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Chromatically Unique Bipartite Graphs with Certain 3-independent Partition Numbers II

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Abstract. For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}_2^{-s}(p,q)$ denote the set of $2-$connected bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges. In this paper, we prove that for any graph $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$ and $1 \leq s \leq q-1$, if the number of 3-independent partitions of G is $2^{p-1}+2^{q-1}+s+4$, then G is chromatically unique. This result extends the similar theorem by Dong et al. [Discrete Math. **224**(2000) 107–124], and the result in [4].

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1. Introduction

All graphs considered here are simple graphs. For a graph G, let $V(G)$, $E(G)$, $\delta(G)$, $\Delta(G)$ and $P(G,\lambda)$ be the vertex set, edge set, minimum degree, maximum degree and the chromatic polynomial of G, respectively. Two graphs G and H are said to be *chromatically equivalent* (or simply $\chi-$*equivalent*), symbolically $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply $\chi-$*unique*) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be $\chi-$*closed*. For two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs, if $P(G_1,\lambda) \neq P(G_2,\lambda)$ for every $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be *chromatically disjoint*, or simply $\chi-$*disjoint*.

For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}^{-s}(p,q)$ (resp. $\mathcal{K}_2^{-s}(p,q)$) denote the set of connected (resp. $2-$connected) bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of s edges.

For a bipartite graph $G = (A,B;E)$ with bipartition A and B and edge set E, let $G' = (A',B';E')$ be the bipartite graph induced by the edge set $E' = \{xy \mid xy \notin$

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E, $x \in A, y \in B$ }, where $A' \subseteq A$ and $B' \subseteq B$. We write $G' = K_{p,q} - G$, where $p = |A|$ and $q = |B|$.

In [1], Dong et al. proved the following result.

Theorem 1.1. *For integers p, q, s with $p \geq q \geq 2$ and $0 \leq s \leq q-1$, $\mathcal{K}_2^{-s}(p,q)$ is $\chi-$closed.*

Throughout this paper, we fix the following conditions for p, q and s:

$$p \geq q \geq 3 \qquad \text{and} \qquad 1 \leq s \leq q-1.$$

For a graph G and a positive integer k, a partition $\{ A_1, A_2, \ldots, A_k \}$ of $V(G)$ is called a k-independent partition in G if each A_i is a non-empty independent set of G. Let $\alpha(G,k)$ denote the number of $k-$*independent partitions* in G.

For any bipartite graph $G = (A, B; E)$, define

$$\alpha'(G,3) = \alpha(G,3) - (2^{|A|-1} + 2^{|B|-1} - 2).$$

In [1], the authors found the following sharp bounds for $\alpha'(G,3)$.

Theorem 1.2. *For $G \in \mathcal{K}^{-s}(p,q)$ with $p \geq q \geq 3$ and $0 \leq s \leq q-1$,*

$$s \leq \alpha'(G,3) \leq 2^s - 1,$$

where $\alpha'(G,3) = s$ iff $\Delta(G') = 1$ and $\alpha'(G,3) = 2^s - 1$ iff $\Delta(G') = s$.

For $t = 0, 1, 2, \ldots$, let $\mathcal{B}(p,q,s,t)$ denote the set of graphs $G \in \mathcal{K}^{-s}(p,q)$ with $\alpha'(G,3) = s + t$. Thus, $\mathcal{K}^{-s}(p,q)$ is partitioned into the following subsets:

$$\mathcal{B}(p,q,s,0), \quad \mathcal{B}(p,q,s,1), \quad \ldots, \mathcal{B}(p,q,s,2^s - s - 1).$$

Assume that $\mathcal{B}(p,q,s,t) = \emptyset$ for $t > 2^s - s - 1$.

Lemma 1.1. (Dong et al. [2]) *For $p \geq q \geq 3$ and $0 \leq s \leq q-1$, if $0 \leq t \leq 2^{q-1} - q - 1$, then*

$$\mathcal{B}(p,q,s,t) \subseteq \mathcal{K}_2^{-s}(p,q).$$

Dong et al. [1] have shown that if G is a 2-connected graph in $\mathcal{B}(p,q,s,0) \cup \mathcal{B}(p,q,s,2^s - s - 1)$, then G is $\chi-$unique. In [2], Dong et al. proved that every 2-connected graph in $\mathcal{B}(p,q,s,t)$ is $\chi-$unique for $1 \leq t \leq 4$. In [4], we extended this result for $t = 5$. In this paper, we prove the chromatic uniqueness of graphs in $\mathcal{B}(p,q,s,6)$.

2. Preliminary results and notation

For any graph G of order n, we have (see [3]):

$$P(G,\lambda) = \sum_{k=1}^{n} \alpha(G,k)\lambda(\lambda-1)\cdots(\lambda-k+1).$$

Thus, we have

Lemma 2.1. *If $G \sim H$, then $\alpha(G,k) = \alpha(H,k)$ for $k = 1, 2, \ldots$.*

By Theorem 1.1, the following two results were obtained in [2].

Theorem 2.1. *The set $\mathcal{B}(p,q,s,t) \cap \mathcal{K}_2^{-s}(p,q)$ is $\chi-$closed for all $t \geq 0$.*

Corollary 2.1. *If* $0 \le t \le 2^{q-1} - q - 1$, *then* $\mathcal{B}(p,q,s,t)$ *is* $\chi-$*closed.*

Let $\beta_i(G)$, or simply β_i, denote the number of vertices in G with degree i, $n_i(G)$ denote the number of i-cycles in G and P_n denote the path with n vertices. Then Dong et al. [2] established the next two results.

Lemma 2.2. *For* $G = (A,B;E) \in \mathcal{K}^{-s}(p,q)$,

(i) *if* $\Delta(G') \le 2$, *then* $\alpha'(G,3) = s + \beta_2(G') + n_4(G')$;
(ii) *if* $\Delta(G') = 3$, *then* $\alpha'(G,3) \ge s + \beta_2(G') + 4\beta_3(G') + n_4(G')$, *where equality holds iff* $|N_{G'}(u) \cap N_{G'}(v)| \le 2$ *for all* $u, v \in A'$ *or* $u, v \in B'$;
(iii) $\alpha'(G,3) \ge 2^{\Delta(G')} + s - 1 - \Delta(G')$.

For two disjoint graphs H_1 and H_2, let $H_1 \cup H_2$ denote the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. Let $kH = \underbrace{H \cup \cdots \cup H}_{k}$ for $k \ge 1$ and let kH be null if $k = 0$.

Lemma 2.3. *Let* $G \in \mathcal{K}^{-s}(p,q)$. *If* $\alpha'(G,3) = s + t \le s + 4$, *then either*

(i) *each component of* G' *is a path and* $\beta_2(G') = t$, *or*
(ii) $G' \cong K_{1,3} \cup (s-3)K_2$.

Now for convenient we define the graphs Y_n and Z_1 as in Figure 1.

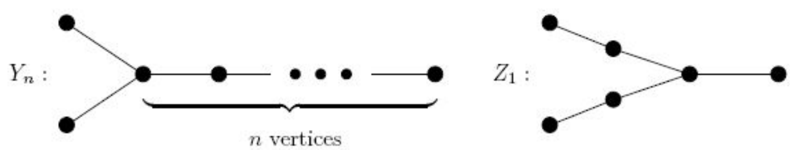

Figure 1. The graphs Y_n and Z_1

The following result is an extension of Lemma 2.3.

Lemma 2.4. *Let* $G \in \mathcal{K}^{-s}(p,q)$. *If* $\alpha'(G,3) = s + 6$, *then one of the following holds.*

(i) *each component of* G' *is a path and* $\beta_2(G') = 6$,
(ii) $G' \cong K_{1,3} \cup P_4 \cup (s-6)K_2$,
(iii) $G' \cong K_{1,3} \cup 2P_3 \cup (s-7)K_2$,
(iv) $G' \cong C_4 \cup P_3 \cup (s-6)K_2$,
(v) $G' \cong Y_4 \cup (s-5)K_2$,
(vi) $G' \cong Y_3 \cup P_3 \cup (s-6)K_2$,
(vii) $G' \cong Z_1 \cup (s-5)K_2$,
(viii) $G' \cong C_6 \cup (s-6)K_2$.

