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On β-I-open Sets and a Decomposition of Almost-I-continuity

[1]E. HATIR AND [2]T. NOIRI

[1]Selçuk Üniversitesi, Eğitim Fakultesi, Matematik Bölümü, 42090, Konya, Turkey
[2]2949-1 Shiokita-cho, Hinagu Yatsushiro-shi, Kumamoto-ken, 869-5142, Japan
[1]hatir10@yahoo.com, [2]t.noiri@nifty.com

Abstract. In this paper, we investigate further properties of β-I-open sets defined in [5] and give a decomposition of almost-I-continuity as the following: a function $f : (X, \tau, I) \to (Y, \sigma)$ is almost-I-continuous if and only if it is β-I-continuous and *-I-continuous.

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1. Introduction

In 1992, Janković and Hamlett [8] introduced the notion of I-open sets in topological spaces via ideals. Abd El-Monsef et al. [2] further investigated I-open sets and I-continuous functions. In 1999, Abd El-Monsef et al. [3] introduced and investigated almost-I-open sets and almost-I-continuous functions. Recently, Hatir and Noiri [5] have introduced the notion of β-I-open sets to obtain certain decompositions of continuity.

In this paper, we obtain the further properties of β-I-open sets and β-I-continuity and give a decomposition of almost-I-continuity.

2. Preliminaries

Throughout the present paper, spaces always mean topological spaces on which no separation properties are assumed unless explicity stated. In a topological space (X, τ), the closure and the interior of any subset A of X will be denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. An ideal is defined as a nonempty collection I of subsets of X satisfying the following two conditions:

(1) if $A \in I$ and $B \subset A$, then $B \in I$;
(2) $A \in I$ and $B \in I$, then $A \cup B \in I$.

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Let (X, τ) be a topological space and I an ideal of subsets of X. An ideal topological space, denoted by (X, τ, I), is a topological space (X, τ) with an ideal I on X. For a subset A of X, $A^*(I) = \{x \in X : U \cap A \notin I$ for each neighborhood U of $x\}$ is called the local function [6] of A with respect to I and τ. We simply write A^* instead of $A^*(I)$ in case there is no chance for confusion. The set $X*$ is often a proper subset of X. It is well known that $\mathrm{Cl}^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$ which is finer than τ. A subset A of (X, τ, I) is called *-dense-in-itself if $A \subset A^*$[6].

Lemma 2.1. [7] *Let (X, τ, I) be an ideal topological space and A, B subsets of X. Then*

(a) *If $A \subset B$, then $A^* \subset B^*$,*
(b) *If $U \in \tau$, then $U \cap A^* \subset (U \cap A)^*$,*
(c) *A^* is closed in (X, τ).*

First we shall recall some definitions used in the sequel.

Definition 2.1. *A subset A of an ideal topological space (X, τ, I) is said to be*

(a) *I-open* [8] *if $A \subset \mathrm{Int}(A^*)$,*
(b) *almost-I-open* [3] *if $A \subset \mathrm{Cl}(\mathrm{Int}(A^*))$,*
(c) *β-I-open* [5] *if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A)))$,*
(d) *β-open* [1] *if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.*

By $\beta IO(X, \tau)$, we denote the family of all β-I-open sets of space (X, τ, I).

3. β-I-open sets

Lemma 3.1. *Every almost-I-open set is β-I-open.*

Proof. Let (X, τ, I) be an ideal topological space and A an almost-I-open set of X. Then $A \subset \mathrm{Cl}(\mathrm{Int}(A^*)) \subset \mathrm{Cl}(\mathrm{Int}(A^* \cup A)) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A)))$. Therefore, A is β-I-open.

The converse of Lemma 3.1 is not necessarily true as shown by the following example. □

Example 3.1. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}\}$ and $I = \{\emptyset, \{c\}\}$. Then $A = \{c\}$ is a β-I-open set which is not almost-I-open.

Lemma 3.2. [5] *(a) Every β-I-open set is β-open but not conversely.*
(b) Every open set is β-I-open but not conversely.

