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The Cyclic Subgroup Separability of Certain HNN Extensions

[1]P.C. WONG AND [2]K.B. WONG
Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia
[1]wongpc@um.edu.my, [2]kbwong@um.edu.my

Abstract. In this note we give characterisations for certain HNN extensions with central associated subgroups to be cyclic subgroup separable. We then apply our results to HNN extensions of polycyclic-by-finite groups and Fuchsian groups.

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1. Introduction

A group G is called cyclic subgroup separable (or π_c for short) if for each cyclic subgroup H and $x \in G\backslash H$, there exists a normal subgroup N of finite index in G such that $x \notin HN$. Clearly a cyclic subgroup separable group is residually finite. The concept of cyclic subgroup separability was introduced by Stebe [14] in 1968 and he used it to prove the residual finiteness of a class of knot groups.

Many classes of groups, including the free groups and the polycyclic-by-finite groups are cyclic subgroup separable ([7], [10]). Also the finite extension of a cyclic subgroup separable group is again cyclic subgroup separable (Stebe [14]). On the other hand, the cyclic subgroup separability of HNN extensions are not much known. For example, the HNN extension $\langle h, t; t^{-1}ht = h^2\rangle$ is residually finite but is not cyclic subgroup separable (see [1]) while another HNN extension, the Baumslag-Solitar group, $\langle h, t; t^{-1}h^2t = h^3\rangle$ is not even residually finite (see [5]).

Kim [8] and Kim and Tang [9] gave characterisations for HNN extensions of cyclic subgroup separable groups with cyclic associated subgroups to be again cyclic subgroup separable. They then apply their results to give characterisations for the HNN extensions of a finitely generated abelian group with cyclic associated subgroups to be cyclic subgroup separable and show that certain HNN extensions of finitely generated torsion-free nilpotent groups with cyclic associated subgroups to be again cyclic subgroup separable.

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Andreadakis, Raptis and Varsos in a series of papers [2], [3], [4] and [11], gave characterisations for HNN extensions of a finitely generated abelian group to be residually finite. Some of these results were extended by Wong and Tang [15] to characterisations for HNN extensions of finitely generated abelian groups to be cyclic subgroup separable.

In this note we investigate the cyclic subgroup separability of HNN extensions with central associated subgroups. More precisely, let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ denote an HNN extension where A is the base group, H, K are the associated subgroups and $\varphi : H \longrightarrow K$ is the associated isomorphism. We shall show that if $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ is an HNN extension where H and K are subgroups in the center of A, $H \neq A \neq K$ and A is subgroup separable, then G is cyclic subgroup separable if and only if its subgroup (HNN extension) $G_1 = \langle t, HK; t^{-1}Ht = K, \varphi\rangle$ is cyclic subgroup separable. Thus we are able to use the results of [11] and [15] to give characterisations for HNN extensions of polycyclic-by-finite groups and Fuchsian groups with central associated subgroups to be cyclic subgroup separable.

More importantly, our result shows that the study of the cyclic subgroup separability of HNN extensions with central associated subgroups can be reduced to that of the cyclic subgroup separability of HNN extensions of abelian groups. Thus the characterisations given in the papers [11] and [15] can be applied to these HNN extensions.

The notation used here is standard. In addition, the following will be used for any group G: $N \lhd_f G$ means N is a normal subgroup of finite index in G.

2. Preliminaries

Definition 2.1. *A group G is called H-separable for the subgroup H if for each $x \in G\backslash H$, there exists $N \lhd_f G$ such that $x \notin HN$. The group G is termed subgroup separable if G is H-separable for every finitely generated subgroup H. The group G is termed cyclic subgroup separable (or π_c for short) if G is H-separable for every cyclic subgroup H.*

It is well known that free groups, polycyclic groups and surface groups are subgroup separable (M. Hall [7], Mal'cev [10], Scott [12]). Since a finite extension of a subgroup separable group is again subgroup separable, polycyclic-by-finite groups and Fuchsian groups (finite extension of surface groups) are subgroup separable.

3. The main results

In this section we will prove our main results, i.e., Theorem 3.2 and Theorem 3.3. To simplify our exposition we will use the term π_c instead of cyclic subgroup separable for the rest of the paper.

The following lemma is an application of a result of Blass and Newman in [6].