Proof. Since $\alpha'(G,3) = s + 6$, $\Delta(G') \le 3$ by Lemma 2.2(iii). If $\Delta(G') \le 3$, by Lemma 2.2(ii), we have $\beta_2(G') = 2$, $n_4(G') = 0$ and $\beta_3(G') = 1$. Thus $G' \cong K_{1,3} \cup P_4 \cup (s-6)K_2$, or $G' \cong K_{1,3} \cup 2P_3 \cup (s-7)K_2$, or $G' \cong Y_4 \cup (s-5)K_2$, or $G' \cong Y_3 \cup P_3 \cup (s-6)K_2$, or $G' \cong Z_1 \cup (s-5)K_2$. If $\Delta(G') = 2$, we have $\beta_2(G') + n_4(G') = 6$ by Lemma 2.2(i), and thus either G' contain no cycles or only have one cycle. Hence, when $\Delta(G')=2$, either each component of G' is a path, and $\beta_2(G')=6$, or $G'\cong C_4\cup P_3\cup(s-6)K_2$, or $G'\cong C_6\cup(s-6)K_2$, by Lemma 2.2(i). $\square$

By Lemma 2.4, we establish the following result.

Theorem 2.2. *Let $G\in\mathcal{K}^{-s}(p,q)$ and $\alpha'(G,3)=s+6$, then*

$$\begin{aligned}G'\in\Big\{\ & P_8\cup(s-7)K_2,\ P_7\cup P_3\cup(s-8)K_2,\ P_6\cup 2P_3\cup(s-9)K_2,\\ & 2P_5\cup(s-8)K_2,\ P_5\cup 3P_3\cup(s-10)K_2,\ P_4\cup P_6\cup(s-8)K_2,\\ & P_4\cup P_5\cup P_3\cup(s-9)K_2,\ 3P_4\cup(s-9)K_2,\ 2P_4\cup 2P_3\cup(s-10)K_2,\\ & P_4\cup 4P_3\cup(s-11)K_2,\ 6P_3\cup(s-12)K_2,\ K_{1,3}\cup P_4\cup(s-6)K_2,\\ & K_{1,3}\cup 2P_3\cup(s-7)K_2,\ C_4\cup P_3\cup(s-6)K_2,\ Y_4\cup(s-5)K_2,\\ & Y_3\cup P_3\cup(s-6)K_2,\ Z_1\cup(s-5)K_2,\ C_6\cup(s-6)K_2\ \Big\}\end{aligned}$$

where $H\cup(s-i)K_2$ does not exist if $s<i$.

For a bipartite graph $G=(A,B;E)$, let

$$\Omega(G)=\{\,Q\mid Q \text{ is an independent sets in } G \text{ with } Q\cap A\neq\emptyset, Q\cap B\neq\emptyset\,\}.$$

For a bipartite graph $G=(A,B;E)$, the number of 4−independent partitions $\{A_1,A_2,A_3,A_4\}$ in G with $A_i\subseteq A$ or $A_i\subseteq B$ for all $i=1,2,3,4$ is

$$\begin{aligned}&(2^{|A|-1}-1)(2^{|B|-1}-1)+\frac{1}{3!}(3^{|A|}-3\cdot 2^{|A|}+3)+\frac{1}{3!}(3^{|B|}-3\cdot 2^{|B|}+3)\\ &\quad=(2^{|A|-1}-2)(2^{|B|-1}-2)+\frac{1}{2}(3^{|A|-1}+3^{|B|-1})-2.\end{aligned}$$

Define

$$\alpha'(G,4)=\alpha(G,4)-\{\,(2^{|A|-1}-2)(2^{|B|-1}-2)+\frac{1}{2}(3^{|A|-1}+3^{|B|-1})-2\,\}.$$

Observe that for $G,\ H\in\mathcal{K}^{-s}(p,q)$,

$$\alpha(G,4)=\alpha(H,4)\quad\text{iff}\quad\alpha'(G,4)=\alpha'(H,4).$$

The following five lemmas (see [2]) will be used to prove our main results.

Lemma 2.5. *For $G=(A,B;E)\in\mathcal{K}^{-s}(p,q)$ with $|A|=p$ and $|B|=q$,*

$$\begin{aligned}\alpha'(G,4)\ =\ &\sum_{Q\in\Omega(G)}(2^{p-1-|Q\cap A|}+2^{q-1-|Q\cap B|}-2)+\\ &\Big|\{\,\{\,Q_1,Q_2\,\}\mid Q_1,Q_2\in\Omega(G),Q_1\cap Q_2=\emptyset\,\}\Big|.\end{aligned}$$

Lemma 2.6. *For a bipartite graph $G=(A,B;E)$, if uvw is a path in G' with $d_{G'}(u)=1$ and $d_{G'}(v)=2$, then for any $k\geq 2$,*

$$\alpha(G,k)=\alpha(G+uv,k)+\alpha(G-\{u,v\},k-1)+\alpha(G-\{u,v,w\},k-1).$$

For a bipartite graph $G = (A, B; E)$, let $\beta_i(G, A)$ (resp., $\beta_i(G, B)$) be the number of vertices in A (resp., B) with degree i.

Lemma 2.7. *For $G \in \mathcal{B}(p, q, s, t)$, if each component of G' is a path, then*

$$\sum_{Q \in \Omega(G)} (2^{p-1-|Q \cap A|} + 2^{q-1-|Q \cap B|} - 2)$$
$$= s(2^{p-2} + 2^{q-2} - 2) + t(2^{p-3} + 2^{q-2} - 2) + (2^{p-3} + 2^{q-3})\beta_2(G', A').$$

Let $p_i(G)$ denote the number of paths P_i in G.

Lemma 2.8. *For $G \in \mathcal{B}(p, q, s, t)$, if each component of G' is a path, then*

$$\{ \{ Q_1, Q_2 \} \mid Q_1, Q_2 \in \Omega(G), Q_1 \cap Q_2 = \emptyset \}| = \binom{s+t}{2} - 3t - 3p_4(G') - p_5(G').$$

For $G \in \mathcal{B}(p, q, s, t)$, define

$$\alpha''(G, 4) \quad = \quad \alpha'(G, 4) - \Big[s(2^{p-2} + 2^{q-2} - 2) + t(2^{p-3} + 2^{q-2} - 2) +$$

$$(s + t)(s + t - 1)/2 - 3t \Big]. \tag{2.1}$$

Observe that for G, $H \in \mathcal{B}(p, q, s, t)$,

$$\alpha''(G, 4) = \alpha''(H, 4) \quad \text{iff} \quad \alpha(G, 4) = \alpha(H, 4).$$

Lemma 2.9. *For $G \in \mathcal{B}(p, q, s, t)$, if each component of G' is a path, then*

$$\alpha''(G, 4) = (2^{p-3} - 2^{q-3})\beta_2(G', A') - 3p_4(G') - p_5(G').$$

3. Main result

Dong et al. [1] have shown that any graph G in $\mathcal{B}(p, q, s, 0) \cup \mathcal{B}(p, q, s, 2^s - s - 1)$, if G is 2-connected, is $\chi-$unique. In [2], Dong et al. proved that every 2-connected graph in $\mathcal{B}(p, q, s, t)$ is $\chi-$unique for $1 \le t \le 4$. In [4], we proved this result for $t = 5$. In this section, we shall prove that every 2-connected graph in $\mathcal{B}(p, q, s, t)$ is $\chi-$unique for $t = 6$.

The following theorem is our main result.

Theorem 3.1. *Let p, q and s be integers with $p \ge q \ge 3$ and $0 \le s \le q - 1$. For every $G \in \mathcal{B}(p, q, s, 6)$, if G is 2-connected, then G is $\chi-$unique.*

Proof. By Theorem 2.1, $\mathcal{B}(p, q, s, t) \cap \mathcal{K}_2^{-s}(p, q)$ is $\chi-$closed for all $t \ge 0$. Hence, to show that every 2-connected graph in $\mathcal{B}(p, q, s, t)$ is $\chi-$unique, it suffices to show that for every two graphs G and H in $\mathcal{B}(p, q, s, t)$, if $G \not\cong H$, then either $\alpha(G, 4) \ne \alpha(H, 4)$ or $\alpha(G, 5) \ne \alpha(H, 5)$. Recall that $\alpha''(G, 4) = \alpha''(H, 4)$ iff $\alpha(G, 4) = \alpha(H, 4)$ and $\alpha'(G, 4) = \alpha'(H, 4)$ iff $\alpha(G, 4) = \alpha(H, 4)$.

