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Some More Properties of $F_{\mathcal{I}}$ and Regular $\mathcal{I}$-Closed Sets in Ideal Topological Spaces

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Abstract. In 1964, Hayashi [8] defined and studied the notions of $\star$-dense in itself sets and $\star$-perfect subsets in ideal topological spaces. In 1999, Dontchev *et al.* [5] have studied the notion of Ideal resolvability through codense and completely codense ideal topological spaces. Recently, in the year 2004, Keskin, Noiri and Yuksel [12] have introduced and studied the concepts of $f_{\mathcal{I}}$-sets and $f_{\mathcal{I}}$-continuity. In this paper, we studied some more properties of $f_{\mathcal{I}}$-sets and $f_{\mathcal{I}}$-continuity with codense and completely codense ideals. We also, continued the study of regular $\mathcal{I}$-closed concepts.

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1. Introduction

Ideals in topological spaces have been considered since 1930. This topic has won its importance by the paper of Vaidyanathaswamy [20]. An *ideal* $\mathcal{I}$ on a topological space (X,τ) is a collection of subsets of X which satisfies (i) A $\in \mathcal{I}$ and B $\subset$ A implies B $\in \mathcal{I}$ and (ii) A $\in \mathcal{I}$ and B $\in \mathcal{I}$ implies A $\cup$ B $\in \mathcal{I}$. Given a topological space (X,τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator () *: $\wp$(X) $\rightarrow \wp(X)$, called a $local function$ [13] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subset X$, $A^\star(\mathcal{I},\tau)$= $\{x \in X \mid U \cap A \notin \mathcal{I}$ for every U $\in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of the basic facts about the local functions [9, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator cl*() for a topology $\tau^\star(\mathcal{I},\tau)$, called the $\star - topology$, finer than τ

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is defined by $\text{cl}^\star(A) = A \cup A^\star(\mathcal{I}, \tau)$[20]. When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I}, \tau)$ and $\tau^\star$ or $\tau^\star(\mathcal{I})$ for $\tau^\star(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an *ideal space*. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X, τ). A subset A of an ideal space $(X, \tau, \mathcal{I})$is $\tau^\star$-*closed* [9] (resp. $\star$-*dense in itself* [8], $\star$-*perfect* [8]) if $A^\star \subset$A (resp. $A \subset A^\star$, A $= A^\star$). Clearly, A is $\star$-perfect if and only if A is $\tau^\star$-closed and $\star$-dense in itself. In ideal topological spaces ,$\mathcal{I}$-open sets [10], almost $\mathcal{I}$-open sets [1] (quasi $\mathcal{I}$-open sets [2]), $\mathcal{I}$-locally closed sets [4], $f_\mathcal{I}$-sets [12] and regular $\mathcal{I}$-closed sets [11] are some of the $\star$-dense in itself sets. In this note, we discuss the properties of the $\star$-dense in itself sets, namely $f_\mathcal{I}$-sets and regular $\mathcal{I}$-closed sets.

2. Preliminaries

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If A $\subset X, cl(A)$ and $\text{int}(A)$ will, respectively, denote the closure and interior of A in (X, τ) and $\text{cl}^\star(A)$ and $\text{int}^\star(A)$ will, respectively, denote the closure and interior of A in $(X, \tau^\star)$. An open subset A of a space (X, τ) is said to be *regular open* if $A = \text{int}(\text{cl}(A))$. The complement of a regular open set is *regular closed*. The family of all regular open (resp. regular closed) set is denoted by $RO(X, \tau)$(resp. $RC(X, \tau)$). A subset A of a space (X, τ) is an α-*open* [16] (resp. *semiopen* [14], *preopen* [15]) if $A \subset \text{int}(\text{cl}(\text{int}(A)))$ (resp. $A \subset \text{cl}(\text{int}(A))$, $A \subset \text{int}(\text{cl}(A))$). The complement of a semiopen (resp. preopen) set is semiclosed (resp. preclosed). The family of all α-open (resp. semiopen, preopen) sets in (X, τ) is denoted by τ^α (resp. SO (X, τ), PO (X, τ)). The smallest preclosed set containing A is called the *preclosure* of A and is denoted by $\text{pcl}(A)$. Also, $\text{pcl}(A) = A \cup \text{cl}(\text{int}(A))$ [3, Theorem 1.5(e)]. The largest preopen set contained in A is called the *preinterior* of A and is denoted by $\text{pint}(A)$. Also, $\text{pint}(A) = A \cap \text{int}(\text{cl}(A))$ [3, Theorem 1.5(f)]. τ^α is a topology finer than τ. The interior of A in (X, τ^α) is denoted by $\text{int}_\alpha(A)$ and $\text{int}_\alpha(A) = A \cap \text{int}(\text{cl}(\text{int}(A)))$ [3, Theorem 1.5(d)]. τ is said to be an α-*topology* [16] if $\tau = \tau^\alpha$. Two topologies τ and σ on X is said to be α-*equivalent* [16] if $\tau^\alpha = \sigma^\alpha$. Recall that, if two topologies on a set X are α-equivalent, then they have the same collection of regular open, semiopen, preopen, dense and nowhere dense sets. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-*open* [10] if $A \subset \text{int}(A^\star)$. The family of all $\mathcal{I}$-open sets is denoted by $IO(X, \tau)$. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-*locally closed* [4] if $A = G \cap V$, where G is open and V is $\star$-perfect. A subset A of X is $\mathcal{I}$-locally closed if and only if $A = G \cap A^\star$ for some open set G [19, Theorem 2.2]. Clearly, every $\star$-perfect set is $\mathcal{I}$-locally closed. Given an ideal space $(X, \tau, \mathcal{I})$, $\mathcal{I}$ is said to be *compatible* with respect to τ [9] (*supercompact* [20]), denoted by $\mathcal{I} \sim \tau$, if for every subset A of X and for each $x \in A$, there exists a neighborhood U of x such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$. $\mathcal{I}$ is said to be *codense* [5] if $\tau \cap \mathcal{I} = \{\emptyset\}$. $\mathcal{I}$ is said to be *completely codense* [5] if $PO(X) \cap \mathcal{I} = \{\emptyset\}$. Every completely codense ideal is codense but not the converse [5]. The following lemmas will be useful in the sequel.

