BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **29***(1) (2006), 101–109*

Generalized Solutions of Volterra Integral Equations of the First Kind

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Abstract. Volterra integral equations of the first kind are studied in terms of generalized functions. The solutions consist of singular and regular components which can be constructed separately. The singular component is constructed as solution of the special linear algebraic system. The regular component is constructed as solution of the special Volterra integral equation of the third kind.

2000 Mathematics Subject Classification: 45D05, 46F99

Key words and phrases: Distribution space, Volterra integral equations, Dirac function, impulses theory.

1. Introduction

Various important problems in electrical engineering [3], in modeling of dynamic impulse systems [11], and in nonlinear dynamic systems identification [6], [2] can be treated in terms of the Volterra integral equations of the first kind which does not have classical continuous solutions. In some cases solutions of the algebraic-differential equations and differential-operator equations with irreversible operator in main part can be also represented as generalized solution of the Volterra integral equations of the first kind. The reader may see chapter 5 in the monograph [8] for details.

In many instances solutions in classes of generalized functions have strict physical interpretations, e.g. see the monographs [10], [3]. Consequently, the problems of existence, deriving and numeric computation of generalized solutions of the Volterra integral equations of the first kind are of interest to both mathematical and engineering communities.

In this paper we concentrate on the structure of the generalized solutions to outline the main steps of the algorithm.

Received: May 16, 2005; *Accepted:* November 28, 2005.

Let us consider the Volterra integral equation of the first kind

$$\int_0^t K(t,s)x(s)ds = f(t), t \geq 0, \tag{1.1}$$

where $K(t,s)$ and $f(t)$ are infinitely differentiable functions. If $f(0) \neq 0$, then equation (1.1) does not have classic solutions. In order to fulfill such requirements we will look for the solution in the distribution space [10]. Distribution space provides existence of solution and follows the physical sense of the problem [10]. For example, the special combination of Dirac functions with deviating arguments can be exploited as test signals for identification of nonlinear dynamical systems [6], [2], [1]. In this case it is useful to construct generalized solutions of the Volterra equations [1]. Generalized solution is the basis of mathematical models formulated in terms of impulses theory [11]. Various well-known electrical engineering problems [3] can be formulated in terms of impulses theory.

Generalized solutions of the Volterra integral equations of the first kind were studied in papers [5], [9], [7]. In paper [4] and in monograph [8] the generalized solutions of the singular differential-operator equations are considered. In these cases such equations are reducible to the Volterra integral equations of the first kind.

We continue these studies in this paper and generalize our results [9], [7].

2. Problem statement

For any function $K(\tau,t) \in C^\infty(R^2)$ and for any generalized function $x(t) \in D'_+$ [10] we define new generalized function $[\Theta(t) * K(\tau,t)x(t)]_{\tau=t}$. This function operates on the base of functions $\phi(t) \in D(R^1)$ and follows the rule:

$$\Big((\Theta(t) * K(\tau,t)x(t))|_{\tau=t}, \phi(t)\Big) = \Big(x(t), \int_t^\infty K(\tau,t)\phi(\tau)d\tau\Big), \tag{2.1}$$

where

$$\Theta(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Function

$$\psi(t) = \int_t^\infty K(\tau,t)\phi(\tau)d\tau$$

does not belong to the class $D(R^1)$ due to $\operatorname{supp}\psi(t) = (-\infty, \tau_1]$, where $\tau_1 = \sup(\operatorname{supp}\phi(\tau))$. But equality (2.1) is correct because we suppose $\operatorname{supp} x(t) \subset [0, +\infty)$. Hence set $\operatorname{supp} x(t) \cap \operatorname{supp}\psi(t)$ is bounded.

In that case function $\psi(t)$ can be replaced with the finite function

$$\psi_1(t) = \int_t^\infty K_1(\tau,t)\phi(\tau)d\tau,$$

where $K_1(\tau,t) \in D(R^2)$, $K_1(\tau,t) = K(\tau,t)$ on the set $\{(\tau,t)|t,\tau \in [0,\tau_1]\}$. Then on this set $\psi_1(t) = \psi(t)$ and the value of the function $(x(t), \psi_1(t))$ is defined. This value does not depend on selection of the function $K_1(\tau,t)$ outside the stated set. Functional $(\Theta(t) * K(\tau,t)x(t))|_{\tau=t}$ belongs to D'_+ since the linearity follows from the properties of linearity of integral and functional $x(t) \in D'_+$. Let us proof the continuity. If $\phi_k(\tau) \to 0$ in $D(R^1)$, then $\exists R > 0 : \operatorname{supp}\phi_k(\tau) \subset [-R,R]$ for $\forall k \in N$. Let $K_1(\tau,t) \in D(R^2)$, $K_1(\tau,t) = K(\tau,t)$ on the set $\{(\tau,t)|t,\tau \in [0,R]\}$. Then the sequence

$$\psi_1^k(t) = \int_t^\infty K_1(\tau,t)\phi_k(\tau)d\tau \to 0 \text{ in } D(R^1).$$

