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Holomorphic Distribution of CR-Submanifolds of a Complex Projective Space

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Abstract. In this paper, we characterize CR-submanifolds of a complex projective space with parallel mean curvature vector and flat normal connection under certain symmetry conditions on the shape operator in the holomorphic distribution.

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1. Introduction

Let N be a Kaehler manifold with complex structure J and Riemannian metric g. A submanifold M of N is called a *CR-submanifold* if there exists on M a holomorphic distribution D such that its complementary orthogonal distribution $D^\perp$ is anti-invariant by J, i.e., $JD_x = D_x$ and $JD_x^\perp \subseteq T_xM^\perp$, for $x \in M$ (cf. [2]). It follows that the normal bundle splits as $TM^\perp = JD^\perp \oplus \nu$, where ν is an invariant subbundle of $TM^\perp$ by J. If $D = \{0\}$ (resp. $D^\perp = \{0\}$) then M is said to be an *anti-invariant* (resp. *invariant*) submanifold.

In what follows, we denote by $\Gamma(\mathbf{V})$ the module of all differentiable sections on a vector bundle $\mathbf{V}$ over M.

For any $X \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$, we denote by PX and FX the tangential part and the normal part of JX respectively, while $t\xi$ and $f\xi$ are the tangential part and the normal part of $J\xi$ respectively.

An important class of CR-submanifolds is the class of all real hypersurfaces of a Kaehler manifold. In particular, real hypersurfaces of a complex space form has been studied extensively (for details see [7]). In [6], Ki and Suh obtained a characterization on real hypersurfaces M of a complex space form $M_n(c)$ with constant holomorphic sectional curvature $4c\neq 0$ under the following conditions:

$$(\nabla_X A)Y = cg(PX, Y)JC,$$

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$$(AP - PA)X = -\tau(X)JC$$

for any $X, Y \in \Gamma(D)$, where C is a unit normal vector field of M, A is the shape operator of M and τ is a 1-form on M. The purpose of this paper is to extend Ki and Suh's result to the setting of CR-submanifolds of a complex projective space CP^n, i.e., we have

Theorem 1.1. *Let M be an m-dimensional complete, connected CR-submanifold of CP^n with flat normal connection and parallel mean curvature vector. Suppose that the isometric immersion of M into CP^n is full and M satisfies the following two conditions:*

$$(\nabla_X A)_\xi Y = cg(PX, Y)t\xi, \tag{1.1}$$

$$(A_\xi P - PA_\xi)X = -\tau(X)\xi \tag{1.2}$$

for any $X, Y \in \Gamma(D)$ and $\xi \in \Gamma(TM^\perp)$, where τ is a 1-form on M. Then M is either

(a) $\pi(S^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_{m+1}))$ of CP^m, where $\sum_{i=1}^{m+1} r_i^2 = 1$ and $n = m$ or

(b) $\pi(S^{n_1}(r_1) \times S^{n_2}(r_2) \times \cdots \times S^{n_k}(r_k))$ of CP^n, where $\sum_{i=1}^{k} r_i^2 = 1$ and $\sum_{i=1}^{k} n_i = m + 1$. Here $n_1, n_2, \cdots, n_k$ are odd numbers except for the case of $n_1 = n_2 = \cdots = n_k = 1$, and $2n + 1 = m + k$.

Remark. The submanifold (a) in Theorem 1.1 has parallel second fundamental form h. On the other hand, the second fundamental form h of the submanifold (b) is *not parallel*. However, it is *cyclic parallel*, namely this second fundamental form h satisfies

$$g((\nabla_X h)Y, Z) + g((\nabla_Y h)Z, X) + g((\nabla_Z h)X, Y) = 0$$

for all vectors X, Y and Z on the submanifold (b). It is well –known that this equation is equivalent to the following:

$$g((\nabla_X h)X, X) = 0$$

for each vector X on the submanifold (b). The spaces in the above theorem are described as follows:
Let $S^n(r)$ be an n-dimensional sphere with radius r center at the origin and CP^n the complex projective space of complex dimension n with constant holomorphic sectional curvature curvature $4c$. Consider the following commutative diagram (cf. [9, page 223]):