Lemma 2.1. [9, Theorem 6.1] *Let $(X, \tau, \mathcal{I})$be an ideal space. Then the following are equivalent.*

(a) $\mathcal{I}$ is codense.

(b) $X = X^\star$.
(c) $G \subset G^\star$ *for every open set* G.
(d) $\text{int}(I) = \emptyset$ *for every* $I \in \mathcal{I}$.

Lemma 2.2. [18, Theorem 5] *Let* $(X, \tau, \mathcal{I})$ *be an ideal space. If* A *is* $\star$*-dense in itself, then* $A^\star = \text{cl}(A) = \text{cl}^\star(A)$.

Lemma 2.3. [18, Corollary 4] *If* $\mathcal{I}$ *is a completely codense ideal of an ideal space* $(X, \tau, \mathcal{I})$*, then*
(a) $\tau \subset \tau^\star \subset \tau^\alpha$,
(b) $SO(X, \tau) = SO(X, \tau^\star) = SO(X, \tau^\alpha)$,
(c) $(\tau^\star)^\alpha = \tau^\alpha$.

Lemma 2.4. [19, Theorem 2.1] *Let* $(X, \tau, \mathcal{I})$ *be an ideal space and* U *and* A *be subsets of* X *such that* $A \subset U \subset A^\star$*. Then* U *is* $\star$*-dense in itself, and* $U^\star$ *and* $A^\star$ *are* $\star$*-perfect.*

Lemma 2.5. [19, Theorem 2.15] *If* $(X, \tau, \mathcal{I})$ *is an ideal space, then* $\mathcal{I}$ *is completely codense if and only if* $PO(X, \tau) = IO(X, \tau)$.

3. More properties of codense ideals and completely codense ideals

The following Theorem 3.1 and its corollary give relationship between codense and completely codense ideals. Given a space (X, τ) and ideals $\mathcal{I}$ and $\Im$ on X, the *extension* of $\mathcal{I}$ via $\Im$ [10], denoted by $\mathcal{I} \star \Im$, is the ideal given by $\mathcal{I} \star \Im = \{A \subset X \mid A^\star(\mathcal{I}) \in \Im\}$. In particular, $\mathcal{I} \star \mathcal{N} = \{A \subset X \mid \text{int}(A^\star(\mathcal{I})) = \phi\}$ is a compatible ideal containing both $\mathcal{I}$ and $\mathcal{N}$ and $\mathcal{I} \star \mathcal{N}$ is usually denoted by $\widetilde{\mathcal{I}}$. Since $\widetilde{\mathcal{I}}$ is compatible, $(A^\star(\widetilde{\mathcal{I}}))^\star(\widetilde{\mathcal{I}}) = A^\star(\widetilde{\mathcal{I}})$ [9, Theorem 4.6(b)]. In Theorem 3.2 below, we discuss the relationship between the α-sets of the topologies τ, $\tau^\star(\mathcal{I})$ and $\tau^\star(\widetilde{\mathcal{I}})$.

Theorem 3.1. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space. Then* $\mathcal{I}$ *is codense in* (X, τ) *if and only if* $\mathcal{I}$ *is completely codense in* $(X, \tau^\star)$.

