

Intuitionistic Hyperideals of Semihypergroups

B. DAVVAZ

Department of Mathematics, Yazd University, Yazd, Iran
davvaz@yazduni.ac.ir

Abstract. The notion of intuitionistic fuzzy sets was introduced by Atanassov as a generalization of the notion of fuzzy sets. In this paper, using Atanassov idea, we establish the intuitionistic fuzzification of the concept of hyperideals in a semihypergroup and investigate some of their properties.

2000 Mathematics Subject Classification: 20N20

Key words and phrases: Semihypergroup, hyperideal, intuitionistic fuzzy set, intuitionistic fuzzy hyperideal.

1. Introduction

After the introduction of fuzzy sets by Zadeh, several researchers were conducted on the generalization of fuzzy sets. As an important generalization of the notion of fuzzy sets on a non-empty set X , Atanassov introduced in [1] the concept of intuitionistic fuzzy sets defined on a non-empty set X as objects having the form

$$A = \{ \langle x, \mu_A(x), \lambda_A(x) \rangle \mid x \in X \},$$

where the functions $\mu_A : X \rightarrow [0, 1]$ and $\lambda_A : X \rightarrow [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\lambda_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$.

Such defined objects are studied by many authors (see for example the papers in “Fuzzy Sets and Systems” and “Notes on Intuitionistic Fuzzy Sets”) and have many interesting applications not only in mathematics (see [Chapter 5, 3]).

Let A and B be two intuitionistic fuzzy sets on X . The following expressions are defined in [1, 2].

- 1) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$,
- 2) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$,
- 3) $A^c = \{ \langle x, \lambda_A(x), \mu_A(x) \rangle \mid x \in X \}$,
- 4) $A \cap B = \{ \langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\} \rangle \mid x \in X \}$,
- 5) $A \cup B = \{ \langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\} \rangle \mid x \in X \}$,
- 6) $\square A = \{ \langle x, \mu_A(x), 1 - \mu_A(x) \rangle \mid x \in X \}$,
- 7) $\diamond A = \{ \langle x, 1 - \lambda_A(x), \lambda_A(x) \rangle \mid x \in X \}$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \lambda_A)$ for intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \lambda_A(x) \rangle \mid x \in X\}$.

2. Semihypergroups

A hypergroupoid is a non-empty set H together with a map $*$ from $H \times H$ into the family of non-empty subsets of H . The image of the pair (x, y) is denoted by $x * y$. If $x \in H$ and A, B be subsets of H , then by $A * B$, $A * x$ and $x * B$ we mean

$$A * B = \bigcup_{a \in A, b \in B} a * b, \quad A * x = A * \{x\}, \quad x * B = \{x\} * B.$$

Definition 2.1. A hypergroupoid $(H, *)$ is called a semihypergroup if

$$(x * y) * z = x * (y * z) \quad \text{for all } x, y, z \in H.$$

The motivating example is the following: Let S be a semigroup and K be any subsemigroup of S . Then $S/K = \{x * K \mid x \in S\}$ becomes a semihypergroup where the hyperoperation is defined in a usual manner $\bar{x} \odot \bar{y} = \{\bar{z} \mid z \in \bar{x} * \bar{y}\}$ where $\bar{x} = x * K$.

Definition 2.2. Let H be a semihypergroup. A non-empty subset I of H is called a left (resp. right) hyperideal if $H * I \subseteq I$ (resp. $I * H \subseteq I$).

A non-empty subset I of H is called a hyperideal (or two-sided hyperideal) if it is both a left hyperideal and right hyperideal.

In [5], Davvaz defined the concept of fuzzy hyperideal of a semihypergroup as follows:

Definition 2.3. (cf. Davvaz [5]). Let H be a semihypergroup and μ be a fuzzy subset of H . Then μ is called

i) a left fuzzy hyperideal of H if

$$\mu(y) \leq \inf_{z \in x * y} \{\mu(z)\} \quad \text{for all } x, y \in H;$$

ii) a right fuzzy hyperideal of H if

$$\mu(x) \leq \inf_{z \in x * y} \{\mu(z)\} \quad \text{for all } x, y \in H;$$

iii) a fuzzy hyperideal (or fuzzy two-sided hyperideal) if it is both a left fuzzy hyperideal and right fuzzy hyperideal.

A recent book [4] contains a wealth of applications of fuzzy sets to the algebraic hyperstructures.

3. Intuitionistic fuzzy hyperideals

Now, we establish the intuitionistic fuzzification of the concept of hyperideals in a semihypergroup and investigate some of their properties.