From this follows $(x(t), \psi_1^k(t)) \to 0$. If $\operatorname{supp}\phi(\tau) \subset (-\infty,0)$, then $\operatorname{supp}\psi(t) \subset (-\infty,0)$ and $\operatorname{supp}x(t) \cap \operatorname{supp}\psi(t) = \emptyset$. Hence

$$\operatorname{supp}\big((\Theta(t) * K(\tau,t)x(t)|_{\tau=t}\big) \subset [0,+\infty).$$

Remark 2.1. If $x(t) \in D'_+$ is regular generalized function, i.e. $x(t) = u(t)\Theta(t)$, where $u(t)$ is locally integrable then

$$\begin{aligned}\big((\Theta(t) * K(\tau,t)x(t))\big|_{\tau=t}, \phi(t)\big) &= \int_0^\infty u(t)\int_t^\infty K(\tau,t)\phi(\tau)d\tau dt \\ &= \int_0^\infty\Big(\int_0^\tau K(\tau,t)u(t)dt\Big)\phi(\tau)d\tau \\ &= \Big(\Big(\int_0^t K(t,s)u(s)ds\Big)\Theta(t), \phi(t)\Big),\end{aligned}$$

i.e.

$$\big(\Theta(t) * K(\tau,t)x(t)\big)\big|_{\tau=t} = \int_0^t K(t,s)u(s)ds\Theta(t).$$

Remark 2.2. If $x(t) = \delta^{(m)}(t)$ then

$$\big(\Theta(t) * K(\tau,t)\delta^{(m)}(t)\big)\big|_{\tau=t} = (-1)^m\frac{\partial^m K(t,0)}{\partial s^m}\Theta(t) +$$

$$+\sum_{i=0}^{m-1}(-1)^i\sum_{l=0}^{i} C_{m-1-l}^{i-l}C_m^l\frac{\partial^i K(0,0)}{\partial t^{i-l}\partial s^l}\delta^{(m-1-i)}(t). \tag{2.2}$$

In fact

$$\Big(\big(\Theta(t) * K(\tau,t)\delta^{(m)}(t)\big)\big|_{\tau=t}, \phi(t)\Big) \stackrel{\text{def}}{=} (-1)^m\Big(\frac{d^m}{dt^m}\int_t^{+\infty} K(\tau,t)\phi(\tau)d\tau\Big)\Big|_{t=0}$$

$$= (-1)^m \Big(\int\limits_t^{+\infty} \frac{\partial^m}{\partial t^m} K(\tau,t)\cdot\phi(\tau)d\tau - \sum_{j=1}^{m} \frac{d^{m-j}}{dt^{m-j}} \Big(\Big(\frac{\partial^{j-1}K(\tau,t)}{\partial t^{j-1}}\phi(\tau)\Big)\Big|_{\tau=t}\Big)\Big)\Big|_{t=0}$$

$$= (-1)^m \int\limits_0^{+\infty} K_{0m}(\tau,0)\phi(\tau)d\tau +$$

$$+(-1)^{m+1}\sum_{i=0}^{m-1}\sum_{j=1}^{i+1} C_{m-j}^{m-1-i}\Big(\frac{d^{i+1-j}}{dt^{i+1-j}}K_{0\,j-1}(t,t)\Big)\Big|_{t=0}\cdot\phi^{(m-1-i)}(0)$$

$$= (-1)^m \int\limits_0^{+\infty} K_{0m}(\tau,0)\phi(\tau)d\tau +$$

$$+(-1)^{m+1}\sum_{i=0}^{m-1}\sum_{j=0}^{i} C_{m-j-1}^{m-1-i}\Big(\frac{d^{i-j}}{dt^{i-j}}K_{0\,j}(t,t)\Big)\Big|_{t=0}\cdot(-1)^{m-1-i}\big(\delta^{(m-1-i)}(t),\phi(t)\big),$$

but

$$\sum_{j=0}^{i} C_{m-j-1}^{m-1-i}\Big(\frac{d^{i-j}}{dt^{i-j}}K_{0\,j}(t,t)\Big)\Big|_{t=0} = \sum_{l=0}^{i} C_{m-1-l}^{i-l}C_m^l K_{i-l\,l}(0;0).$$

Now let us come back to equation (1.1). Let $x(t) \in C_{[0,t_0]}$ is the solution of equation (1.1). If function $x(t)$ is continued by zero for $t < 0$, then it is the generalized solution of equation

$$\big(\Theta(t) * K(\tau,t)x(t)\Theta(t)\big)|_{\tau=t} = f(t)\Theta(t).$$

Definition (2.1) is used here.