Proof. If $\mathcal{I}$ is completely codense in $(X, \tau^\star)$, then $\mathcal{I}$ is codense in $(X, \tau^\star)$ and so $\mathcal{I}$ is codense in (X, τ). Conversely, suppose $\mathcal{I}$ is codense in (X, τ). Let $A \in PO(X, \tau^\star) \cap \mathcal{I}$. $A \in PO(X, \tau^\star) \cap \mathcal{I} \Rightarrow A \in PO(X, \tau^\star)$ and $A \in \mathcal{I}$. $A \in PO(X, \tau^\star) \Rightarrow A \subset \text{int}^\star(\text{cl}^\star(A))$. $A \in \mathcal{I} \Rightarrow \text{int}^\star(A) = \emptyset$, by Lemma 2.1(d) and A is $\tau^\star$-closed, by [9, Lemma 2.7]. Therefore, $\text{int}^\star(\text{cl}^\star(A))= \text{int}^\star(A) = \emptyset$ which implies that A $= \emptyset$. Therefore, $\mathcal{I}$ is completely codense in $(X, \tau^\star)$. □

Corollary 3.1. *If* $(X, \tau, \mathcal{I})$ *is an ideal space, then the following are equivalent.*
(a) $\mathcal{I}$ *is codense in* (X, τ).
(b) $\widetilde{\mathcal{I}}$ *is codense in* (X, τ).
(c) $\widetilde{\mathcal{I}}$ *is completely codense in* $(X, \tau^\star)$.
(d) $\mathcal{I}$ *is completely codense in* $(X, \tau^\star)$.

Proof. (a) and (b) are equivalent by [10, Theorem 3.5]. (b) and (c) are equivalent by Theorem 3.1. (a) and (d) are equivalent by Theorem 3.1.

□

Theorem 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then*

(a) $\tau^\star(\widetilde{\mathcal{I}}) = (\tau^\star(\mathcal{I}))^\star(\widetilde{\mathcal{I}})$.

(b) If $\mathcal{I}$ is codense, then $(\tau^\star(\widetilde{\mathcal{I}}))^\alpha = (\tau^\star(\mathcal{I}))^\alpha$.

(c) If $\mathcal{I}$ is completely codense, then $(\tau^\star(\widetilde{\mathcal{I}}))^\alpha = \tau^\star(\widetilde{\mathcal{I}}) = (\tau^\star(\mathcal{I}))^\alpha = \tau^\alpha$.

Proof. (a) Since $\mathcal{I} \subset \widetilde{\mathcal{I}}$, $\tau^\star(\mathcal{I}) \subset \tau^\star(\widetilde{\mathcal{I}})$ which implies that $(\tau^\star(\mathcal{I}))^\star(\widetilde{\mathcal{I}}) \subset (\tau^\star(\widetilde{\mathcal{I}}))^\star(\widetilde{\mathcal{I}})$ and so $(\tau^\star(\mathcal{I}))^\star(\widetilde{\mathcal{I}}) \subset \tau^\star(\widetilde{\mathcal{I}})$. Suppose $A \in \tau^\star(\widetilde{\mathcal{I}})$. For each $x \in A$, there exists $U \in \tau$ and $I \in \widetilde{\mathcal{I}}$ such that $x \in U - I \subset A$. Since $U \in \tau^\star(\mathcal{I})$, $A \in (\tau^\star(\mathcal{I}))^\star(\widetilde{\mathcal{I}})$ and so $\tau^\star(\widetilde{\mathcal{I}}) \subset (\tau^\star(\mathcal{I}))^\star(\widetilde{\mathcal{I}})$. Therefore, $\tau^\star(\widetilde{\mathcal{I}}) = (\tau^\star(\mathcal{I}))^\star(\widetilde{\mathcal{I}})$.

(b) If $\mathcal{I}$ is codense, by Corollary 3.2, $\widetilde{\mathcal{I}}$ is completely codense in $(X, \tau^\star)$ and so by [5, Theorem 4.13], $\widetilde{\mathcal{I}} \subset \mathcal{N}(\tau^\star(\mathcal{I}))$. Therefore, $(\tau^\star(\mathcal{I}))^\star(\widetilde{\mathcal{I}}) \subset (\tau^\star(\mathcal{I}))^\star(\mathcal{N}(\tau^\star(\mathcal{I})))$. By (a), we have, $\tau^\star(\widetilde{\mathcal{I}}) \subset (\tau^\star(\mathcal{I}))^\alpha$. Since $\tau^\star(\mathcal{I}) \subset \tau^\star(\widetilde{\mathcal{I}})$, we have $\tau^\star(\mathcal{I}) \subset \tau^\star(\widetilde{\mathcal{I}}) \subset (\tau^\star(\mathcal{I}))^\alpha$ and so $(\tau^\star(\widetilde{\mathcal{I}}))^\alpha = (\tau^\star(\mathcal{I}))^\alpha$ [16, Proposition 10].