Lemma 3.1. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension. If $K = A$ and $H \neq A$, then G is not H-separable.*

Remark 3.1. In this note we consider the HNN extension $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ where H and K are subgroups in the center of A. Note that if $H = A = K$, then A is abelian. Furthermore A is normal in G and $G/A \cong \langle t\rangle$. Hence G is polycyclic and by [10], G is subgroup separable and hence π_c. Now by Lemma 3.1, if $K = A$, $H \neq A$ and H is cyclic, then G is not π_c. So we shall only consider the HNN extensions where $H \neq A \neq K$.

Theorem 3.1. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension. Suppose H and K are subgroups in the center of A and $H \neq A \neq K$. Let $\Delta = \{N \lhd_f A\ ;\ \varphi(N \cap H) = N \cap K\}$. Then G is π_c if and only if $\bigcap_{N\in\Delta} NH = H$, $\bigcap_{N\in\Delta} NK = K$ and $\bigcap_{N\in\Delta} N\langle a\rangle = \langle a\rangle, \forall a \in A$.*

Proof. Suppose G is π_c. Then G is residually finite. Suppose $A = HK$. Since H and K are in the center of A, the subgroup HK satisfies the nontrivial identity $W(x, y) = x^{-1}y^{-1}xy$. Furthermore $H \nsubseteq K$ and $K \nsubseteq H$ because $H \neq A = HK \neq K$. Therefore by [13, Theorem 3'], $\bigcap_{N\in\Delta} NH = H$ and $\bigcap_{N\in\Delta} NK = K$.

Suppose $A \neq HK$. Let $c \in A - HK$. Then the subgroup generated by c and HK, i.e., $\langle c, HK\rangle$, is abelian and satisfies the nontrivial identity $W(x, y) = x^{-1}y^{-1}xy$. Furthermore HK is properly contained in $\langle c, HK\rangle$. Therefore by [13, Theorem 3], $\bigcap_{N\in\Delta} NH = H$ and $\bigcap_{N\in\Delta} NK = K$.

So in both cases we have $\bigcap_{N\in\Delta} NH = H$ and $\bigcap_{N\in\Delta} NK = K$.

Next we show that $\bigcap_{N\in\Delta} N\langle a\rangle = \langle a\rangle$. Let $a \in A$ and $x \notin \langle a\rangle$. Since G is π_c, there exists $M \lhd_f G$ such that $x \notin M\langle a\rangle$. Note that $(M \cap A) \in \Delta$ and $x \notin (M \cap A)\langle a\rangle$. This implies that $\bigcap_{N\in\Delta} N\langle a\rangle = \langle a\rangle$.

The converse follows from [8, Theorem 2.2]. □

By using Theorem 3.1, we shall prove Theorem 3.2 and Theorem 3.3. Before that, we state Lemma 3.2, whose proof is immediate from Theorem 3.1 and the result of [6].

Lemma 3.2. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension where H and K are subgroups in the center of A and $H \neq A \neq K$. Suppose G is π_c. If $H \subseteq K$ or $K \subseteq H$ then $H = K$.*

Theorem 3.2. *Let $G = \langle t, A; t^{-1}Ht = H, \varphi\rangle$ be an HNN extension where H is a finitely generated subgroup in the center of A and $H \neq A$. If A is $H^n\langle a\rangle$-separable for every $a \in A$ and every positive integer n, then G is π_c.*

Proof. Let $N \in \Delta$ where $\Delta = \{N \lhd_f A\ ;\ \varphi(N \cap H) = N \cap H\}$. By Theorem 3.1, it is sufficient to show that $\bigcap_{N\in\Delta} NH = H$ and $\bigcap_{N\in\Delta} N\langle a\rangle = \langle a\rangle, \forall a \in A$.

First we show that $\bigcap_{N\in\Delta} NH = H$. Let $a \in A - H$. Since A is H-separable, there exists $M_a \lhd_f A$ such that $a \notin M_aH$. Note that $M_aH \lhd_f A$ and $M_aH \in \Delta$. This implies that $\bigcap_{a\in A-H} M_aH = H$ and hence $\bigcap_{N\in\Delta} NH = H$.

Next we show that $\bigcap_{N\in\Delta} N\langle a\rangle = \langle a\rangle, \forall a \in A$. But first we construct a subgroup $N_n \in \Delta$ for each $n \geq 2$. Let $h_0, h_1, \ldots, h_m$ be coset representatives of H^n in H where $h_0 = 1$ and $n \geq 2$. Since A is H^n-separable, there exists $M_n \lhd_f A$ such that

$h_i \notin M_n H^n$ for $1 \leq i \leq m$. Let $N_n = M_n H^n$. Then $N_n \lhd_f A$. We claim that $N_n \in \Delta$, that is, $N_n \cap H = H^n$. Clearly we only need to show that $N_n \cap H \subseteq H^n$. Suppose $a \in (N_n \cap H) - H^n$. Since $a \notin H^n$, we have $a = h_i \bar{h}$ where $h_i \neq 1$ is a coset representative of H^n in H and $\bar{h} \in H^n$. On the other hand, since $a \in N_n = M_n H^n$, $a = m\tilde{h}$ where $m \in M_n$ and $\tilde{h} \in H^n$. But then $h_i \in M_n H^n$, a contradiction. Hence $N_n \cap H \subseteq H^n$. Therefore $N_n \in \Delta$ for each $n \geq 2$.