Definition 3.1. Let H be a semihypergroup. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in H is called a left (resp. right) intuitionistic fuzzy hyperideal of H if

$$1) \mu_A(y) \leq \inf_{z \in x * y} \{\mu_A(z)\} \quad (\text{resp. } \mu_A(x) \leq \inf_{z \in x * y} \{\mu_A(z)\}) \quad \text{for all } x, y \in H,$$

$$2) \sup_{z \in x*y} \{\lambda_A(z)\} \leq \lambda_A(y) \text{ (resp. } \sup_{z \in x*y} \{\lambda_A(z)\} \leq \lambda_A(x) \text{ for all } x, y \in H.$$

Example 3.1. Let $H = \{a, b, c, d\}$ be a semihypergroup with the following Cayley table (Table 1):

Table 1. Cayley table

*	a	b	c	d
a	a	b	c	d
b	b	{a, c}	{b, c}	d
c	c	{b, c}	{a, b}	d
d	d	d	d	H

It is not difficult to see that a fuzzy set μ of H is a hyperideal if and only if $\mu(x) = \mu(y)$ for all $x, y \in H$. Define an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in H by

$$\mu_A(x) = t, \lambda_A(x) = 1 - t \text{ for all } x \in H,$$

where $0 \leq t \leq 1$. By routine calculation, we can check that $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy hyperideal of H .

For all the results formulated in this paper, we only describe proof for the left hyperideals. For right hyperideals similar results hold as well.

Proposition 3.1. *If $\{A_i\}_{i \in \Lambda}$ is a family of left (resp. right) intuitionistic fuzzy hyperideals of a semihypergroup H , then $\bigcap_{i \in \Lambda} A_i$ is a left (resp. right) intuitionistic fuzzy hyperideal of H .*

Proof. Suppose $B = \bigcap_{i \in \Lambda} A_i$ and $x, y \in H$. Then we have

$$\begin{aligned} \mu_B(y) &= \inf_{i \in \Lambda} \{\mu_{A_i}(y)\} \leq \inf_{i \in \Lambda} \{ \inf_{z \in x*y} \{\mu_{A_i}(z)\} \} \\ &= \inf_{z \in x*y} \{ \inf_{i \in \Lambda} \{\mu_{A_i}(z)\} \} = \inf_{z \in x*y} \{\mu_B(z)\}. \end{aligned}$$

Also, we have

$$\begin{aligned} \sup_{z \in x*y} \{\lambda_B(z)\} &= \sup_{z \in x*y} \{ \sup_{i \in \Lambda} \{\lambda_{A_i}(z)\} \} = \sup_{i \in \Lambda} \{ \sup_{z \in x*y} \{\lambda_{A_i}(z)\} \} \\ &\leq \sup_{i \in \Lambda} \{\lambda_{A_i}(y)\} = \lambda_B(y). \end{aligned}$$

□

Proposition 3.2. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ is a left (resp. right) intuitionistic fuzzy hyperideal of H if and only if the fuzzy sets μ_A and λ_A^c are left (resp. right) fuzzy hyperideals.*

Proof. Assume that $A = (\mu_A, \lambda_A)$ is a left intuitionistic fuzzy hyperideal of H . Clearly μ_A is a left fuzzy hyperideal of H . For $x, y \in H$ we have

$$\begin{aligned} \inf_{z \in x*y} \{\lambda_A^c(z)\} &= \inf_{z \in x*y} \{1 - \lambda_A(z)\} = 1 - \sup_{z \in x*y} \{\lambda_A(z)\} \\ &\geq 1 - \lambda_A(y) = \lambda_A^c(y). \end{aligned}$$

Hence λ_A^c is a left fuzzy hyperideal of H .

Conversely, suppose μ_A and λ_A^c are left fuzzy hyperideals of H . For every $x, y \in H$, we have $\mu_A(y) \leq \inf_{z \in x * y} \{\mu_A(z)\}$ and we get

$$\begin{aligned} \sup_{z \in x * y} \{\lambda_A(z)\} &= \sup_{z \in x * y} \{1 - \lambda_A^c(z)\} = 1 - \inf_{z \in x * y} \{\lambda_A^c(z)\} \\ &\leq 1 - \lambda_A^c(y) = \lambda_A(y). \end{aligned}$$

Hence $A = (\mu_A, \lambda_A)$ is a left intuitionistic fuzzy hyperideal of H . □

Corollary 3.1. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set in H . Then $A = (\mu_A, \lambda_A)$ is a left intuitionistic fuzzy hyperideal of H if and only if $\Box A$ and $\Diamond A$ are left intuitionistic fuzzy hyperideals of H . For right hyperideals similar result holds as well.*

For any $t \in [0, 1]$ and a fuzzy set μ of H , the set

$$U(\mu; t) = \{x \in H \mid \mu(x) \geq t\} \quad (\text{resp. } L(\mu; t) = \{x \in H \mid \mu(x) \leq t\})$$

is called an upper (resp. lower) t -level cut of μ .