(c) If $\mathcal{I}$ is completely codense,by [18, Theorem 7(c)], $\widetilde{\mathcal{I}} = \mathcal{N}$ and so $\tau^\star(\widetilde{\mathcal{I}}) = \tau^\star(\mathcal{N}) = \tau^\alpha$. By Lemma 2.3(c), $(\tau^\star(\mathcal{I}))^\alpha = \tau^\alpha$. Therefore, (c) follows. □

We have the following.

Corollary 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $\mathcal{I}$ be codense. Then $\tau^\star(\widetilde{\mathcal{I}})$and $\tau^\star(\mathcal{I})$ are α-equivalent and so they have the same collection of regular open, semiopen, preopen, dense and nowhere dense sets.*

Proof. By Theorem 3.2(b), $\tau^\star(\widetilde{\mathcal{I}})$ and $\tau^\star(\mathcal{I})$ are α-equivalent and so, they have the same collection of regular open, semiopen, preopen, dense and nowhere dense sets. □

Corollary 3.3. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $\mathcal{I}$ be completely codense. Then*

(a) $\tau^\star(\widetilde{\mathcal{I}})$ is an α-topology.

(b) $\tau^\star(\widetilde{\mathcal{I}}) = \tau^\star(\mathcal{N})$, and

(c) τ, $\tau^\star(\mathcal{I})$and $\tau^\star(\widetilde{\mathcal{I}})$ are α-equivalent and so they have the same collection of regular open, semiopen, preopen, dense and nowhere dense sets.

Proof. (a) Since $(\tau^\star(\widetilde{\mathcal{I}}))^\alpha = \tau^\star(\widetilde{\mathcal{I}})$, by Theorem 3.2(c) , $\tau^\star(\widetilde{\mathcal{I}})$ is an α-topology.
(b)Since $\tau^\alpha = \tau^\star(\mathcal{N})$ [9, Example 2.10], by Theorem 3.2(c), $\tau^\star(\widetilde{\mathcal{I}}) = \tau^\star(\mathcal{N})$.
(c) follows from Theorem 3.2(c).

□

4. Characterizations of $f_{\mathcal{I}}$-sets

A subset A of an ideal space $(X, \tau, \mathcal{I})$ is called an $f_{\mathcal{I}}$-set [12] if $A \subset (\text{int}(A))^\star$. The family of all $f_{\mathcal{I}}$-sets in $(X, \tau, \mathcal{I})$ is denoted by $\mathcal{F}(\tau, \mathcal{I})$. Clearly, if A is any non-empty $f_{\mathcal{I}}$-set, then $\text{int}(A) \neq \emptyset$ and if $\mathcal{I}$ is not codense, then X is not an $f_{\mathcal{I}}$-set. In addition to this, $\mathcal{F}(\tau, \mathcal{I})$ has the following nice property.

Theorem 4.1. *If $(X, \tau, \mathcal{I})$ is an ideal space, then $\mathcal{F}(\tau, \mathcal{I}) \cap \mathcal{I} = \{\emptyset\}$.*

Proof. If $A \in \mathcal{F}(\tau, \mathcal{I}) \cap \mathcal{I}$, then $A \in \mathcal{F}(\tau, \mathcal{I})$ and $A \in \mathcal{I}$. $A \in \mathcal{I} \Rightarrow A^\star = \emptyset$ and $A \in \mathcal{F}(\tau, \mathcal{I}) \Rightarrow A \subset (\text{int}(A))^\star \subset A^\star = \emptyset$. Therefore, $A = \emptyset$ which completes the proof. □

Every $f_{\mathcal{I}}$-set is a semiopen set [12, Remark 2] but not the converse [12, Example 3]. The following Theorem 4.2 and its Corollary 4.1, characterizes codense ideals in terms of $f_{\mathcal{I}}$-sets and Theorem 4.2 shows that for codense ideals, semiopen sets and $f_{\mathcal{I}}$-sets coincide.

Theorem 4.2. *Let $(X, \tau, \mathcal{I})$ be an ideal space, then the following are equivalent.*

(a) $\mathcal{I}$ is codense.
(b) $SO(X, \tau) = \mathcal{F}(\tau, \mathcal{I})$.
(c) $\tau \subset \mathcal{F}(\tau, \mathcal{I})$.