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\iota} & S^{2n+1} \\ \pi \downarrow & & \tilde{\pi} \downarrow \\ M & \xrightarrow{\tilde{\iota}} & CP^n \end{array}$$

where $\tilde{\pi}$ is the Hopf fibration from the unit sphere S^{2n+1} onto CP^n. If $\widetilde{M}$ is a contact CR-submanifold in S^{2n+1} then the submersion $\pi = \pi|_{\widetilde{M}}$ induced a CR-submanifold $M = \pi(\widetilde{M})$ on CP^n. In particular, when we put $\widetilde{M} = S^{n_1}(r_1) \times$

$S^{n_2}(r_2) \times \cdots \times S^{n_k}(r_{m+1})$, where $\sum_{i=1}^{k} r_i^2 = 1$, $n_1, n_2, \cdots, n_k$ are odd numbers, $\sum_{i=1}^{k} n_i = m+1$ and $m+k$=$2n+1$. Then $M = \pi(\widetilde{M})$ is a CR-submanifold of CP^n with parallel mean curvature vector and with flat normal connection.

2. Preliminaries

Let N be a Kaehler manifold with complex structure J and with Riemannian metric g. Suppose M is an m-dimensional Riemannian manifold isometrically immersed in N. We denote by the same g the Riemannian metric induced on M, $\widetilde{\nabla}$ and ∇ respectively the Levi-Civita connection on N and the connection induced on M. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$, where $\nabla^\perp$ denotes the normal connection induced in the normal bundle $TM^\perp$ of M and h the second fundamental form of M. The shape operator A_ξ is related to h by

$$g(A_\xi X, Y) = g(h(X, Y), \xi).$$

We define the covariant derivative of A by

$$(\nabla_X A)_\xi Y = \nabla_X (A_\xi Y) - A_{\nabla^\perp_X \xi} Y - A_\xi \nabla_X Y.$$

Then we have

$$g((\nabla_X A)_\xi Y, Z) = g((\nabla_X h)(Y, Z), \xi)$$

for any $X, Y, Z \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$. Let $R^\perp$ be the curvature tensor associated with $\nabla^\perp$. If $R^\perp$=0 then we say that the normal connection $\nabla^\perp$ is *flat.* A normal vector field ξ is said to be *parallel* if we have $\nabla^\perp \xi$=0. Let M be a CR-submanifold of a Kaehler manifold N. Then we have the following identities [3]

$$(\nabla_X P)Y = A_{FY} X + th(X, Y) \tag{2.1}$$

$$(\nabla_X F)Y = -h(X, PY) + fh(X, Y) \tag{2.2}$$

$$(\nabla_X t)\xi = A_{f\xi} X - PA_\xi X \tag{2.3}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(TM^\perp)$.

When the ambient space N is the complex projective space CP^n of constant holomorphic sectional curvature $4c > 0$, the Codazzi equation is given as follows

$$(\nabla_X A)_\xi Y - (\nabla_Y A)_\xi X = c\{g(FX, \xi)PY - g(FY, \xi)PX + 2g(PX, Y)t\xi\}. \tag{2.4}$$

Finally, we state some known results for later use.

Theorem 2.1. [8] *Let M be an m-dimensional complete, connected CR-submanifold of CP^n with flat normal connection and parallel mean curvature vector. If $A_\xi P = PA_\xi$, for any $\xi \in \Gamma(TM^\perp)$, then M is either*
(a) $\pi(S^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_{m+1}))$; $\sum_{i=1}^{m+1} r_i^2 = 1$ *of some CP^m in CP^n ; or*
(b) $\pi(S^{n_1}(r_1) \times S^{n_2}(r_2) \times \cdots \times S^{n_k}(r_k))$; $\sum_{i=1}^{k} r_i^2 = 1$; $\sum_{i=1}^{k} n_i = m+1$ *where $n_1, n_2, \cdots, n_k$ are odd numbers and $2n+1 = m+k$.*

Lemma 2.1. [1] *Let M be an m-dimensional complete, connected, totally geodesic submanifold of CP^n . Then M is either*
(a) an invariant complex projective space $CP^{m/2}$; or;
(b) an anti-invariant real projective space RP^m.