Theorem 3.1. *A subset A of a space (X, τ, I) is β-I-open if and only if $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A)))$.*

Proof. Let A be a β-I-open set. Then we have $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A)))$ and hence $\mathrm{Cl}(A) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A))) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))) \subset \mathrm{Cl}(A)$. Therefore, we have $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A)))$. The converse is obvious. □

The intersection of even two β-I-open sets need not be β-I-open as shown by the following example due to Dontchev [4].

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$ and $I = \{\emptyset, \{c\}\}$. Set $A = \{a, c\}$ and $B = \{b, c\}$. Since $A^* = B^* = X$, then both A and B are β-I-open. But on the other hand $A \cap B = \{c\} \notin \beta IO(X, \tau)$.

Theorem 3.2. *Let (X, τ, I) be an ideal topological space and $\{A_\alpha : \alpha \in \Delta\}$ a family of subsets of X, where Δ is an arbitrary index set. Then,*

(a) *If $\{A_\alpha : \alpha \in \Delta\} \subset \beta IO(X, \tau)$, then $\cup\{A_\alpha : \alpha \in \Delta\} \in \beta IO(X, \tau)$.*
(b) *If $A \in \beta IO(X, \tau)$ and $U \in \tau$, then $A \cap U \in \beta IO(X, \tau)$.*

Proof. (a) Since $\{A_\alpha : \alpha \in \Delta\} \subset \beta IO(X, \tau)$, then $A_\alpha \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A_\alpha)))$ for each $\alpha \in \Delta$. Then we have

$$\begin{aligned}\cup_{\alpha\in\Delta} A_\alpha &\subset \cup_{\alpha\in\Delta} \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A_\alpha)))\\ &\subset \mathrm{Cl}(\mathrm{Int}(\cup_{\alpha\in\Delta} \mathrm{Cl}^*(A_\alpha)))\\ &\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(\cup_{\alpha\in\Delta} A_\alpha))).\end{aligned}$$

This shows that $\cup_{\alpha\in\Delta} A_\alpha \in \beta IO(X, \tau)$.

(b) By the assumption, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A)))$ and $U = \mathrm{Int}(U)$. Thus using Lemma 2.1, we have

$$\begin{aligned}A \cap U &\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A))) \cap \mathrm{Int}(U)\\ &\subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A)) \cap \mathrm{Int}(U))\\ &= \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A) \cap U))\\ &= \mathrm{Cl}(\mathrm{Int}((A^* \cup A) \cap U))\\ &= \mathrm{Cl}(\mathrm{Int}((A^* \cap U) \cup (A \cap U)))\\ &\subset \mathrm{Cl}(\mathrm{Int}((A \cap U)^* \cup (A \cap U)))\\ &= \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A \cap U))).\end{aligned}$$

This shows that $A \cap U \in \beta IO(X, \tau)$. □

Definition 3.1. *A subset F of a space (X, τ, I) is said to be β-I-closed if its complement is β-I-open.*

Theorem 3.3. *A subset A of a space (X, τ, I) is β-I-closed if and only if*

$$\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}^*(A))) \subset A.$$

Proof. Let A be a β-I-closed set of (X, τ, I). Then $X - A$ is β-I-open and hence

$$X - A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(X - A))) = X - \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}^*(A))).$$

Therefore, we have $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}^*(A))) \subset A$.

Conversely, let $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}^*(A))) \subset A$. Then $X - A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(X - A)))$ and hence $X - A$ is β-I-open. Therefore, A is β-I-closed. □

Remark 3.1. For a subset A of a space (X, τ, I), we have

$$X - \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(A))) \neq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(X - A)))$$

as shown by the following example.

Example 3.3. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $I = \{\emptyset, \{a\}\}$. Then if we put $A = \{a, c\}$, $X - \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(A))) = \{b, c\}$ and $\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(X - A))) = \emptyset$.