Proof. $(a) \Rightarrow (b)$. If $A \in SO(X, \tau)$, then $A \subset \mathrm{cl}(\mathrm{int}(A)) = (\mathrm{int}(A))^\star$, by Lemma 2.1(c) and Lemma 2.2 and so $A \in \mathcal{F}(\tau, \mathcal{I})$. If $A \in \mathcal{F}(\tau, \mathcal{I})$, then $A \subset (\mathrm{int}(A))^\star = \mathrm{cl}(\mathrm{int}(A))$ and so $A \in SO(X, \tau)$.
$(b) \Rightarrow (c)$ is clear.
$(c) \Rightarrow (a)$. Follows from Theorem 4.1. □

Corollary 4.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent.*

(a) $\mathcal{I}$ is codense.
(a) $SO(X, \tau^\star) = \mathcal{F}(\tau^\star, \mathcal{I})$.
(a) $\tau^\star \subset \mathcal{F}(\tau^\star, \mathcal{I})$.

Proof. Since $\mathcal{I}$ is codense in (X, τ) if and only if $\mathcal{I}$ is codense in $(X, \tau^\star)$ by the remark below Theorem 6.1 of [9], the proof follows from Theorem 4.2. □

Corollary 4.2. [18, Theorem 1] *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $\mathcal{I}$ is codense if and only if $SO(X, \tau) \cap \mathcal{I} = \{\emptyset\}$.*

Since $\tau \subset \tau^\star$, $\mathcal{F}(\tau, \mathcal{I}) \subset \mathcal{F}(\tau^\star, \mathcal{I})$. The following Example 4.1, shows that the reverse direction is not true in general and Theorem 4.3 below shows that the two collection of sets are equal if the ideal $\mathcal{I}$ is completely codense.

Example 4.1. [12, Example 1] Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{c\}, \{a, d\}, \{a, c, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}, \{d\}, \{b, d\}\}$. If $A = \{a\}$, then $\mathrm{int}(A) = \emptyset$ and so $A \notin \mathcal{F}(\tau, \mathcal{I})$. Since $\mathrm{int}^\star(A) = A$ and $(\mathrm{int}^\star(A))^\star = \{a, b, d\}$, $A \in \mathcal{F}(\tau^\star, \mathcal{I})$.

Theorem 4.3. *Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ be completely codense. Then $SO(X, \tau) = SO(X, \tau^\star) = SO(X, \tau^\alpha) = \mathcal{F}(\tau, \mathcal{I}) = \mathcal{F}(\tau^\star, \mathcal{I}) = \mathcal{F}(\tau^\alpha, \mathcal{I})$.*

Proof. Since $\mathcal{I}$ is completely codense, by Lemma 2.3(b), $SO(X, \tau) = SO(X, \tau^\star) = SO(X, \tau^\alpha)$. Since $\mathcal{N}$ is codense, $\mathcal{I} \subset \mathcal{N}$ and $\tau^\star(\mathcal{N}) = \tau^\alpha$, by Theorem 4.2, $SO(X, \tau^\alpha) = \mathcal{F}(\tau^\alpha, \mathcal{I})$. Therefore, the proof follows from Theorem 4.2 and Corollary 4.1. □

Corollary 4.3. [12, Proposition 3(a)]. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If $\mathcal{I} = \{\emptyset\}$ or $\mathcal{N}$, then $SO(X, \tau) = \mathcal{F}(\tau, \mathcal{I})$.*

Corollary 4.4. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If $\mathcal{I}$ is completely codense, then $\tau^\alpha = \mathcal{F}(X, \tau) \cap IO(X, \tau)$.*

Proof. We know that $\tau^\alpha = SO(X, \tau) \cap PO(X, \tau)$ [17]. Since $\mathcal{I}$ is completely codense, by Lemma 2.5, $PO(X, \tau) = IO(X, \tau)$ and by Theorem 4.2, $SO(X, \tau) = \mathcal{F}(X, \tau)$. Therefore, the proof follows. □

Definition 4.1. *A subset A of an ideal space $(X,\tau,\mathcal{I})$ is called a regular $\mathcal{I}$-closed set [11] if $A = (\text{int}(A))^\star$. Every regular $\mathcal{I}$-closed set is an $f_{\mathcal{I}}$-set* [12, Proposition 5].

The following Theorems 4.4 and 4.5, give some properties of $f_{\mathcal{I}}$-sets.