The set $\mathcal{B}(p, q, s, 6)$ contains 59 graphs by Theorem 2.2, named as $G_{6,1}$, $G_{6,2}$, $G_{6,3}, \ldots, G_{6,59}$ (see Table 1). These graphs are shown in this table with the values $\alpha''(G_{6,1}, 4), \alpha''(G_{6,2}, 4), \ldots, \alpha''(G_{6,59}, 4)$. For each graph $G_{6,i}$, if every component of $G'_{6,i}$ is a path, then $\alpha''(G_{6,i}, 4)$ can be obtained by Lemma 2.9; otherwise, we must find $\alpha'(G_{6,i}, 4)$ by Lemma 2.5, and then we find $\alpha''(G_{6,i}, 4)$ by using Equation (2.1).

Partition $\mathcal{B}(p,q,s,6)$ into 13 subsets:

$$\begin{aligned}
\mathcal{T}_1 &= \{\ G_{6,1}, G_{6,2}, G_{6,3}\ \}\\
\mathcal{T}_2 &= \{\ G_{6,4}, G_{6,5}G_{6,6}, G_{6,7}\ \}\\
\mathcal{T}_3 &= \{\ G_{6,8}, G_{6,9}, G_{6,10}, G_{6,11}\ \}\\
\mathcal{T}_4 &= \{\ G_{6,12}, G_{6,13}, G_{6,14}\ \}\\
\mathcal{T}_5 &= \{\ G_{6,15}, G_{6,16}, G_{6,17}, G_{6,18}\ \}\\
\mathcal{T}_6 &= \{\ G_{6,19}, G_{6,20}, G_{6,21}, G_{6,22}, G_{6,23}, G_{6,24}, G_{6,25}\ \}\\
\mathcal{T}_7 &= \{\ G_{6,26}, G_{6,27}, G_{6,28}, G_{6,29}, G_{6,30}, G_{6,31}, G_{6,32}, G_{6,33}\ \}\\
\mathcal{T}_8 &= \{\ G_{6,34}, G_{6,35}, G_{6,36}\ \}\\
\mathcal{T}_9 &= \{\ G_{6,37}, G_{6,38}, G_{6,39}, G_{6,40}, G_{6,41}\ \}\\
\mathcal{T}_{10} &= \{\ G_{6,42}, G_{6,43}, G_{6,44}, G_{6,45}, G_{6,46}, G_{6,47}, G_{6,48}\ \}\\
\mathcal{T}_{11} &= \{\ G_{6,49}, G_{6,50}, G_{6,51}, G_{6,52}\ \}\\
\mathcal{T}_{12} &= \{\ G_{6,53}, G_{6,54}, G_{6,55}\ \}\\
\mathcal{T}_{13} &= \{\ G_{6,56}, G_{6,57}, G_{6,58}, G_{6,59}\ \}
\end{aligned}$$

For non-empty sets $W_1, W_2, \ldots, W_k$ of graphs, let $\eta(W_1, W_2, \ldots, W_k) = 0$ if $\alpha(G_1, 4) \neq \alpha(G_2, 4)$ for every two graphs $G_1 \in W_i$ and $G_2 \in W_j$, where $i \neq j$, and let $\eta(W_1, W_2, \ldots, W_k) = 1$ otherwise.

The values of $\alpha''(G_{6,i}, 4)$ for $i = 2, 3;\ 20, \ldots, 25;\ 49, \ldots, 59$ are not given by Lemma 2.9, but they can be obtained by Lemma 2.5 and Equation (2.1). For example, we show a detail computation of $\alpha''(G_{6,2}, 4)$ below.

$$\begin{aligned}
\text{(i)}\ \ \alpha''(G_{6,2},4) &= \Bigg[\ s(2^{p-2}+2^{q-2}-2)+3(2^{p-3}+2^{q-2}-2)+2(2^{q-3}+2^{p-2}-2)+\\
&\qquad (2^{p-3}+2^{q-3}-2)+\left\{\ 12(s-6)+\binom{s-6}{2}+29\ \right\}\Bigg]-\\
&\qquad \left[\ s(2^{p-2}+2^{q-2}-2)+6(2^{p-3}+2^{q-2}-2)+\binom{s+6}{2}-18\ \right]\\
&= 2^{p-2}-3\cdot 2^{q-3}-19.
\end{aligned}$$

Similarly, we obtained $\alpha''(G_{6,i}, 4)$ for other values of i.

(ii) $\alpha''(G_{6,3}, 4) = 3 \cdot 2^{p-3} - 2^{q-1} - 19.$
(iii) $\alpha''(G_{6,20}, 4) = -2^{p-4} - 9.$
(iv) $\alpha''(G_{6,21}, 4) = 2^{p-4} - 2^{q-3} - 9.$
(v) $\alpha''(G_{6,22}, 4) = 3 \cdot 2^{p-4} - 2^{q-2} - 9.$
(vi) $\alpha''(G_{6,23}, 4) = 2^{p-1} - 9 \cdot 2^{q-4} - 9.$
(vii) $\alpha''(G_{6,24}, 4) = 5 \cdot 2^{p-3} - 11 \cdot 2^{q-4} - 9.$
(viii) $\alpha''(G_{6,25}, 4) = 6 \cdot 2^{p-3} - 13 \cdot 2^{q-4} - 9.$
(ix) $\alpha''(G_{6,49}, 4) = 3 \cdot 2^{p-4} - 2^{q-2} - 28.$
(x) $\alpha''(G_{6,50}, 4) = 4 \cdot 2^{p-3} - 9 \cdot 2^{q-4} - 28.$
(xi) $\alpha''(G_{6,51}, 4) = 2^{p-4} - 2^{q-3} - 12.$
(xii) $\alpha''(G_{6,52}, 4) = 5 \cdot 2^{p-3} - 11 \cdot 2^{q-4} - 12.$
(xiii) $\alpha''(G_{6,53}, 4) = 3 \cdot 2^{p-3} - 3 \cdot 2^{q-3} - 24.$

(xiv) $\alpha''(G_{6,54}, 4) = 2^{p-4} - 2^{q-3} - 24$.
(xv) $\alpha''(G_{6,55}, 4) = 5 \cdot 2^{p-3} - 11 \cdot 2^{q-4} - 24$.
(xvi) $\alpha''(G_{6,56}, 4) = 2^{p-4} - 2^{q-3} - 18$.
(xvii) $\alpha''(G_{6,57}, 4) = 5 \cdot 2^{p-3} - 11 \cdot 2^{q-4} - 18$.
(xiix) $\alpha''(G_{6,58}, 4) = 3 \cdot 2^{p-4} - 2^{q-2} - 18$.
(xix) $\alpha''(G_{6,59}, 4) = 2^{p-1} - 9 \cdot 2^{q-4} - 18$.

and we showed them in Table 1.

Claim 1. $\eta(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \ldots, \mathcal{T}_{13}) = 0$.

Proof of Claim 1. Note that if 2^k (k is an integer ≥ 1) is not a factor of x, then 2^h is also not a factor of x for any integer $h \geq k$. Similarly, if 2^k (k is an integer ≥ 1) is a factor of x, then 2^h is also a factor of x for any integer $1 \leq h \leq k$.