Theorem 2.2. [5] *Let M be an invariant submanifold of a complex space form $M_n(c)$. Then the normal connection is flat if and only if $c = 0$ and M is totally geodesic.*

3. Certain lemmas

In order to prove our Theorem, we prepare some lemmas in this section. We first prove the following:

Lemma 3.1. *Let M be a CR-submanifold of a Kaehler manifold N. If M satisfies the condition (1.2) then $A_\xi X = 0$, for any $X \in \Gamma(D)$ and $\xi \in \Gamma(\nu)$.*

Proof. By putting $X \in \Gamma(D)$, $\xi \in \Gamma(\nu)$ in (1.2) and taking inner product with $Y \in \Gamma(TM)$, we obtain

$$g((A_\xi P - PA_\xi)X, Y) = -\tau(X)g(t\xi, Y) = 0 \tag{3.1}$$

since $t\xi = 0$. If we let $Y \in \Gamma(D^\perp)$ then we have $g(A_\xi PX, Y) = 0$, which means that $A_\xi D \perp D^\perp$. Next, by putting $X, Y \in \Gamma(D)$ in (2.2) and taking inner product with $\xi \in \Gamma(\nu)$ we have

$$0 = g(-h(X, PY) + fh(X, Y), \xi) = -g(A_\xi X, PY) - g(A_{f\xi}X, Y).$$

Since $A_{f\xi}$ is symmetric with respect to X and Y, the above equation and (3.1) imply that $g(A_\xi PX, Y) = 0$ for any $X, Y \in \Gamma(D)$ and $\xi \in \Gamma(\nu)$. Hence $A_\xi D \perp$ D and this completes the proof. □

Now let M be an m-dimensional CR-submanifold of CP^n with flat normal connection. Then there is a local field of orthonormal frames $\{\xi_a\}$ in $\Gamma(TM^\perp)$ such that each ξ_a, ($1 \le a \le 2n - m$) is parallel, (cf. [4, page 99]). In the following we put $A_a = A_{\xi_a}$ and

$$(\nabla_X A)_a = (\nabla_X A)_{\xi_a}$$

to simplify the notation.

Lemma 3.2. *Let M be a CR-submanifold of CP^n with flat normal connection. If M satisfies the conditions (1.1) and (1.2) then*

$$\tau(Y)g(A_aX, PZ) + \tau(PY)g(A_aX, Z) + \tau(Z)g(A_aX, PY) + \tau(PZ)g(A_aX, Y) = 0$$

for any $X, Y, Z \in \Gamma(D)$.

Proof. For any $Y, Z \in \Gamma(D)$, (1.2) implies that $g((A_aP - PA_a)Y, Z) = 0$. Differentiating this equation with respect to $X \in \Gamma(D)$, we get

$$g((\nabla_X A)_a Z, PY) + g((\nabla_X A)_a Y, PZ) + g((\nabla_X P)Z, A_aY) + g((\nabla_X P)Y, A_aZ)$$

$$+g(\nabla_X Z, (A_aP - PA_a)Y) + g(\nabla_X Y, (A_aP - PA_a)Z) = 0. \tag{3.2}$$

On the other hand, by using (1.2), we have

$$g(\nabla_X Z, (A_aP - PA_a)Y) = -\tau(Y)g(\nabla_X Z, t\xi_a) = \tau(Y)g(Z, (\nabla_X t)\xi_a).$$

Together with (2.3) and Lemma 3.1, yields

$$g(\nabla_X Z, (A_aP - PA_a)Y) = \tau(Y)g(Z, A_{f_a}X - PA_aX) = \tau(Y)g(PZ, A_aX).$$

Similarly, we have

$$g(\nabla_X Y, (A_aP - PA_a)Z) = \tau(Z)g(PY, A_aX).$$

It follows from (1.1), (2.1) and these equations that (3.2) reduces to

$$g(th(X,Z), A_aY) + g(th(X,Y), A_aZ)$$

$$+ \tau(Y)g(A_aX, PZ) + \tau(Z)g(A_aX, PY) = 0. \tag{3.3}$$

Note that the condition (1.2) implies that $FA_aX = \tau(PX)Ft\xi_a$. From which, together with Lemma 3.1 and the fact that

$$\xi_a = -Ft\xi_a - f^2\xi_a$$

we get

$$g(th(X,Z), A_aY) = \tau(PY)g(A_aX, Z),$$

and

$$g(th(X,Y), A_aZ) = \tau(PZ)g(A_aX, Y).$$

From these equations and (3.3), we obtain the lemma. □

4. Proof of the theorem

First of all, let us suppose that $t\xi_a = 0$ for all a. Then M is an invariant submanifold of CP^n, but this contradicts Theorem 2.2. Hence we may assume that $t\xi_a \neq 0$ for some a and define a local vector field $V_a = -\frac{1}{\|t\xi_a\|^2}t\xi_a$. Now we decompose A_aV_a into

$$A_aV_a = \beta_aU_a + \mu_aW_a \tag{4.1}$$

where U_a and W_a are two unit vector fields in $\Gamma(D)$ and $\Gamma(D^\perp)$ respectively.

