then M can be written as a product of range symmetric matrices in $\mathcal{M}$.*

Proof. Since M is of the form (2.1) and M is range symmetric in $\mathcal{M}$, by Lemma 2.1, A is range symmetric in $\mathcal{M}$ and

$$M = \begin{bmatrix} A & AX \\ -G_1X^{\sim}A & -G_1X^{\sim}AX \end{bmatrix}.$$

Since M is range symmetric in $\mathcal{M}$ by Theorem 1.5(ii), GM in EPr, where

$$GM = \begin{bmatrix} G_1 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} A & AX \\ -G_1X^{\sim}A & -G_1X^{\sim}AX \end{bmatrix} = \begin{bmatrix} G_1A & G_1AX \\ G_1X^{\sim}A & G_1X^{\sim}AX \end{bmatrix}.$$

Consider

$$\begin{aligned} P &= \begin{bmatrix} G_1AA^+G_1 & G_1AA^+G_1X \\ X^*G_1AA^+G_1 & X^*G_1AA^+G_1X \end{bmatrix}, \\ L &= \begin{bmatrix} G_1A & 0 \\ 0 & -I \end{bmatrix}, \\ Q &= \begin{bmatrix} A^+A & A^+AX \\ X^*A^+A & X^*A^+AX \end{bmatrix}. \end{aligned}$$

By using (1.1),

$$\begin{aligned} P^* &= \begin{bmatrix} (G_1AA^+G_1)^* & (X^*G_1AA^+G_1)^* \\ (G_1AA^+G_1X)^* & (X^*G_1AA^+G_1X)^* \end{bmatrix} \\ &= \begin{bmatrix} G_1(AA^+)^*G_1 & G_1(AA^+)^*G_1X \\ X^*G_1(AA^+)^*G_1 & X^*G_1(AA^+)^*G_1X \end{bmatrix} \\ &= \begin{bmatrix} G_1AA^+G_1 & G_1AA^+G_1X \\ X^*G_1AA^+G_1 & X^*G_1AA^+G_1X \end{bmatrix} = P \end{aligned}$$

Similarly $Q^* = Q$ can be proved. Thus $P = P^*$ and $Q = Q^*$ and therefore P, Q are EPr. Since A is range symmetric by Theorem 1.5(iii) G_1A is EPr and hence L is EPr.

Now

$$PLQ$$

$$= \begin{bmatrix} G_1AA^+G_1 & G_1AA^+G_1X \\ X^*G_1AA^+G_1 & X^*G_1AA^+G_1X \end{bmatrix} \begin{bmatrix} G_1A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^+A & A^+AX \\ X^*A^+A & X^*A^+AX \end{bmatrix}$$

$$= \begin{bmatrix} G_1AA^+G_1G_1A & 0 \\ X^*G_1AA^+G_1G_1A & 0 \end{bmatrix} \begin{bmatrix} A^+A & A^+AX \\ X^*A^+A & X^*A^+AX \end{bmatrix}$$

$$= \begin{bmatrix} G_1A & 0 \\ X^*G_1A & 0 \end{bmatrix} \begin{bmatrix} A^+A & A^+AX \\ X^*A^+A & X^*A^+AX \end{bmatrix}$$

$$= \begin{bmatrix} G_1AA^+A & G_1AA^+AX \\ X^*G_1AA^+A & X^*G_1AA^+AX \end{bmatrix}$$

$$= \begin{bmatrix} G_1A & G_1AX \\ X^*G_1A & X^*G_1AX \end{bmatrix}$$

$$= \begin{bmatrix} A & AX \\ -G_1X^{\sim}A & -G_1X^{\sim}AX \end{bmatrix}$$

$$= GM.$$

By using (1.1), $M = GPLQ = (GP)(LG)(GQ)$. Since P, Q, L and EPr, by Theorem 1.5 (ii) and (iii), it follow that GP, LG, GQ are range symmetric $\mathcal{M}$. Thus a range symmetric matrix M in $\mathcal{M}$ is expressed as a product of range symmetric matrices in $\mathcal{M}$. □

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