Theorem 4.4. *If A is an $f_{\mathcal{I}}$-set of an ideal space $(X,\tau,\mathcal{I})$, then*

(a) A *and* $\text{int}(A)$ *are $\star$-dense in itself.*
(b) $(\text{int}(A))^\star$ *is $\star$-dense in itself.*
(c) $A^\star = (\text{int}(A))^\star = ((\text{int}(A))^\star)^\star$.
(d) $A^\star$ *is $\star$-perfect and $\mathcal{I}$-locally closed.*
(e) $(\text{int}(A))^\star$ *is $\star$-perfect and $\mathcal{I}$-locally closed.*
(f) $A^\star(\widetilde{\mathcal{I}})$ *is $\star$-dense in itself and* $A^\star = \text{cl}(\text{int}(A)) = \text{cl}(\text{int}(A^\star)) = A^\star(\widetilde{\mathcal{I}})$.
(g) $A^\star = A^\star(\widetilde{\mathcal{I}})$ *is regular closed and* $A \subset A^\star(\mathcal{N})$.
(h) $A^\star$ *is regular $\mathcal{I}$-closed and hence an $f_{\mathcal{I}}$-set.*

Proof. (a) A is $\star$-dense in itself by Corollary 1 of [12]. Since $\text{int}(A) \subset A \subset (\text{int}(A))^\star$, $\text{int}(A)$ is $\star$-dense in itself.
(b) Since $A \subset (\text{int}(A))^\star \subset A^\star$, by Lemma 2.4, $(\text{int}(A))^\star$ is $\star$-dense in itself.
(c) Since $A \subset (\text{int}(A))^\star \subset A^\star$, $A^\star \subset ((\text{int}(A))^\star)^\star \subset (\text{int}(A))^\star \subset (A^\star)^\star \subset A^\star$ and so $A^\star = ((\text{int}(A))^\star)^\star = (\text{int}(A))^\star = (A^\star)^\star$.
(d) Since $(A^\star)^\star = A^\star$, $A^\star$ is $\star$-perfect and so is $\mathcal{I}$-locally closed.
(e) By (c), $(\text{int}(A))^\star$ is $\star$-perfect and so $\mathcal{I}$-locally closed.
(f) Since $A^\star = (\text{int}(A))^\star \subset \text{cl}(\text{int}(A)) \subset \text{cl}(\text{int}(A^\star)) = A^\star(\widetilde{\mathcal{I}})$ by [10, Theorem 3.2] and $A^\star(\widetilde{\mathcal{I}}) \subset A^\star$, $A^\star(\widetilde{\mathcal{I}})$ is $\star$-dense in itself and $A^\star = \text{cl}(\text{int}(A)) = \text{cl}(\text{int}(A^\star)) = A^\star(\widetilde{\mathcal{I}})$.
(g) Since $A^\star(\widetilde{\mathcal{I}})$ is regular closed by [18, Theorem 10(b)], by (f), $A^\star$ is regular closed. Since $\mathcal{N} \subset \widetilde{\mathcal{I}}$, $A \subset A^\star(\mathcal{N})$.
(h) Since $A \subset (\text{int}(A))^\star \subset (\text{int}(A^\star))^\star \subset A^\star$, we have $A^\star = (\text{int}(A^\star))^\star$ and so $A^\star$ is regular $\mathcal{I}$-closed and so is an $f_{\mathcal{I}}$-set. □

Theorem 4.5. *Let $(X,\tau,\mathcal{I})$ be an ideal space and A be an $f_{\mathcal{I}}$-subset of X. Then*

(a) $\text{pcl}(A) = \text{cl}(\text{int}(A)) = \text{cl}(A) = A^\star$.
(b) $\text{pint}(A) = \text{int}_\alpha(A)$.
(c) $\text{pint}(\text{pcl}(A)) = \text{int}(\text{pcl}(A)) = \text{int}(A^\star)$.

Proof. (a) $A \in \mathcal{F}(\tau,\mathcal{I}) \Rightarrow A \subset (\text{int}(A))^\star \subset \text{cl}(\text{int}(A)) \Rightarrow A \cup \text{cl}(\text{int}(A)) = \text{cl}(\text{int}(A)) \Rightarrow \text{pcl}(A) = \text{cl}(\text{int}(A))$. By Theorem 4.4(f), $\text{pcl}(A) = A^\star = \text{cl}(A)$.
(b) $\text{pint}(A) = A \cap \text{int}(\text{cl}(A)) = A \cap \text{int}(A^\star)$, since A is $\star$-dense in itself. By (a), $\text{pint}(A) = A \cap \text{int}(\text{cl}(\text{int}(A)))$ and so $\text{pint}(A) = \text{int}_\alpha(A)$.
(c) $\text{pint}(\text{pcl}(A)) = \text{pint}(\text{cl}(A))$, by (a) and so $\text{pint}(\text{pcl}(A)) = \text{cl}(A) \cap \text{int}(\text{cl}(\text{cl}(A))) = \text{int}(\text{cl}(A)) = \text{int}(\text{pcl}(A)) = \text{int}(A^\star)$. □

We recall the following.