(a) For $s \leq 5$, only $\mathcal{T}_{11}$ and $\mathcal{T}_{12}$ are non-empty. Observe that $\alpha''(G_{6,i}, 4) \neq \alpha''(G_{6,j}, 4)$ for $G_{6,i} \in \mathcal{T}_{11}$ ($i = 49, 50$) and $G_{6,j} \in \mathcal{T}_{12}$ ($j = 54, 55$). Hence $\eta(\mathcal{T}_{11}, \mathcal{T}_{12}) = 0$.
(b) For $s \geq 6$, $\alpha''(G, 4)$ is odd if $G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_4 \cup \mathcal{T}_6 \cup \mathcal{T}_7 \cup \mathcal{T}_9$ and even if $G \in \mathcal{T}_3 \cup \mathcal{T}_5 \cup \mathcal{T}_8 \cup \mathcal{T}_{10} \cup \mathcal{T}_{11} \cup \mathcal{T}_{12} \cup \mathcal{T}_{13}$. Hence $\eta(\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_4 \cup \mathcal{T}_6 \cup \mathcal{T}_7 \cup \mathcal{T}_9, \mathcal{T}_3 \cup \mathcal{T}_5 \cup \mathcal{T}_8 \cup \mathcal{T}_{10} \cup \mathcal{T}_{11} \cup \mathcal{T}_{12} \cup \mathcal{T}_{13}) = 0$.
(c) For $s \geq 6$, 2^4 is a factor of $\alpha''(G, 4) + 19$ for $G \in \mathcal{T}_1 \cup \mathcal{T}_9$, 2^4 is not a factor of $\alpha''(G, 4) + 19$ for $G \in \mathcal{T}_2 \cup \mathcal{T}_4 \cup \mathcal{T}_6 \cup \mathcal{T}_7$. Hence $\eta(\mathcal{T}_1 \cup \mathcal{T}_9, \mathcal{T}_2 \cup \mathcal{T}_4 \cup \mathcal{T}_6 \cup \mathcal{T}_7) = 0$.
(d) For $s \geq 6$, 2^5 is a factor of $\alpha''(G, 4) + 3$ for $G \in \mathcal{T}_9$, but 2^5 is not a factor of $\alpha''(G, 4) + 3$ for $G \in \mathcal{T}_1$. Hence $\eta(\mathcal{T}_1, \mathcal{T}_9) = 0$
(e) For $s \geq 6$, 2^4 is a factor of $\alpha''(G, 4) + 15$ for $G \in \mathcal{T}_2$, 2^2 is not a factor of $\alpha''(G, 4) + 15$ for $G \in \mathcal{T}_6$, and 2^2 is a factor of $\alpha''(G, 4) + 15$ but 2^4 is not for $G \in \mathcal{T}_4 \cup \mathcal{T}_7$. Hence $\eta(\mathcal{T}_2, \mathcal{T}_6, \mathcal{T}_4 \cup \mathcal{T}_7) = 0$.
(f) For $s \geq 6$, 2^3 is a factor of $\alpha''(G, 4) + 11$ for $G \in \mathcal{T}_4$ but 2^3 is not a factor of $\alpha''(G, 4) + 11$ for $G \in \mathcal{T}_7$. Hence $\eta(\mathcal{T}_4, \mathcal{T}_7) = 0$.
(g) For $s \geq 6$, 2^2 is a factor of $\alpha''(G, 4) + 28$ for $G \in \mathcal{T}_{10} \cup \mathcal{T}_{11} \cup \mathcal{T}_{12}$, but 2^2 is not a factor of $\alpha''(G, 4) + 28$ for $G \in \mathcal{T}_3 \cup \mathcal{T}_5 \cup \mathcal{T}_8 \cup \mathcal{T}_{13}$. Hence $\eta(\mathcal{T}_{10} \cup \mathcal{T}_{11} \cup \mathcal{T}_{12}, \mathcal{T}_3 \cup \mathcal{T}_5 \cup \mathcal{T}_8 \cup \mathcal{T}_{13}) = 0$.
(h) For $s \geq 6$, 2^4 is a factor of $\alpha''(G, 4)$ for $G \in \mathcal{T}_{10}$, 2^3 is not a factor of $\alpha''(G, 4)$ for $G \in \mathcal{T}_{11}$, and 2^3 is a factor of $\alpha''(G, 4)$ but 2^4 is not for $G \in \mathcal{T}_{12}$. Hence $\eta(\mathcal{T}_{10}, \mathcal{T}_{11}, \mathcal{T}_{12}) = 0$.
(i) For $s \geq 6$, 2^3 is a factor of $\alpha''(G, 4) + 18$ for $G \in \mathcal{T}_5 \cup \mathcal{T}_{13}$ but 2^3 is not a factor of $\alpha''(G, 4) + 18$ for $G \in \mathcal{T}_3 \cup \mathcal{T}_8$. Hence $\eta(\mathcal{T}_5 \cup \mathcal{T}_{13}, \mathcal{T}_3 \cup \mathcal{T}_8) = 0$.
(j) For $s \geq 6$, 2^4 is a factor of $\alpha''(G, 4) + 10$ for $G \in \mathcal{T}_5$ but 2^4 is not a factor of $\alpha''(G, 4) + 10$ for $G \in \mathcal{T}_{13}$. Hence $\eta(\mathcal{T}_5, \mathcal{T}_{13}) = 0$.
(k) For $s \geq 6$, 2^4 is a factor of $\alpha''(G, 4) + 14$ for $G \in \mathcal{T}_3$ but it is not for every $G \in \mathcal{T}_8$. Hence $\eta(\mathcal{T}_3, \mathcal{T}_8) = 0$.

By (a) – (k), Claim 1 holds.

The remaining work is to compare every two graphs in each $\mathcal{T}_i$ for $1 \leq i \leq 13$. Since this comparison process is standard, long and rather repetitive, we shall not discuss all here. In the following we only show detail camparisons of every two graphs in $\mathcal{T}_i$ for $i = 1, 2$ and 3. The reader may refer to [5] for complete comparisons.

(1) $\mathcal{T}_1$

(1.1) When $p = q$, $G_{6,2} \cong G_{6,3}$ and

$$\begin{aligned}\alpha''(G_{6,1},4) - \alpha''(G_{6,2},4) &\\ &= (3\cdot 2^{p-3} - 3\cdot 2^{q-3} - 19) - (2^{p-2} - 3\cdot 2^{q-3} - 19)\\ &= 2^{p-3} > 0.\end{aligned}$$

(1.2) When $p > q$, we have

$$\begin{aligned}\alpha''(G_{6,2},4) - \alpha''(G_{6,3},4) &\\ &= (2^{p-2} - 3\cdot 2^{q-3} - 19) - (3\cdot 2^{p-3} - 2^{q-1} - 19)\\ &= -2^{p-3} + 2^{q-3} < 0\end{aligned}$$

$$\begin{aligned}\alpha''(G_{6,3},4) - \alpha''(G_{6,1},4) &\\ &= (3\cdot 2^{p-3} - 2^{q-1} - 19) - (3\cdot 2^{p-3} - 3\cdot 2^{q-3} - 19)\\ &= -2^{q-3} < 0.\end{aligned}$$

Therefore,

$$\alpha''(G_{6,2},4) < \alpha''(G_{6,3},4) < \alpha''(G_{6,1},4).$$

(2) $\mathcal{T}_2$

(2.1) When $p = q$, $G_{6,4} \cong G_{6,7}$, $G_{6,5} \cong G_{6,6}$ and we can easily see that $\alpha''(G_{6,4},4) = \alpha''(G_{6,5},4)$. Thus, we need to compare $\alpha(G_{6,4},5) - \alpha(G_{6,5},5)$. By using Lemma 2.6, we have

$$\begin{aligned}&\alpha(G_{6,4},5) - \alpha(G_{6,5},5)\\ &= \Big[\ \alpha(G_{6,4}+a_1b_1,5) + \alpha(G_{6,4}-\{a_1,b_1\},4) + \alpha(G_{6,4}-\{a_1,b_1,c_1\},4)\ \Big] -\\ &\qquad \Big[\ \alpha(G_{6,5}+a_2b_2,5) + \alpha(G_{6,5}-\{a_2,b_2\},4) + \alpha(G_{6,5}-\{a_2,b_2,c_2\},4)\ \Big]\\ &= \alpha(G_{6,4}-\{a_1,b_1,c_1\},4) - \alpha(G_{6,5}-\{a_2,b_2,c_2\},4)\\ &\qquad \text{since}\quad G_{6,4}+a_1b_1 \cong G_{6,5}+a_2b_2, \quad\text{and}\quad G_{6,4}-\{a_1,b_1\} \cong G_{6,5}-\{a_2,b_2\}\\ &= \alpha''(G_{6,4}-\{a_1,b_1,c_1\},4) - \alpha''(G_{6,5}-\{a_2,b_2,c_2\},4).\end{aligned}$$

Since

$$\begin{aligned}&G_{6,4}-\{a_1,b_1,c_1\} \in \mathcal{B}(p-1,q-2,s-2,5), \quad\text{and}\\ &G_{6,5}-\{a_2,b_2,c_2\} \in \mathcal{B}(p-2,q-1,s-2,5),\end{aligned}$$

by Lemma 2.9, we have

$$\begin{aligned}\alpha(G_{6,4},5) - \alpha(G_{6,5},5) &\\ &= \alpha''(G_{6,4}-\{a_1,b_1,c_1\},4) - \alpha''(G_{6,5}-\{a_2,b_2,c_2\},4)\\ &= \Big[\ 3(2^{p-4} - 2^{q-5}) - 3\cdot 4 - 3\ \Big] - \Big[\ 3(2^{p-5} - 2^{q-4}) - 3\cdot 4 - 3\ \Big]\\ &= 3\cdot 2^{q-4} \quad (\text{ since } p = q).\end{aligned}$$

Thus,

$$\alpha(G_{6,4},5) > \alpha(G_{6,5},5).$$

(2.2) When $p > q$, from Table 1, we can easily see that

(a) $\alpha''(G_{6,7},4) < \alpha''(G_{6,6},4)$,
(b) $\alpha''(G_{6,6},4) < \alpha''(G_{6,4},4)$,
(c) $\alpha''(G_{6,5},4) = \alpha''(G_{6,6},4)$.