Theorem 3.1. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ is a left (resp. right) intuitionistic fuzzy hyperideal of H if and only if for all $s, t \in [0, 1]$, the sets $U(\mu_A; t)$ and $L(\lambda_A; s)$ are either empty or left (resp. right) hyperideal of H .*

Proof. Assume that all non-empty level sets $U(\mu_A; t)$ and $L(\lambda_A; s)$ are left (resp. right) hyperideals of H . Let $x, y \in H$. If $t_0 = \mu_A(y)$ and $t_1 = \lambda_A(y)$ then $y \in U(\mu_A; t_0)$ and $y \in L(\lambda_A; t_1)$. So $x * y \subseteq U(\mu_A; t_0)$ and $x * y \subseteq L(\lambda_A; t_1)$.

Therefore for all $z \in x * y$ we have $\mu_A(z) \geq t_0$ and $\lambda_A(z) \leq t_1$, and so

$$\inf_{z \in x * y} \{\mu_A(z)\} \geq \mu_A(y) \quad \text{and} \quad \sup_{z \in x * y} \{\lambda_A(z)\} \leq \lambda_A(y).$$

Hence $A = (\mu_A, \lambda_A)$ is a left intuitionistic fuzzy hyperideal of H .

Conversely, let $A = (\mu_A, \lambda_A)$ be a left intuitionistic fuzzy hyperideal of H . Let $x \in H$ and $y \in U(\mu_A; t)$. We have $\inf_{z \in x * y} \{\mu_A(z)\} \geq \mu_A(y) \geq t$. Therefore for all $z \in x * y$ we have $z \in U(\mu_A; t)$, and so $x * y \subseteq U(\mu_A; t)$.

Now, let $y \in L(\lambda_A; s)$. We have $\sup_{z \in x * y} \{\lambda_A(z)\} \leq \lambda_A(y) \leq s$. Therefore for all $z \in x * y$ we have $z \in L(\lambda_A; s)$, and so $x * y \subseteq L(\lambda_A; s)$. This completes the proof. □

Corollary 3.2. *Let I be a left (resp. right) hyperideal of a semihypergroup H . If fuzzy sets μ and λ are defined on H by*

$$\mu(x) = \begin{cases} a_0 & \text{if } x \in I \\ a_1 & \text{if } x \in H \setminus I \end{cases} \quad \text{and} \quad \lambda(x) = \begin{cases} b_0 & \text{if } x \in I \\ b_1 & \text{if } x \in H \setminus I \end{cases}$$

where $0 \leq a_1 < a_0$, $0 \leq b_0 < b_1$ and $a_i + b_i \leq 1$ for $i = 0, 1$. Then $A = (\mu, \lambda)$ is a left intuitionistic fuzzy hyperideal of H and $U(\mu, a_0) = I = L(\lambda, b_0)$.

Corollary 3.3. *Let χ_I be the characteristic function of a left (resp. right) hyperideal I of H . Then $I = (\chi_I, \chi_I^c)$ is a left (resp. right) fuzzy hyperideal of H .*

Theorem 3.2. *If $A = (\mu, \lambda)$ is a left (resp. right) intuitionistic fuzzy hyperideal of H , then for all $x \in H$ we have*

$$\mu_A(x) = \sup\{t \in [0, 1] \mid x \in U(\mu_A; t)\}$$

and

$$\lambda_A(x) = \inf\{s \in [0, 1] \mid x \in L(\mu_A; s)\}.$$

Acknowledgement. The author is highly grateful to the referee for his / her valuable comments and suggestions for improving the paper.

References

- [1] K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, **20** (1986), 87–96.
- [2] K. Atanassov, New operations defined over the intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, **61** (1994), 137–142.
- [3] K. Atanassov, *Intuitionistic Fuzzy Sets, Theory and Applications*, Studies in Fuzziness and Soft Computing, Vol. 35, Physica-Verlag, Heidelberg, 1999.
- [4] P. Corsini and V. Leoreanu, *Applications of Hyperstructures Theory*, Advanced in Mathematics, Kluwer Academic Publisher, 2003.
- [5] B. Davvaz, Fuzzy hyperideals in semihypergroups, *Italian J. Pure Appl. Math.* **8** (2000), 67–74.