Definition 4.2. *A mapping $f\colon (X,\tau,\mathcal{I}) \to (Y,\sigma)$ is said to be $f_{\mathcal{I}}$-continuous* [12] *(resp. semicontinuous* [14]*) if for every $V \in \sigma$, $f^{-1}(V)$ is an $f_{\mathcal{I}}$-set (resp. semiopen set).*

Corollary 4.5. *Let $(X,\tau,\mathcal{I})$ be an ideal space where $\mathcal{I}$ be codense. Then $f\colon (X,\tau,\mathcal{I}) \rightarrow (Y,\sigma)$ is $f_{\mathcal{I}}$-continuous if and only if f is semicontinuous.*

Proof. Follows from Theorem 4.2. □

Corollary 4.6. *Let $(X,\tau,\mathcal{I})$ be an ideal space where $\mathcal{I}$ be completely codense. Then the following are equivalent.*

(a) $f\colon (X,\tau,\mathcal{I}) \rightarrow (Y,\sigma)$ *is $f_{\mathcal{I}}$-continuous.*
(b) $f\colon (X,\tau^{\star},\mathcal{I}) \rightarrow (Y,\sigma)$ *is $f_{\mathcal{I}}$-continuous.*
(c) $f\colon (X,\tau^{\alpha},\mathcal{I}) \rightarrow (Y,\sigma)$ *is $f_{\mathcal{I}}$-continuous.*
(d) $f\colon (X,\tau,\mathcal{I}) \rightarrow (Y,\sigma)$ *is semicontinuous.*
(e) $f\colon (X,\tau^{\star},\mathcal{I}) \rightarrow (Y,\sigma)$ *is semicontinuous.*
(f) $f\colon (X,\tau^{\alpha},\mathcal{I}) \rightarrow (Y,\sigma)$ *is semicontinuous.*

Proof. Proof follows from Theorem 4.3. □

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is called an *$\alpha - \mathcal{I}$-open set* [7] if $A \subset \mathrm{int}(\mathrm{cl}^{\star}(\mathrm{int}(A)))$. $\alpha - \mathcal{I}$-openness and $\mathcal{I}$-openness are independent concepts [7, Remark 2.1]. In [18, Corollary 1(iii)], it was established that every $\star$-dense in itself, $\alpha - \mathcal{I}$-open subset is $\mathcal{I}$-open. The following Theorem 4.6, shows that the two kinds of sets are equivalent for the collection of $f_{\mathcal{I}}$-sets.

Theorem 4.6. *Let $(X,\tau,\mathcal{I})$ be an ideal space and A be an $f_{\mathcal{I}}$-subset of X. Then A is $\alpha - \mathcal{I}$-open if and only if A is $\mathcal{I}$-open.*

Proof. Suppose A is $\alpha - \mathcal{I}$-open. Since A is an $f_{\mathcal{I}}$-subset, A is $\star$-dense in itself and so by [18, Corollary 1(iii)], A is $\mathcal{I}$-open. Conversely, suppose A is $\mathcal{I}$-open. Then $A \subset \mathrm{int}(A^{\star}) = \mathrm{int}((\mathrm{int}(A))^{\star})$ by Theorem 4.4(c), and so $A \subset \mathrm{int}(\mathrm{cl}^{\star}(\mathrm{int}(A)))$. Therefore, A is $\alpha - \mathcal{I}$-open. □

We end this section with the following characterization of $f_{\mathcal{I}}$-sets in terms of open sets.

Theorem 4.7. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then A is an $f_{\mathcal{I}}$-subset of X if and only if there exists an open set G such that $G \subset A \subset G^{\star}$.*

Proof. Suppose A is an $f_{\mathcal{I}}$-subset of X. Let G $= \mathrm{int}(A)$. Then G is the required open set such that $G \subset A \subset G^{\star}$. Conversely, suppose there is an open set G such that $G \subset A \subset G^{\star}$. Now $G \subset A \Rightarrow G \subset \mathrm{int}(A) \Rightarrow G^{\star} \subset (\mathrm{int}(A))^{\star} \Rightarrow A \subset (\mathrm{int}(A))^{\star}$ and so A is an $f_{\mathcal{I}}$-subset. □

Corollary 4.7. *If A is an $f_{\mathcal{I}}$-subset of an ideal space $(X,\tau,\mathcal{I})$, then there exists an open set $G \subset A$ such that $A^{\star} = G^{\star}$.*

5. Properties of regular $\mathcal{I}$-closed sets

We will denote the family of all regular $\mathcal{I}$-closed sets in $(X,\tau,\mathcal{I})$ by $\mathcal{R}(\tau,\mathcal{I})$. If the ideal $\mathcal{I}$ is not codense, then X is regular closed in $(X,\tau,\mathcal{I})$ but not regular $\mathcal{I}$-closed and so regular closed sets need not be regular $\mathcal{I}$-closed. But every regular $\mathcal{I}$-closed set is a regular closed set by Theorem 5.4(e) below. The easy proof of the following

Theorems 5.1 and 5.2. are omitted. Theorem 5.2 below gives a characterization of codense ideals.