Since $\alpha''(G_{6,5},4) = \alpha''(G_{6,6},4)$, we need to compare $\alpha(G_{6,5},5)$ with $\alpha(G_{6,6},5)$. By using Lemma 2.6, we have

$$\begin{aligned}
&\alpha(G_{6,5},5)\\
&= \alpha(G_{6,5}+a_1'b_1',5) + \alpha(G_{6,5}-\{a_1',b_1'\},4) + \alpha(G_{6,5}-\{a_1',b_1',c_1'\},4)\\
&= \Big[\ \alpha(G_{6,5}+a_1'b_1'+b_1'c_1',5) + \alpha(G_{6,5}-\{b_1',c_1'\},4) + \alpha(G_{6,5}-\{b_1',c_1',d_1'\},4)\ \Big] +\\
&\quad \alpha(G_{6,5}-\{a_1',b_1'\},4) + \alpha(G_{6,5}-\{a_1',b_1',c_1'\},4)\\
&= \Big[\ \alpha(G_{6,5}+a_1'b_1'+b_1'c_1'+c_1'd_1',5) + \alpha(G_{6,5}-\{c_1',d_1'\},4) +\\
&\quad \alpha(G_{6,5}-\{c_1',d_1',e_1'\},4) + \alpha(G_{6,5}-\{b_1',c_1'\},4) + \alpha(G_{6,5}-\{b_1',c_1',d_1'\},4) +\\
&\quad \alpha(G_{6,5}-\{a_1',b_1'\},4) + \alpha(G_{6,5}-\{a_1',b_1',c_1'\},4)\\
&= \Big[\ \alpha(G_{6,5}+a_1'b_1'+b_1'c_1'+c_1'd_1'+d_1'e_1',5) + \alpha(G_{6,5}-\{d_1',e_1'\},4) +\\
&\quad \alpha(G_{6,5}-\{d_1',e_1',f_1'\},4)\ \Big] + \alpha(G_{6,5}-\{c_1',d_1'\},4) + \alpha(G_{6,5}-\{c_1',d_1',e_1'\},4) +\\
&\quad \alpha(G_{6,5}-\{b_1',c_1'\},4) + \alpha(G_{6,5}-\{b_1',c_1',d_1'\},4) + \alpha(G_{6,5}-\{a_1',b_1'\},4) +\\
&\quad \alpha(G_{6,5}-\{a_1',b_1',c_1'\},4).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\alpha(G_{6,6},5)\\
&= \Big[\ \alpha(G_{6,6}+a_2'b_2'+b_2'c_2'+c_2'd_2'+d_2'e_2',5) + \alpha(G_{6,6}-\{d_2',e_2'\},4) +\\
&\quad \alpha(G_{6,6}-\{d_2',e_2',f_2'\},4)\ \Big] + \alpha(G_{6,6}-\{c_2',d_2'\},4) + \alpha(G_{6,6}-\{c_2',d_2',e_2'\},4) +\\
&\quad \alpha(G_{6,6}-\{b_2',c_2'\},4) + \alpha(G_{6,6}-\{b_2',c_2',d_2'\},4) + \alpha(G_{6,6}-\{a_2',b_2'\},4) +\\
&\quad \alpha(G_{6,6}-\{a_2',b_2',c_2'\},4).
\end{aligned}$$

Observe that

$$\begin{aligned}
&G_{6,5} + a_1'b_1' + b_1'c_1' + c_1'd_1' + d_1'e_1' \cong G_{6,6} + a_2'b_2' + b_2'c_2' + c_2'd_2' + d_2'e_2',\\
&G_{6,5} - \{d_1', e_1'\} \cong G_{6,6} - \{d_2', e_2'\},\\
&G_{6,5} - \{d_1', e_1', f_1'\} \cong G_{6,6} - \{d_2', e_2', f_2'\},\\
&G_{6,5} - \{c_1', d_1'\} \cong G_{6,6} - \{c_2', d_2'\},\\
&G_{6,5} - \{b_1', c_1', d_1'\} \cong G_{6,6} - \{b_2', c_2', d_2'\},
\end{aligned}$$

Since $G_{6,5} - \{a_1', b_1'\}$ and $G_{6,6} - \{a_2', b_2'\}$ belong to $\mathcal{B}(p-1, q-1, s-2, 4)$, by Lemmas 2.6, 2.8 and 2.9, we have

$$\begin{aligned}
&\alpha(G_{6,5} - \{a_1', b_1'\}, 4) - \alpha(G_{6,6} - \{a_2', b_2'\}, 4)\\
&= \alpha'(G_{6,5} - \{a_1', b_1'\}, 4) - \alpha'(G_{6,6} - \{a_2', b_2'\}, 4)\\
&= \Bigg[(s-2)(2^{p-3} + 2^{q-3} - 2) + 4(2^{p-4} + 2^{q-3} - 2) + 2(2^{p-4} - 2^{q-4}) +\\
&\qquad \left\{\binom{s+2}{2} - 19\right\}\Bigg] - \Bigg[(s-2)(2^{p-3} + 2^{q-3} - 2) + 4(2^{p-4} + 2^{q-3} - 2) +\\
&\qquad 2(2^{p-4} - 2^{q-4}) + \left\{\binom{s+2}{2} - 19\right\}\Bigg]\\
&= 0. \qquad (3.1)
\end{aligned}$$

Similarly, since $G_{6,5} - \{b_1', c_1'\}$ and $G_{6,6} - \{b_2', c_2'\}$ belong to $\mathcal{B}(p-1, q-1, s-3, 3)$, by Lemma 2.6, 2.8 and 2.9, we have

$$\begin{aligned}
&\alpha(G_{6,5} - \{b_1', c_1'\}, 4) - \alpha(G_{6,6} - \{b_2', c_2'\}, 4)\\
&= \alpha'(G_{6,5} - \{b_1', c_1'\}, 4) - \alpha'(G_{6,6} - \{b_2', c_2'\}, 4)\\
&= \Bigg[(s-3)(2^{p-3} + 2^{q-3} - 2) + 3(2^{p-4} + 2^{q-3} - 2) + (2^{p-4} - 2^{q-4}) +\\
&\qquad \left\{\binom{s}{2} - 12\right\}\Bigg] - \Bigg[(s-3)(2^{p-3} + 2^{q-3} - 2) + 3(2^{p-4} + 2^{q-3} - 2) +\\
&\qquad 2(2^{p-4} - 2^{q-4}) + \left\{\binom{s}{2} - 12\right\}\Bigg]\\
&= -(2^{p-4} - 2^{q-4}). \qquad (3.2)
\end{aligned}$$

Since

$$\begin{aligned}
&G_{6,5} - \{c_1', d_1', e_1'\} \in \mathcal{B}(p-1, q-2, s-4, 1),\\
&G_{6,6} - \{c_2', d_2', e_2'\} \in \mathcal{B}(p-2, q-1, s-4, 1),
\end{aligned}$$

by Lemmas 2.5, 2.7, and 2.8, we have

$$\begin{aligned}
&\alpha(G_{6,5}-\{c_1',d_1',e_1'\},4)-\alpha(G_{6,6}-\{c_2',d_2',e_2'\},4)\\
&=\alpha'(G_{6,5}-\{c_1',d_1',e_1'\},4)-\alpha'(G_{6,6}-\{c_2',d_2',e_2'\},4)\\
&=\Bigg[\,(s-4)(2^{p-3}+2^{q-4}-2)+(2^{p-4}+2^{q-4}-2)+\\
&\quad\left\{\binom{s-3}{2}-3\right\}-\Bigg[\,(s-4)(2^{p-4}+2^{q-3}-2)+\\
&\quad(2^{p-5}+2^{q-3}-2)+(2^{p-5}-2^{q-4})+\left\{\binom{s-3}{2}-3\right\}\Bigg]\Bigg]\\
&=2^{p-5}\Big[\,2^2(s-4)+2-2(s-4)-1-1\,\Big]+\\
&\quad 2^{q-4}\Big[\,(s-4)+1-2(s-4)-2+1\,\Big]\\
&=(s-4)(2^{p-4}-2^{q-4}). \qquad (3.3)
\end{aligned}$$