Theorem 3.4. *If a subset A of a space (X, τ, I) is β-I-closed, then*

$$\mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(A))) \subset A.$$

Proof. Let A be any β-I-closed set of (X, τ, I). Since $\tau * (I)$ is finer than τ, we have

$$\mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(A))) \subset \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}^*(A))) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}^*(A))).$$

Therefore, by Theorem 3.3, we obtain $\mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(A))) \subset A$. □

Corollary 3.1. *Let A be a subset of a space (X, τ, I) such that*

$$X - \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(A))) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(X - A))).$$

Then A is β-I-closed if and only if $\mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(A))) \subset A$.

Proof. This is an immediate consequence of Theorem 3.3. □

4. β-I-continuous functions

Definition 4.1. *A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be β-I-continuous* [5] *(resp. almost- I -continuous* [3]*, β-continuous* [1]*) if $f^{-1}(V)$ is β-I-open (resp. almost-I-open, β-open) in (X, τ, I) for each open set V of (Y, σ).*

Remark 4.1. It is obvious from Lemmas 3.1 and 3.2 that almost-I-continuity implies β-I-continuity and β-I-continuity implies β-continuity.

Theorem 4.1. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following conditions are equivalent:*

(a) *f is β-I-continuous,*
(b) *For each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $U \in \beta IO(X, \tau)$ containing x such that $f(U) \subset V$,*
(c) *The inverse image of each closed set in Y is β-I-closed.*

Proof. Straightforward. □

Definition 4.2. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be β-I-irresolute if $f^{-1}(V)$ is β-I-open for every β-J-open set V of (Y, σ, J).*

Theorem 4.2. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ and $g : (Y, \sigma, J) \to (Z, \eta)$ be two functions, where I and J are ideals on X and Y respectively. Then*

(a) *gof is β-I-continuous if f is β-I-continuous and g is continuous,*
(b) *gof is β-I-continuous if f is β-I-irresolute and g is β-I-continuous.*

If (X, τ, I) is an ideal topological space and A is subset of X, we denote by $\tau_{|A}$ the relative topology on A and $I_{|A} = \{A \cap I | I \in I\}$ is obviously an ideal on A.

Lemma 4.1. [7] *Let (X, τ, I) be an ideal topological space and B, A subsets of X such that $B \subset A$. Then $B^*(\tau_{|A}, I_{|A}) = B^*(\tau, I) \cap A$.*

Theorem 4.3. *Let (X, τ, I) be an ideal topological space. If $U \in \tau$ and $A \in \beta IO(X, \tau)$, then $U \cap A \in \beta IO(U, \tau_{|U}, I_{|U})$.*

Proof. Since $U \in \tau$ and $A \in \beta IO(X, \tau)$, by Theorem 3.2 we have

$$A \cap U \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A \cap U))$$

and hence

$$\begin{aligned} A \cap U &\subset U \cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A \cap U))) \\ &\subset \mathrm{Cl}(U \cap \mathrm{Int}(\mathrm{Cl}^*(A \cap U))) \\ &\subset \mathrm{Cl}(\mathrm{Int}[U \cap \mathrm{Cl}^*(A \cap U)]) \\ &= \mathrm{Cl}(\mathrm{Int}_U(U \cap \mathrm{Cl}^*(A \cap U))). \end{aligned}$$

Since $U \in \tau \subset \tau*$, we obtain

$$A \cap U \subset U \cap \mathrm{Cl}(\mathrm{Int}_U(\mathrm{Cl}^*_U(A \cap U))) = \mathrm{Cl}_U(\mathrm{Int}_U(\mathrm{Cl}^*_U(A \cap U))).$$

This shows that $A \cap U \in \beta IO(U, \tau_{|U}, I_{|U})$. □

Theorem 4.4. *Let $f : (X, \tau, I) \to (Y, \sigma)$ be β-I-continuous function and $U \in \tau$. Then the restriction $f_{|U} : (U, \tau_{|U}, I_{|U}) \to (Y, \sigma)$ is β-I-continuous.*