We call the problem of construction of the solution $x(t) \in D'_+$ of the equation

$$(\Theta(t) * K(\tau,t)x(t))|_{\tau=t} = f(t)\Theta(t), \tag{2.3}$$

as problem of solvability of initial equation (1.1) in the class D'_+.

3. Generalized solutions construction

Now let us introduce the basic condition to be used below

(A) $K_{t^i}^{(i)}(t,s)|_{s=t} = 0,\ i = 0,1,...,n-1,$

$$K_{t^n}^{(n)}(t,s)|_{s=t} \sim\ at^m,\ a \neq 0 \text{ for } t \to 0, m \geq 0.$$

Taylor formula gives us $K(t,s) = (t-s)^n Q(t,s)$, $K_{t^n}^{(n)}(t,s)|_{s=t} = n!Q(t,t)$, where

$$Q(t,s) = Q_1(t,s) + \sum_{i+k\geq m} a_{ik}t^i s^k,$$

$$Q_1(t,t) = 0,\ \sum_{i+k=m} a_{ik} = a.$$

If $K(t,s)$ is not a convolution, then we can consider the most interesting case $m \geq 1$. In this case the conditions of existence and uniqueness of the generalized solutions of equation (1.1) are not well studied. We follow the paper [9] and look for the solution as the following series

$$x = c_0\delta(t) + ... + c_n\delta^{(n)}(t) + u(t)\Theta(t), \tag{3.1}$$

where $\delta(t)$ is the Dirac function and $u(t)$ is regular function.

Due to formula (2.2) and condition **(A)** for $j \leq n$ the following equalities are correct

$$(\Theta(t) * K(\tau, t)\delta^{(j)}(t))|_{\tau=t} = (-1)^j K^{(j)}_{s^j}(t, 0)\Theta(t).$$

From these equalities follows that regular item $u(t)$ should satisfy the equation

$$\int_0^t K(t, s)u(s)ds = F(t, c), \tag{3.2}$$

where

$$F(t, c) = f(t) - \sum_{j=0}^{n} (-1)^j K^{(j)}_{s^j}(t, 0)c_j. \tag{3.3}$$

Let vector $c = (c_0, ..., c_n)'$ satisfies the equalities

$$\sum_{j=0}^{n} (-1)^j \frac{\partial^{i+j} K(0, 0)}{\partial t^i \partial s^j} c_j = f^{(i)}(0), \;\; i = 0, 1, ..., n. \tag{3.4}$$

Then equation (3.2) is an equivalent of the Volterra integral equation of the third kind:

$$n!Q(t, t)u(t) + \int_0^t \frac{\partial^{n+1} K(t, s)}{\partial t^{n+1}} u(s)ds = F^{(n+1)}(t, c), \tag{3.5}$$

where $Q(t, t) \sim \;\; at^m$. For existence of the regular solution $u(t)$ of equation (3.5) due to condition **(A)** it is necessary the equality $F^{(n+i)}_{t^{n+1}}(0, c) = 0, i = 1, ..., m$ to be hold.

That is the reason why the sought vector c should satisfy the following system

$$\sum_{j=0}^{n} (-1)^j \frac{\partial^{i+j} K(0, 0)}{\partial t^i \partial s^j} c_j = f^{(i)}(0), \;\; i = 0, 1, ..., n + m. \tag{3.6}$$

If system (3.6) is not solvable, then equation (1.1) does not have the generalized solutions (3.1) with singularity order n.

Lemma 3.1. *Let the following conditions be fulfilled*

$$\frac{\partial^{n+m} K(0, 0)}{\partial t^i \partial s^{m+n-i}} \neq 0, \; i = m, m + 1, ..., m + n,$$

$Q_1(t, s) = 0$, *condition* **(A)** *and* $f^{(i)}(0) = 0$, $i = 0, 1, ..., m - 1$. *Then system (3.6) has unique solution.*

To prove this lemma it is enough to note that system (3.6) in the conditions of this lemma is following

$$\sum_{j=0}^{n}(-1)^j \frac{\partial^{m+k+j}K(0,0)}{\partial t^{m+k}\partial s^j}c_j = f^{(m+k)}(0),\ k = 0,1,...,n. \tag{3.7}$$