Similarly, since

$$\begin{aligned}
G_{6,5}-\{a_1',b_1',c_1'\}&\in\mathcal{B}(p-1,q-2,s-3,3),\\
G_{6,6}-\{a_2',b_2',c_2'\}&\in\mathcal{B}(p-2,q-1,s-3,3),
\end{aligned}$$

by Lemmas 2.5, 2.7, and 2.8, we have

$$\begin{aligned}
&\alpha(G_{6,5}-\{a_1',b_1',c_1'\},4)-\alpha(G_{6,6}-\{a_2',b_2',c_2'\},4)\\
&=\alpha'(G_{6,5}-\{a_1',b_1',c_1'\},4)-\alpha'(G_{6,6}-\{a_2',b_2',c_2'\},4)\\
&=\Bigg[\,(s-3)(2^{p-3}+2^{q-4}-2)+3(2^{p-4}+2^{q-4}-2)+(2^{p-4}-2^{q-5})+\\
&\quad\left\{\binom{s}{2}-12\right\}-\Bigg[\,(s-3)(2^{p-4}+2^{q-3}-2)+3(2^{p-5}+2^{q-3}-2)+\\
&\quad 2(2^{p-5}-2^{q-4})+\left\{\binom{s}{2}-12\right\}\Bigg]\Bigg]\\
&=2^{p-5}\Big[\,2^2(s-3)+3\cdot 2+2-2(s-3)-3-2\,\Big]+\\
&\quad 2^{q-5}\Big[\,2(s-3)+3\cdot 2-1-2^2(s-3)-3\cdot 2^2+2\cdot 2\,\Big]\\
&=(2s-3)(2^{p-5}-2^{q-5}). \qquad (3.4)
\end{aligned}$$

By (3.1) – (3.4), we have

$$\begin{aligned}
&\alpha(G_{6,5},5)-\alpha(G_{6,6},5)\\
&\quad=-(2^{p-4}-2^{q-4})+(s-4)(2^{p-4}-2^{q-4})+(2s-3)(2^{p-5}-2^{q-5})\\
&\quad=(4s-13)(2^{p-5}-2^{q-5}).
\end{aligned}$$

Since $p>q$ and $s\geq 8$, we have

$$\alpha(G_{6,5},5)>\alpha(G_{6,6},5)$$

(3) $\mathcal{T}_{\mathbf{3}}$

(3.1) When $p=q$, $G_{6,8}\cong G_{6,10}$ and from Table 1, we can easily see that

(a) $\alpha''(G_{6,8},4)-\alpha''(G_{6,9},4)=0$,
(b) $\alpha''(G_{6,9},4)-\alpha''(G_{6,11},4)=0$.

Thus, we need to compare for each pair of $\alpha(G_{6,i},5)$, where $i=8$, 9 and 11. By using Lemma 2.6,

$$\begin{aligned}
&\alpha(G_{6,8},5)-\alpha(G_{6,9},5)\\
&\quad=\Big[\ \alpha(G_{6,8}+a_3b_3,5)+\alpha(G_{6,8}-\{a_3,b_3\},4)+\alpha(G_{6,8}-\{a_3,b_3,c3\},4)\ \Big]-\\
&\qquad\quad\Big[\ \alpha(G_{6,9}+a_3'b_3',5)+\alpha(G_{6,9}-\{a_3',b_3'\},4)+\alpha(G_{6,9}-\{a_3',b_3',c3'\},4)\ \Big]\\
&\quad=\alpha(G_{6,8}-\{a_3,b_3,c3\},4)-\alpha(G_{6,9}-\{a_3',b_3',c3'\},4)\\
&\qquad\qquad\text{since } G_{6,8}+a_3b_3\cong G_{6,9}+a_3'b_3',\ \text{ and } G_{6,8}-\{a_3,b_3\}\cong G_{6,9}-\{a_3',b_3'\}\\
&\quad=\alpha''(G_{6,8}-\{a_3,b_3,c3\},4)-\alpha''(G_{6,9}-\{a_3',b_3',c3'\},4).
\end{aligned}$$

Since

$$\begin{aligned}
G_{6,8}-\{a_3,b_3,c_3\}&\in\mathcal{B}(p-2,q-1,s-3,3),\\
G_{6,9}-\{a_3',b_3',c_3'\}&\in\mathcal{B}(p-1,q-2,s-3,3),
\end{aligned}$$

by using Lemma 2.9, we have

$$\begin{aligned}
&\alpha(G_{6,8},5)-\alpha(G_{6,9},5)\\
&\quad=\alpha''(G_{6,8}-\{a_3,b_3,c_3\},4)-\alpha''(G_{6,9}-\{a_3',b_3',c_3'\},4)\\
&\quad=\Big[\ 1(2^{(p-2)-3}-2^{(q-1)-3})-3\cdot 2-1\ \Big]-\Big[\ 1(2^{(p-1)-3}-2^{(q-2)-3})-3\cdot 2-1\ \Big]\\
&\quad=-2^{p-5}-2^{q-5}<0\quad(\text{ since } p=q).
\end{aligned}$$

Similarly, by Lemma 2.6, we have

$$\begin{aligned}
&\alpha(G_{6,9},5)-\alpha(G_{6,11},5)\\
&=\Big[\alpha(G_{6,9}+u_1v_1,5)+\alpha(G_{6,9}-\{u_1,v_1\},4)+\alpha(G_{6,9}-\{u_1,v_1,w_1\},4)\Big]-\\
&\quad\Big[\alpha(G_{6,11}+u_2v_2,5)+\alpha(G_{6,11}-\{u_2,v_2\},4)+\alpha(G_{6,11}-\{u_2,v_2,w_2\},4)\Big]\\
&=\Big[\alpha(G_{6,9}-\{u_1,v_1\},4)+\alpha(G_{6,9}-\{u_1,v_1,w_1\},4)\Big]-\\
&\quad\Big[\alpha(G_{6,11}-\{u_2,v_2\},4)+\alpha(G_{6,11}-\{u_2,v_2,w_2\},4)\Big]\\
&\quad\text{since } G_{6,9}+u_1v_1\cong G_{6,11}+u_2v_2\\
&=\Big[\alpha(G_{6,9}-\{u_1,v_1\},4)-\alpha(G_{6,11}-\{u_2,v_2\},4)\Big]+\\
&\quad\Big[\alpha(G_{6,9}-\{u_1,v_1,w_1\},4)-\alpha(G_{6,11}-\{u_2,v_2,w_2\},4)\Big]\\
&=\Big[\alpha''(G_{6,9}-\{u_1,v_1\},4)-\alpha''(G_{6,11}-\{u_2,v_2\},4)\Big]+\\
&\quad\Big[\alpha''(G_{6,9}-\{u_1,v_1,w_1\},4)-\alpha''(G_{6,11}-\{u_2,v_2,w_2\},4)\Big].
\end{aligned}$$

Observe that

$$\begin{aligned}
&G_{6,9}-\{u_1,v_1\}\in\mathcal{B}(p-1,q-1,s-2,4),\\
&G_{6,11}-\{u_2,v_2\}\in\mathcal{B}(p-1,q-1,s-2,4),\\
&G_{6,9}-\{u_1,v_1,w_1\}\in\mathcal{B}(p-2,q-1,s-3,3),\\
&G_{6,11}-\{u_2,v_2,w_2\}\in\mathcal{B}(p-2,q-1,s-3,3).
\end{aligned}$$