Theorem 5.1. *If $(X,\tau,\mathcal{I})$ is an ideal space, then $\mathcal{R}(\tau,\mathcal{I})\cap\mathcal{I}=\{\emptyset\}$.*

Theorem 5.2. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then $\mathcal{I}$ is codense if and only if X is regular $\mathcal{I}$-closed.*

Theorem 5.3. *If $(X,\tau,\mathcal{I})$ is an ideal space where $\mathcal{I}$ is codense, then $\mathcal{R}(\tau,\mathcal{I}) = RC(X,\tau)$.*

Proof. $A\in\mathcal{R}(\tau,\mathcal{I}) \Leftrightarrow A=(\mathrm{int}(A))^\star \Leftrightarrow A=\mathrm{cl}(\mathrm{int}(A))$, since $\mathcal{I}$ is codense $\Leftrightarrow A\in RC(X,\tau)$. □

Corollary 5.1. *If $(X,\tau,\mathcal{I})$ is an ideal space where $\mathcal{I}$ is codense, then the following are equivalent.*

(a) $A\in RC(X,\tau)$.
(b) $A\in\mathcal{R}(\tau,\mathcal{I})$.
(c) $A\in\mathcal{F}(\tau,\mathcal{I})$ *and* A *is* $\tau^\star$*-closed.*
(d) $A\in SO(X,\tau)$ *and* A *is* $\tau^\star$*-closed.*

Proof. Proof follows from Theorem 5.3, [12, Proposition 5] and Theorem 4.2. □

The following Theorem 5.4 gives some properties of regular $\mathcal{I}$-closed sets. Also, it is established that every regular $\mathcal{I}$-closed set is $\mathcal{I}$-locally closed.

Theorem 5.4. *If A is a regular $\mathcal{I}$-closed set of an ideal space $(X,\tau,\mathcal{I})$, then*

(a) A *and* $\mathrm{int}(A)$ *are* $\star$*-dense in itself.*
(b) $A^\star=(\mathrm{int}(A))^\star=((\mathrm{int}(A))^\star)^\star=A$.
(c) A *is* $\star$*-perfect and* $\mathcal{I}$*-locally closed.*
(d) $(\mathrm{int}(A))^\star$ *is* $\star$*-perfect and* $\mathcal{I}$*-locally closed.*
(e) $A=\mathrm{cl}(\mathrm{int}(A))=A^\star(\widetilde{\mathcal{I}})$ *and so* A *is regular closed.*

Proof. (a) Since $\mathrm{int}(A)\subset A=(\mathrm{int}(A))^\star\subset A^\star$, $\mathrm{int}(A)$ and A are $\star$-dense in itself.
(b) Since $A=(\mathrm{int}(A))^\star\subset A^\star$, $A^\star=((\mathrm{int}(A))^\star)^\star\subset(\mathrm{int}(A))^\star=A\subset A^\star$ and so $A^\star=((\mathrm{int}(A))^\star)^\star=(\mathrm{int}(A))^\star=A$.
(c) Since $A=A^\star$, A is $\star$-perfect and so is $\mathcal{I}$-locally closed.
(d) By (b), $(\mathrm{int}(A))^\star$ is $\star$-perfect and so $\mathcal{I}$-locally closed.
(e) Since $A=(\mathrm{int}(A))^\star\subset\mathrm{cl}(\mathrm{int}(A))=\mathrm{cl}(\mathrm{int}(A^\star))=A^\star(\widetilde{\mathcal{I}})$ by Theorem 3.2 of [10] and $A^\star(\widetilde{\mathcal{I}})\subset A^\star=A$, $A=\mathrm{cl}(\mathrm{int}(A))=A^\star(\widetilde{\mathcal{I}})$ and so A is regular closed. □

We end this section with the following characterization of regular $\mathcal{I}$-closed sets in terms of open sets.

Theorem 5.5. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then A is a regular $\mathcal{I}$-closed subset of X if and only if there exists an open set G such that $G\subset A=G^\star$.*

Proof. Suppose A is a regular $\mathcal{I}$-closed subset of X. Let G = $\mathrm{int}(A)$. Then G is the required open set such that $G\subset A=G^\star$. Conversely, suppose that there is an open set G such that $G\subset A=G^\star$. Now $G\subset A\Rightarrow G\subset\mathrm{int}(A)\Rightarrow G^\star\subset(\mathrm{int}(A))^\star\Rightarrow A\subset(\mathrm{int}(A))^\star$ and $(\mathrm{int}(A))^\star\subset A^\star=G^\star=A$. Therefore, A is regular $\mathcal{I}$-closed. □

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