In this system the matrix is low triangular and

$$|\det\triangle| = \prod_{i=0}^{n}\left|\frac{\partial^{m+n}K(0,0)}{\partial t^{m+i}\partial s^{n-i}}\right| \neq 0.$$

Remark 3.1. Lemma 3.1 is still correct if $Q(t,s) = \sum_{i,k} a_{ik}t^i s^k$ where $a_{ik} = 0$ for $i + k \leq m-1,\ \ i \leq m-1,\ \ k \leq n.$

Let vector c satisfies system (3.6). Then $\forall t$ we have

$$\sum_{i=0}^{m+n}\left(\sum_{j=0}^{n}\frac{\partial^{i+j}K(0,0)}{\partial t^i\partial s^j}c_j - f^{(i)}(0)\right)\frac{t^i}{i!} = 0$$

and we can rewrite the right hand side of equation (3.2) as follows

$$F(t,c) = f(t) - \sum_{i=0}^{n+m} f^{(i)}(0)\frac{t^i}{i!} - \sum_{j=0}^{n}(-1)^j\left(\frac{\partial^j K(t,0)}{\partial s^j} - \sum_{i=0}^{n+m}\frac{\partial^{i+j}K(0,0)}{\partial t^i\partial s^j}\frac{t^i}{i!}\right)c_j.$$

Due to the foregoing formulae on the base of Taylor formula we have $F(t,c) = O(t^{n+m+1})$. Finally

$$\lim_{t\to 0}\frac{F^{(n+1)}(t,c)}{Q(t,t)} = 0, \tag{3.8}$$

if $Q_1(t,s) = 0$.

In addition, due to condition **(A)** in the area $0 < s \leq t \leq t_0$ we can guarantee the following estimate:

$$\frac{\frac{\partial^{n+1}K(t,s)}{\partial t^{n+1}}}{Q(t,t)} = O\left(\frac{1}{t}\right). \tag{3.9}$$

Integral equation (3.5) has the regular singularity in zero due to estimate (3.9).

Further, let the homogeneous equation which corresponds to equation (1.1) has only zero solution. The formal solution of integral equation (3.5) can be can constructed in the form of series

$$u(t) \sim \sum_{1}^{\infty} u_i t^i \tag{3.10}$$

by the method of unknown coefficients.

Remark 3.2. Due to condition **(A)** for $Q_1(t,s) = 0$ we have

$$\frac{\partial^{n+1}K(t,s)}{\partial t^{n+1}} = \sum_{i+k=m-1} b_{ik}t^i s^k + O((t+s)^m).$$

Then uniqueness of the solution of the homogeneous equation is equivalent of the condition

$$a + \sum_{i+k=m-1} b_{ik} \frac{1}{k+l} \neq 0$$

for $l = 0, 1, ...$

Theorem 3.1. *Let the homogeneous equation which correspond to equation (1.1) has only zero formal solution (3.10) and the conditions of Lemma 3.1 are fulfilled. Then equation (1.1) has the unique solution (3.1) in the class D'_+.*

Proof. We define the vector c in expansion (3.1) from system (3.7) by substituting it in the right hand side of equation (3.5). We can find N first coefficients u_i of formal solution (3.10).

Let $u = \sum_{i=1}^{N} u_i t^i + v(t)$ in equation (3.5). Then to define $v(t)$ we get the integral equation

$$v(t) + \int_0^t L(t,s) v(s) ds = b(t),$$

where

$$L(t,s) = \frac{K^{(n+1)}_{t^{n+1}}(t,s)}{n! Q(t,t)},$$

$$b(t) = \frac{F^{(n+1)}(t,c)}{n! Q(t,t)} - \sum_{i=0}^{N} \Big(u_i t^i + \int_0^t L(t,s) u_i s^i ds \Big).$$

Taking into account (3.9) we can note that $b(t) = O(t^{N+1})$. Kernel $\frac{c}{t}$, $c > 0$ has the resolvent $\frac{c}{t}\left(\frac{t}{s}\right)^c$. Because of estimate (3.9) the kernel $L(t,s)$ in the area $0 < s \leq t \leq t_0$ for small enough t_0 also has resolvent $R(t,s)$ with similar estimate. But in this case for a big enough N the integral $\int_0^t R(t,s) b(s) ds$ is converging and we can define function $v(t)$ by known formula

$$v(t) = b(t) + \int_0^t R(t,s) b(s) ds.$$

□

Remark 3.3. If the conditions of Theorem 3.1 are fulfilled and $f^{(i)}(0) = 0$, $i = m, ..., m+n$, then $c_0 = ... = c_n = 0$ and solution (3.1) is classical.