By using Lemma 2.9, we have

$$\begin{aligned}
&\alpha''(G_{6,9}-\{u_1,v_1\},4)-\alpha''(G_{6,11}-\{u_2,v_2\},4)\\
&\quad=\Big[2(2^{(p-1)-3}-2^{(q-1)-3})-3\cdot 2-1\Big]-\\
&\qquad\Big[2(2^{(p-1)-3}-2^{(q-1)-3})-3\cdot 2\Big]=-1
\end{aligned}$$

and

$$\begin{aligned}
&\alpha''(G_{6,9}-\{u_1,v_1,w_1\},4)-\alpha''(G_{6,11}-\{u_2,v_2,w_2\},4)\\
&\quad=\Big[2(2^{(p-2)-3}-2^{(q-1)-3})-3\cdot 2-1\Big]-\\
&\qquad\Big[2(2^{(p-2)-3}-2^{(q-1)-3})-3\cdot 1\Big]=-4
\end{aligned}$$

Thus, we have

$$\alpha(G_{6,9},5) < \alpha(G_{6,11},5). \tag{3.5}$$

(3.2) When $p > q$, from Table 1, we can easily see that

(a) $\alpha''(G_{6,8},4) - \alpha''(G_{6,9},4) < 0$,
(b) $\alpha''(G_{6,9},4) - \alpha''(G_{6,10},4) < 0$,
(c) $\alpha''(G_{6,9},4) - \alpha''(G_{6,11},4) = 0$.

Since $\alpha''(G_{6,9},4) = \alpha''(G_{6,11},4)$, we need to calculate $\alpha(G_{6,9},5) - \alpha(G_{6,11},5)$. By Equation (3.5), we have $\alpha(G_{6,9},5) - \alpha(G_{6,11},5) \neq 0$.

For the remaining work in comparing every two graphs in $\mathcal{T}_{\mathbf{4}}$ to $\mathcal{T}_{\mathbf{13}}$, the reader may refer to [5].

This completes the proof of Theorem 3.1. □

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TABLE 1: $\mathbf{t} = \mathbf{6}$ (1 of 8)

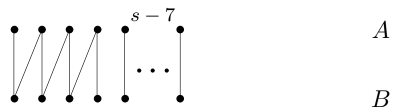
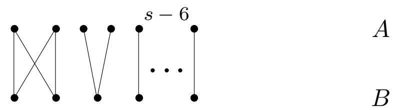
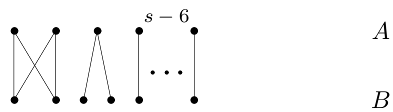
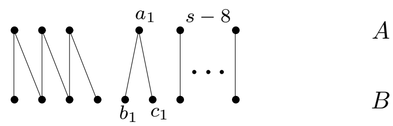
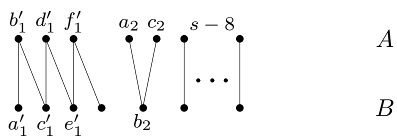
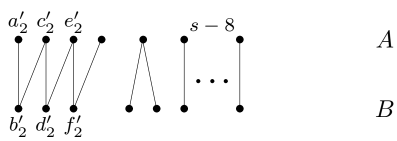
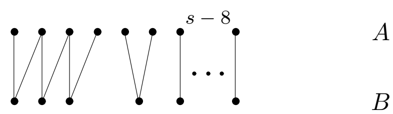
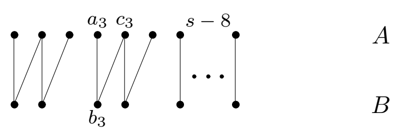
Name of Graph, $G_{6,i}$	Graphs $G'_{6,i}$ ($G'_{6,i} = K_{p,q} - G_{6,i}$) $\lvert A\rvert = p$, $\lvert B\rvert = q$	$\alpha''(G_{6,i}, 4)$	Conditions on s
$G_{6,1}$		$3(2^{p-3} - 2^{q-3}) - 19$	$7 \le s \le q-1$
$G_{6,2}$		$2^{p-2} - 3 \cdot 2^{q-3} - 19$	$6 \le s \le q-1$
$G_{6,3}$		$3 \cdot 2^{p-3} - 2^{q-1} - 19$	$6 \le s \le q-1$
$G_{6,4}$		$4(2^{p-3} - 2^{q-3}) - 15$	$8 \le s \le q-1$
$G_{6,5}$		$3(2^{p-3} - 2^{q-3}) - 15$	$8 \le s \le q-1$
$G_{6,6}$		$3(2^{p-3} - 2^{q-3}) - 15$	$8 \le s \le q-1$
$G_{6,7}$		$2(2^{p-3} - 2^{q-3}) - 15$	$8 \le s \le q-1$
$G_{6,8}$		$2(2^{p-3} - 2^{q-3}) - 14$	$8 \le s \le q-1$

TABLE 1: $\mathbf{t} = \mathbf{6}$ (2 of 8)

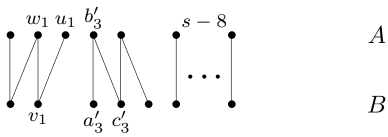
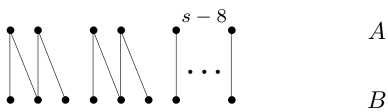
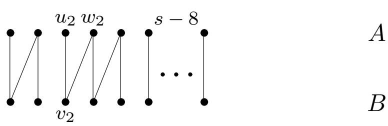
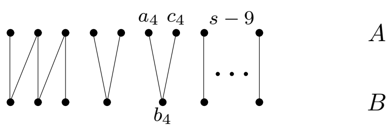
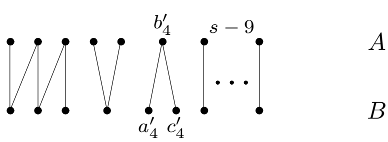
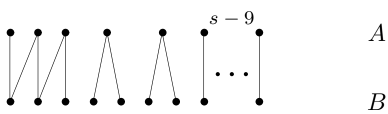
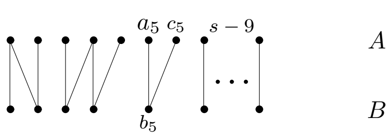
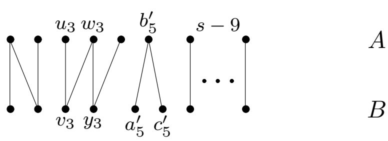
Name of Graph, $G_{6,i}$	Graphs $G'_{6,i}$ ($G'_{6,i} = K_{p,q} - G_{6,i}$) $\|A\| = p,\ \|B\| = q$	$\alpha''(G_{6,i}, 4)$	Conditions on s
$G_{6,9}$		$3(2^{p-3} - 2^{q-3}) - 14$	$8 \leq s \leq q-1$
$G_{6,10}$		$4(2^{p-3} - 2^{q-3}) - 14$	$8 \leq s \leq q-1$
$G_{6,11}$		$3(2^{p-3} - 2^{q-3}) - 14$	$8 \leq s \leq q-1$
$G_{6,12}$		$2(2^{p-3} - 2^{q-3}) - 11$	$9 \leq s \leq q-1$
$G_{6,13}$		$3(2^{p-3} - 2^{q-3}) - 11$	$9 \leq s \leq q-1$
$G_{6,14}$		$4(2^{p-3} - 2^{q-3}) - 11$	$9 \leq s \leq q-1$
$G_{6,15}$		$2(2^{p-3} - 2^{q-3}) - 10$	$9 \leq s \leq q-1$
$G_{6,16}$		$3(2^{p-3} - 2^{q-3}) - 10$	$9 \leq s \leq q-1$

TABLE 1: $\mathbf{t} = \mathbf{6}$ (3 of 8)