Let $a \in A$ and $b \in A - \langle a \rangle$. We claim that $b \notin H^n\langle a \rangle$ for some $n \geq 1$. If $b \notin H\langle a \rangle$, then we are done. So we may assume that $b \in H\langle a \rangle$. Let $b = ha^j$ for some $h \in H$ and integer j. Clearly $h \notin H \cap \langle a \rangle$. Since H is finitely generated abelian, there exists an integer $n \geq 1$, such that $h \notin H^n(H \cap \langle a \rangle)$. This implies that $h \notin H^n\langle a \rangle$, and therefore $b \notin H^n\langle a \rangle$.

Since A is $H^n\langle a \rangle$-separable, there exists $M_1 \lhd_f A$ such that $b \notin M_1 H^n \langle a \rangle$. Note that for sufficiently large m, $H^m \subseteq M_1$. As above, we can constructs $N_m \in \Delta$ such that $N_m \cap H = H^m$. Let $M = N_m \cap M_1$. Then $M \in \Delta$ since $M \cap H = H^m$. Furthermore $b \notin MH^n\langle a \rangle$. This implies that $b \notin M\langle a \rangle$ and so $\bigcap_{N \in \Delta} N\langle a \rangle = \langle a \rangle$. The proof is now completed. □

Theorem 3.3. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where H and K are subgroups in the center of A and $H \neq A \neq K$. Suppose $H \nsubseteq K, K \nsubseteq H$ and A is $M\langle a \rangle$-separable for every subgroup $M \lhd_f HK$ and $a \in A$. Then G is π_c if and only if $G_1 = \langle t, HK; t^{-1}Ht = K, \varphi \rangle$ is π_c.*

Proof. Suppose G is π_c. Since G_1 is a subgroup of G, G_1 is π_c.

Suppose G_1 is π_c. Since $H \neq HK \neq K$, then by Theorem 3.1, $\bigcap_{M \in \Delta_1} MH = H$, $\bigcap_{M \in \Delta_1} MK = K$ and $\bigcap_{M \in \Delta_1} M\langle x \rangle = \langle x \rangle, \forall x \in HK$, where $\Delta_1 = \{M \lhd_f HK \ ; \ \varphi(M \cap H) = M \cap K\}$.

Let $\Delta = \{N \lhd_f A \ ; \ \varphi(N \cap H) = N \cap K\}$. By Theorem 3.1, it is sufficient to show that $\bigcap_{N \in \Delta} NH = H$, $\bigcap_{N \in \Delta} NK = K$ and $\bigcap_{N \in \Delta} N\langle a \rangle = \langle a \rangle$, $\forall a \in A$.

We begin by constructing a subgroup $N_{(b,a,M)} \in \Delta$ for each $M \in \Delta_1$, $a \in A$ and $b \in A - M\langle a \rangle$. Let $M \in \Delta_1$, $a \in A$ and $b \in A - M\langle a \rangle$. Let $h_0, h_1, \ldots, h_m$ be coset representatives of M in HK where $h_0 = 1$. Since A is M-separable, there exists $P_M \lhd_f A$ such that $h_i \notin P_M M$ for $1 \leq i \leq m$. Since A is $M\langle a \rangle$-separable, there exists $P_{(b,a)} \lhd_f A$ such that $b \notin P_{(b,a)} M\langle a \rangle$. Let $P_{(b,a,M)} = P_{(b,a)} \cap P_M$ and $N_{(b,a,M)} = P_{(b,a,M)} M$. Then $N_{(b,a,M)} \lhd_f A$ and $b \notin N_{(b,a,M)}\langle a \rangle$. Next we claim that $N_{(b,a,M)} \cap HK = M$. Clearly we need only to show that $N_{(b,a,M)} \cap HK \subseteq M$. Suppose $y \in (N_{(b,a,M)} \cap HK) - M$. Since $y \notin M$, then $y = h_i m_1$ where $h_i \neq 1$ is a coset representative of M in HK and $m_1 \in M$. On the other hand, since $y \in N_{(b,a,M)} = P_{(b,a,M)} M$, we have $y = pm_2$ where $p \in P_{(b,a,M)}$ and $m_2 \in M$. But this implies that $h_i \in P_{(b,a,M)} M \subseteq P_M M$, a contradiction. Thus $N_{(b,a,M)} \cap HK = M$. This implies that $N_{(b,a,M)} \cap H = M \cap H$ and $N_{(b,a,M)} \cap K = M \cap K$. Hence $N_{(b,a,M)} \in \Delta$.