Proof. Let V be any open set of (Y, σ). Since f is β-I-continuous, we have $f^{-1}(V) \in \beta IO(X, \tau)$. Since $U \in \tau$, by Theorem 4.3 $U \cap f^{-1}(V) \in \beta IO(U, \tau_{|U}, I_{|U})$. On the other hand, $(f_{|U})^{-1}(V) = U \cap f^{-1}(V)$ and $(f_{|U})^{-1}(V) \in \beta IO(U, \tau_{|U}, I_{|U})$. This shows that $f_{|U} : (U, \tau_{|U}, I_{|U}) \to (Y, \sigma)$ is β-I-continuous. □

Theorem 4.5. *A function $f : (X, \tau, I) \to (Y, \sigma)$ be β-I-continuous if and only if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is β-I-continuous.*

Proof. Necessity. Suppose that f is β-I-continuous. Let $x \in X$ and W be any open set of $X \times Y$ containing $g(x)$. Then there exists a basic open set $U \times V$ such that $g(x) = (x, f(x)) \in U \times V \subset W$. Since f is β-I-continuous, there exists a β-I-open set U_o of X containing x such that $f(U_o) \subset V$. By Theorem 3.2, $U_o \cap U \in \beta IO(X, \tau)$ and $g(U_o \cap U) \subset U \times V \subset W$. This shows that g is β-I-continuous.

Sufficiency. Suppose that g is β-I-continuous. Let $x \in X$ and V be any open set of Y containing $f(x)$. Then $X \times V$ is open in $X \times Y$ and by β-I-continuity of g, there exists $U \in \beta IO(X, \tau)$ containing x such that $g(U) \subset X \times V$. Therefore, we obtain $f(U) \subset V$. This shows that f is β-I-continuous. □

5. A decomposition of almost-I-continuity

Definition 5.1. *A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be *-I-continuous* [4] *if the preimage of every open set in (Y, σ) is *-dense-in-itself in (X, τ, I).*

Theorem 5.1. *For a subset A of an ideal topological space (X, τ, I), the following conditions are equivalent:*

(a) *A is almost-I-open,*
(b) *A is β-I-open and *-dense-in-itself.*

Proof. $(a) \Rightarrow (b)$. By Lemma 3.1, every almost-I-open set is β-I-open. On the other hand, by Lemma 2.1 we have $A \subset \mathrm{Cl}(\mathrm{Int}(A^*)) \subset \mathrm{Cl}(A^*) = A^*$. This shows that A is *-dense-in-itself.

$(b) \Rightarrow (a)$. By the assumption,

$$A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(A))) = \mathrm{Cl}(\mathrm{Int}(A^* \cup A)) = \mathrm{Cl}(\mathrm{Int}(A^*)).$$

This shows that A is almost-I-open. □

Thus we have the following decomposition of almost-I-continuity.

Theorem 5.2. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following conditions are equivalent:*

(a) *f is almost-I-continuous,*
(b) *f is β-I-continuous and *-I-continuous.*

Proof. This is an immediate consequence from Theorem 5.1. □

Remark 5.1. β-I-continuity and *-I-continuity are independent notions as shown by the following example due to Dontchev [4].

Example 5.1. Let $X = \{a, b, c\}$, $I = \{\emptyset, \{c\}\}$, $\tau = \{\emptyset, X, \{b\}\}$, $\sigma = \{\emptyset, X, \{c\}\}$ and $\gamma = \{\emptyset, X, \{a\}\}$. The identity function $f : (X, \tau, I) \to (X, \gamma, I)$ is *-I-continuous but neither almost-I-continuous nor β-I-continuous since $f^{-1}(\{a\}) = \{a\}$ and $\{a\}^* = \{a, c\}$. On the other hand, the identity function $g : (X, \sigma, I) \to (X, \sigma, I)$ is β-I-continuous but neither almost-I-continuous nor *-I-continuous since $f^{-1}(\{c\}) = \{c\}$ and $\{c\}^* = \emptyset$.

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