Next we shall show that $\beta_a = 0$. For this purpose, we suppose that there exists a non empty open set G in M on which β_a is nowhere zero. Then taking inner product with $t\xi_a$ in (1.2) and using the above equation, we obtain

$$\tau(X) = -\beta_a g(X, PU_a) \tag{4.2}$$

for any $X \in \Gamma(D)$. From which we can see that $\tau(PU_a) = -\beta_a \neq 0$ and $\tau(U_a) = 0$. By putting $Z = U_a$ in Lemma 3.2, we obtain

$$\tau(Y)g(A_aX, PU_a) + \tau(PY)g(A_aX, U_a) + \tau(PU_a)g(A_aX, Y) = 0.$$

By putting $Y = U_a$ and then $Y = PU_a$ in this equation we obtain

$$g(A_a PU_a, X) = g(A_a U_a, X) = 0$$

for any $X \in \Gamma(D)$, or equivalently, $PA_aPU_a = PA_aU_a = 0$. Together with (1.2) and (4.2) yields

$$A_a PU_a = 0$$

and

$$A_a U_a = -\beta_a t\xi_a. \tag{4.3}$$

Now by using the Codazzi equation, we get

$$g((\nabla_{V_a} A)_a U_a, PU_a) - g((\nabla_{U_a} A)_a V_a, PU_a) = cg(FV_a, \xi_a) = c.$$

On the other hand, (4.3) implies that

$$\begin{aligned} g((\nabla_{V_a} A)_a U_a, PU_a) &= g(\nabla_{V_a} A_a U_a, PU_a) - g(A_a \nabla_{V_a} U_a, PU_a) \\ &= -\beta_a g((\nabla_{V_a} t)\xi_a, PU_a). \end{aligned} \tag{4.4}$$

By using (2.3), (4.1) and Lemma 3.1, this equation gives

$$g((\nabla_{V_a} A)_a U_a, PU_a) = \beta_a^2.$$

Moreover, it follows from (1.1) that we can see

$$g((\nabla_{U_a} A)_a V_a, PU_a) = g((\nabla_{U_a} A)_a PU_a, V_a) = cg(t\xi_a, V_a) = -c.$$

From these equations, we obtain $\beta_a^2 = 0$, which is a contradiction. Consequently, we conclude that $\beta_a = 0$ and so (4.2) implies that $\tau(X) = 0$ for $X \in \Gamma(D)$. Since $(A_aP - PA_a)D^\perp = 0$, we have $A_aP - PA_a = 0$. Thus our Theorem follows from Theorem 2.1.

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References

[1] K. Abe, Applications of a Riccati type differential equation to Riemannian manifolds with totally geodesic distributions, *Tohoku Math. J.* (2) **25** (1973), 425–444.

[2] A. Bejancu, CR-submanifolds of a Kaehler manifold I, *Proc. Amer. Math. Soc.* **69** (1978), 135–142.

[3] B. Y. Chen, CR-submanifolds of a Kaehler manifold I, *J. Differential Geom.* **16** (1981), 305–322.

[4] B. Y. Chen, *Geometry of submanifolds*, Dekker, New York, 1973.

[5] B. Y. Chen and K. Ogiue, Some extrinsic results for Kaehler submanifolds, *Tamkang J. Math.* **4** (1973), 207–213.

[6] U. H. Ki and Y. J. Suh, On a characterization of real hypersurfaces of type A in a complex space form, *Canad. Math. Bull.* **37** (1994), 238–244.

[7] R. Niebergall and P. J. Ryan, Real hypersurfaces in complex space forms, in *Tight and Taut Submanifolds (Berkeley, CA, 1994)*, 233–305, Cambridge Univ. Press, Cambridge.

[8] K. Yano and M. Kon, CR submanifolds of a complex projective space, *J. Differential Geom.* **16**(3) (1981), 431–444

[9] K. Yano and M. Kon, *Structures on manifolds*, World Sci. Publishing, Singapore, 1984.