Next we show that $\bigcap_{N \in \Delta} NH = H$. Let $a \in A - H$. Suppose $a \in A - HK$. Since A is HK-separable, there exists $M_a \lhd_f A$ such that $a \notin M_a HK$. Note that $M_a HK \lhd_f A$ and $M_a HK \in \Delta$. Suppose $a \in HK$. Since $a \in HK - H$, there exists

$M \in \Delta_1$ such that $a \notin MH$. As above, we can construct a subgroup $N_{(a,1,M)}$ for $M \in \Delta_1$, $1 \in A$ and $a \in A - M\langle 1\rangle$. We claim that $a \notin N_{(a,1,M)}H$. Suppose $a \in N_{(a,1,M)}H$. Then $a = nh$ for some $n \in N_{(a,1,M)}$ and $h \in H$. This implies that $n \in N_{(a,1,M)} \cap HK$. But $N_{(a,1,M)} \cap HK = M$ by its construction above. Hence $a \in MH$, a contradiction. Therefore $a \notin N_{(a,1,M)}H$ and thus $\bigcap_{N\in\Delta} NH = H$. Similarly $\bigcap_{N\in\Delta} NK = K$.

Finally we show that $\bigcap_{N\in\Delta} N\langle a\rangle = \langle a\rangle, \forall a \in A$. Let $a \in A$ and $b \in A - \langle a\rangle$. Suppose $b \notin HK\langle a\rangle$. Since A is $HK\langle a\rangle$-separable, there exists $M \lhd_f A$ such that $b \notin MHK\langle a\rangle$. Note that $MHK \lhd_f A$ and $MHK \in \Delta$. Suppose $b \in HK\langle a\rangle$. Then $b = xa^i$ for some $x \in HK$ and integer i. Clearly $x \notin HK \cap \langle a\rangle$. Therefore there exists $M_1 \in \Delta_1$ such that $x \notin M_1(HK \cap \langle a\rangle)$. This implies that $b \notin M_1\langle a\rangle$. As above, we can construct a subgroup $N_{(b,a,M_1))}$ for $M_1 \in \Delta_1$, $a \in A$ and $b \in A - M_1\langle a\rangle$. From this construction, we have $N_{(b,a,M)} \in \Delta$ and $b \notin N_{(b,a,M_1)}\langle a\rangle$. Hence $\bigcap_{N\in\Delta} N\langle a\rangle = \langle a\rangle$. The proof is now completed. □

4. Applications

In this section will apply the results in section 3 to HNN extensions of polycyclic-by-finite groups and Fuchsian groups. But first we have the following lemma.

Lemma 4.1. *Let A be a group and H and K be isomorphic finitely generated subgroups in the center of A such that $\varphi : H \longrightarrow K$ is an isomorphism from H onto K. Let $\Delta_1 = \{M \lhd_f HK \ ; \ \varphi(M \cap H) = M \cap K\}$. Suppose $\bigcap_{M\in\Delta_1} MH = H$ and $\bigcap_{M\in\Delta_1} MK = K$. Then there exists $N \in \Delta_1$ such that $N^n \in \Delta_1$ for all $n \geq 1$.*

Proof. Let $i_{HK}(H) = \{b \in HK; b^n \in H$ for some positive integer n$\}$. Then $i_{HK}(H)$ is a group and H is of finite index in $i_{HK}(H)$. Similarly let $i_{HK}(K) = \{b \in HK; b^n \in K$ for some positive integer n$\}$. Then $i_{HK}(K)$ is a group and K has finite index in $i_{HK}(K)$. Since $\bigcap_{M\in\Delta_1} MH = H$, $\bigcap_{M\in\Delta_1} MK = K$ and H and K are of finite index in $i_{HK}(H)$ and $i_{HK}(K)$ respectively, there exists $N \in \Delta_1$ such that $N \cap i_{HK}(H) = N \cap H$ and $N \cap i_{HK}(K) = N \cap K$. Furthermore $N^n \cap H = (N \cap H)^n$ and $N^n \cap K = (N \cap K)^n$. This implies that $N^n \in \Delta_1$ for all $n \geq 1$. □