Now we consider the generalized solutions of equation (1.1) for $\sum_{i=0}^{m+n} |f^{(i)}(0)| \neq 0$. We will proof that in this case the generalized solutions with the highest singularity order can exist.

We use the following condition below

$$\text{(B)} \qquad \frac{\partial^{i+j} K(0,0)}{\partial t^i \partial s^j} = \begin{cases} \text{null}, & 0 \leq i+j \leq n+m-1 \\ \text{non-null}, & i+j = n+m \end{cases}$$

If condition **(A)** is fulfilled then

$$Q(t,s) = \sum_{i+k\geq m} a_{ik} t^i s^k.$$

We look for the solution of equation (1.1) as follows

$$x(t) = c_0\delta(t) + ... + c_{n+m}\delta^{(n+m)}(t) + u(t)\Theta(t). \tag{3.11}$$

The vector $(c_0, ..., c_{n+m})'$ can be defined from the system

$$\Xi c = \beta, \tag{3.12}$$

where

$$\Xi = \left\| (-1)^j \frac{\partial^{i+j} K(0,0)}{\partial t^i \partial s^j} \right\|_{i,j=0,\overline{n+m}},$$

$$\beta = (f(0), f'(0), ..., f^{(n+m)}(0))'.$$

Due to condition **(B)** matrix of system (3.6) is lower triangular and nonsingular. To define the regular component $u(t)$ we again have equation (3.2) where $F(t,c) = f(t) - \sum\limits_{j=0}^{n+m} (-1)^j K_{s^j}^{(j)}(t,0) c_j$.

The solution c of system (3.12) for $\forall t$ obviously satisfies the equality

$$\sum_{j=0}^{n+m} (-1)^j \sum_{i=n+m-j}^{n+m} \frac{\partial^{i+j} K(0,0)}{\partial t^i \partial s^j} \frac{t^i}{i!} c_j = \sum_{i=0}^{n+m} f^{(i)}(0) \frac{t^i}{i!}.$$

From condition **(A)**, $\frac{F^{(n+1)}(t,c)}{Q(t,t)} = O(t)$ and from the aforesaid proof of Theorem 3.1 we have the following theorem.

Theorem 3.2. *Let the homogeneous equation which correspond to equation (1.1) has only zero solution and conditions* **(A)** *and* **(B)** *are fulfilled. Then for* $\forall f(t) \in C^\infty[0,t_0)$ *equation (1.1) has the unique generalized solution (3.11).*

Remark 3.4. If the conditions of the Theorem 3.2 are fulfilled and $f(0) = f'(0) = ... = f^{(m-1)}(0) = 0$ then $c_{n+1} = ... = c_{n+m} = 0$ and the result of the Theorem 3.1 is obtained.

4. Remark on the families of parametric generalized solutions

Let the conditions of Theorem 3.2 are fulfilled and some of $[n+m]$th derivatives of the kernel $K(t,s)$ in the point (0,0) are zeros. Then matrix Ξ in system (3.12) is degenerate. If in this case system (3.12) remains solvable then equation (1.1) has $n+m+1-r$- parametric family of generalized solutions (3.11) where $r = \mathrm{rank}\Xi$. If in this case we allow homogeneous equation of (1.1) has d nontrivial solutions for $d \leq n+m+1-r$, then d arbitrary parameters in vector c can be defined by the construction of formal series (3.10). But in this case the coefficients u_i of the formal series (3.10) remains arbitrary and we again get $n+m+1-r$ parametric family of generalized solutions (3.11).

As a footnote, let us note that if system (3.11) is not solvable then there are no generalized solutions of equation (1.1).

5. Conclusion

Novel approach to the construction of the generalized solutions of the Volterra integral equations of the first kind is proposed. This approach along with the methods presented in paper [4] and in the monograph [8] (chapter 6) provide basement for construction of the theory of generalized solutions of the conventional Volterra integral equations of the first kind in the Banach spaces.

The generalized solutions of the Volterra equations of the first kind can be constructed based on two-stage analytical-numerical scheme. The solutions have singular and regular components, which are supposed to be constructed separately. On the first stage the singular component can be constructed as solution of the special linear algebraic system. On the second stage the regular component can be constructed as solution of the special Volterra integral equation of the third kind.

Our current work involves regularized numeric methods [1] incorporation into the described method for nonlinear dynamic systems identification [6, 2].

Acknowledgments. This work is partly funded by NATO Programme for Security Through Science. It is carried out within the project "Computational and Visual Methods of Identification in Uncertain Environment". D.N.S. is supported by NATO Grant No RIG981276 and by RFBR Grant No 05-01-00336. Authors are grateful to referees for useful comments and suggestions.

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