Theorem 4.1. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension where H and K are finitely generated subgroups in the center of A. Suppose A is subgroup separable and $H \neq A \neq K$. Then G is π_c if and only if one of the following holds:*

(a) *$H = K$*

(b) *$H \nsubseteq K, K \nsubseteq H$ and there exists a torsion free subgroup $N \lhd_f HK$ such that $\varphi(N \cap H) = N \cap K$ and $N \cap K, N \cap H$ are isolated in N.*

Proof. Suppose G is π_c and suppose $H \neq K$. If $H \subset K$, then by Lemma 3.2, $H = K$, a contradiction. Therefore $H \neq HK$ and similarly $K \neq HK$. So by Theorem 3.3, $G_1 = \langle t, HK; t^{-1}Ht = K, \varphi\rangle$ is π_c and therefore residually finite. Then by [11, Theorem], there exists a torsion free subgroup $N \lhd_f HK$ such that $\varphi(N \cap H) = N \cap K$ and $N \cap K, N \cap H$ are isolated in N.

Conversely suppose $H = K$. Then G is π_c by Theorem 3.2. Next suppose $H \neq K$. Then $H \neq HK \neq K$. If there exists a torsion free subgroup $N \lhd_f HK$ such that $\varphi(N \cap H) = N \cap K$ and $N \cap K, N \cap H$ are isolated in N, then $G_1 = \langle t, HK; t^{-1}Ht = K, \varphi\rangle$ is residually finite by [11, Theorem]. We will show that G_1 is π_c. By Theorem 3.1, it is sufficient to show that $\bigcap_{M\in\Delta_1} MH = H$, $\bigcap_{M\in\Delta_1} MK = K$ and $\bigcap_{M\in\Delta_1} M\langle x\rangle = \langle x\rangle, \forall x \in HK$, where $\Delta_1 = \{M \lhd_f HK\ ;\ \varphi(M\cap H) = M \cap K\}$.

First we show that $\bigcap_{N\in\Delta_1} NH = H$ and $\bigcap_{N\in\Delta_1} NK = K$. Since H and K are in the center of A, the subgroup HK satisfies the nontrivial identity $W(x,y) = x^{-1}y^{-1}xy$. Therefore by [13, Theorem 3'], $\bigcap_{N\in\Delta_1} NH = H$ and $\bigcap_{N\in\Delta_1} NK = K$.

Next we will show that $\bigcap_{M\in\Delta_1} M\langle x\rangle = \langle x\rangle, \forall x \in HK$. Let $y \in HK - \langle x\rangle$. Since HK is abelian and so subgroup separable, there exists $M \lhd_f HK$ such that $y \notin M\langle x\rangle$. By Lemma 4.1, there exists $N \in \Delta_1$ such that $N^n \in \Delta_1$ for all $n \geq 1$. Since M is of finite index in HK, there exists a positive integer n such that $N^n \subseteq M$. Thus $y \notin N^n\langle x\rangle$ and so $\bigcap_{M\in\Delta_1} M\langle x\rangle = \langle x\rangle$. Hence we have shown that G_1 is π_c. Now by Theorem 3.3, G is π_c. □

Corollary 4.1. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension where A is a polycyclic-by-finite group or a Fuchsian group. Suppose H and K are finitely generated subgroups in the center of A and $H \neq A \neq K$. Then G is π_c if and only if one of the following holds:*

(a) $H = K$

(b) $H \nsubseteq K, K \nsubseteq H$ *and there exists a torsion free subgroup* $N \lhd_f HK$ *such that* $\varphi(N\cap H) = N \cap K$ *and* $N \cap K, N \cap H$ *are isolated in* N.

Proof. Since polycyclic-by-finite groups and Fuchsian groups are subgroup separable, the corollary follows from Theorem 4.1. □

Another application from Section 3 is the following result.

Corollary 4.2. *Let $G = \langle t, A; t^{-1}Ht = K, \varphi\rangle$ be an HNN extension where A is a polycyclic-by-finite group or a Fuchsian group. Suppose H and K are finitely generated subgroups in the center of A such that $H \cap K$ is finite. Then G is π_c.*

Proof. By Theorem 3.3, G is π_c if and only if $G_1 = \langle t, HK; t^{-1}Ht = K, \varphi\rangle$ is π_c. But by [15, Theorem 2], G_1 is π_c. Hence G is π_c. □

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