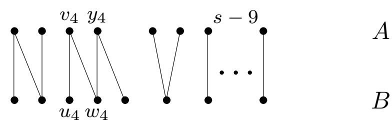
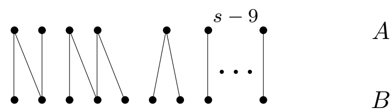
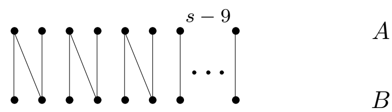
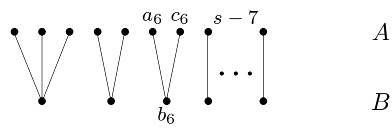
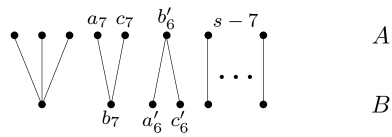
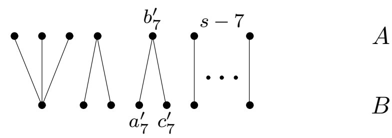
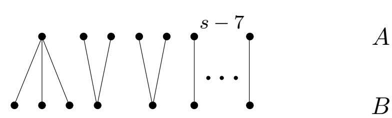
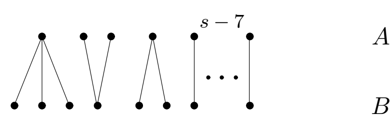
Name of Graph, $G_{6,i}$	Graphs $G'_{6,i}$ ($G'_{6,i} = K_{p,q} - G_{6,i}$) $\lvert A\rvert = p$, $\lvert B\rvert = q$	$\alpha''(G_{6,i}, 4)$	Conditions on s
$G_{6,17}$		$3(2^{p-3} - 2^{q-3}) - 10$	$9 \leq s \leq q-1$
$G_{6,18}$		$4(2^{p-3} - 2^{q-3}) - 10$	$9 \leq s \leq q-1$
$G_{6,19}$		$3(2^{p-3} - 2^{q-3}) - 9$	$9 \leq s \leq q-1$
$G_{6,20}$		$-2^{p-4} - 9$	$7 \leq s \leq q-1$
$G_{6,21}$		$2^{p-4} - 2^{q-3} - 9$	$7 \leq s \leq q-1$
$G_{6,22}$		$3 \cdot 2^{p-4} - 2^{q-2} - 9$	$7 \leq s \leq q-1$
$G_{6,23}$		$2^{p-1} - 9 \cdot 2^{q-4} - 9$	$7 \leq s \leq q-1$
$G_{6,24}$		$5 \cdot 2^{p-3} - 11 \cdot 2^{q-4} - 9$	$7 \leq s \leq q-1$

TABLE 1: $\mathbf{t} = \mathbf{6}$ (4 of 8)

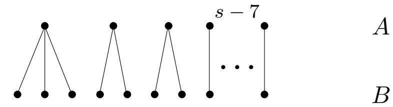
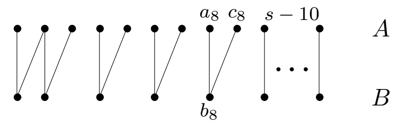
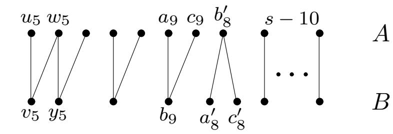
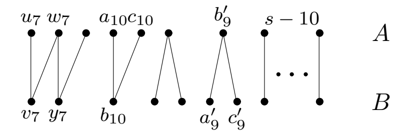
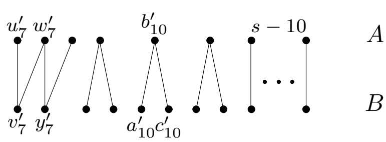
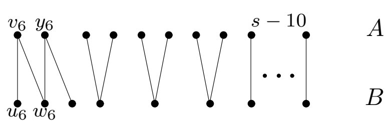
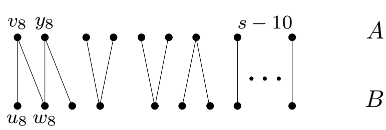
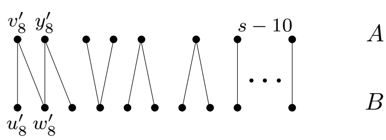
Name of Graph, $G_{6,i}$	Graphs $G'_{6,i}$ ($G'_{6,i} = K_{p,q} - G_{6,i}$) $\lvert A\rvert = p,\ \lvert B\rvert = q$	$\alpha''(G_{6,i}, 4)$	Conditions on s
$G_{6,25}$		$6 \cdot 2^{p-3} - 13 \cdot 2^{q-4} - 9$	$7 \le s \le q-1$
$G_{6,26}$		$2^{p-3} - 2^{q-3} - 7$	$10 \le s \le q-1$
$G_{6,27}$		$2(2^{p-3} - 2^{q-3}) - 7$	$10 \le s \le q-1$
$G_{6,28}$		$3(2^{p-3} - 2^{q-3}) - 7$	$10 \le s \le q-1$
$G_{6,29}$		$4(2^{p-3} - 2^{q-3}) - 7$	$10 \le s \le q-1$
$G_{6,30}$		$2(2^{p-3} - 2^{q-3}) - 7$	$10 \le s \le q-1$
$G_{6,31}$		$3(2^{p-3} - 2^{q-3}) - 7$	$10 \le s \le q-1$
$G_{6,32}$		$4(2^{p-3} - 2^{q-3}) - 7$	$10 \le s \le q-1$

TABLE 1: $\mathbf{t} = \mathbf{6}$ (5 of 8)

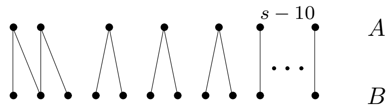
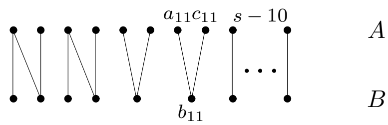
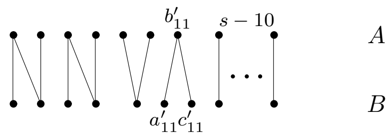
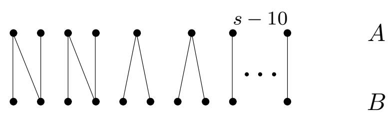
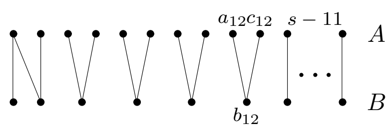
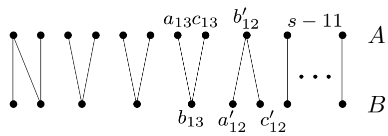
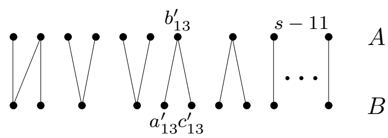
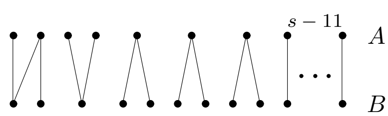
Name of Graph, $G_{6,i}$	Graphs $G'_{6,i}$ ($G'_{6,i} = K_{p,q} - G_{6,i}$) $\|A\| = p$, $\|B\| = q$	$\alpha''(G_{6,i}, 4)$	Conditions on s
$G_{6,33}$		$5(2^{p-3} - 2^{q-3}) - 7$	$10 \le s \le q-1$
$G_{6,34}$		$2(2^{p-3} - 2^{q-3}) - 6$	$10 \le s \le q-1$
$G_{6,35}$		$3(2^{p-3} - 2^{q-3}) - 6$	$10 \le s \le q-1$
$G_{6,36}$		$4(2^{p-3} - 2^{q-3}) - 6$	$10 \le s \le q-1$
$G_{6,37}$		$2^{p-3} - 2^{q-3} - 3$	$11 \le s \le q-1$
$G_{6,38}$		$2(2^{p-3} - 2^{q-3}) - 3$	$11 \le s \le q-1$
$G_{6,39}$		$3(2^{p-3} - 2^{q-3}) - 3$	$11 \le s \le q-1$
$G_{6,40}$		$4(2^{p-3} - 2^{q-3}) - 3$	$11 \le s \le q-1$

TABLE 1: $\mathbf{t} = \mathbf{6}$ (6 of 8)

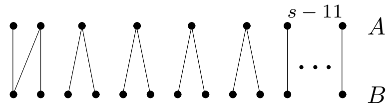
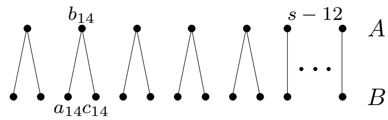
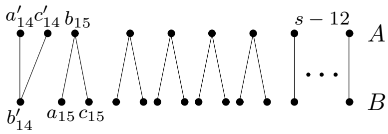
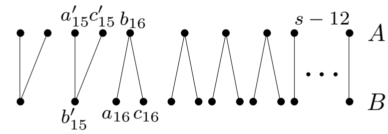
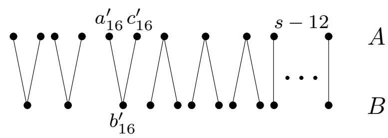
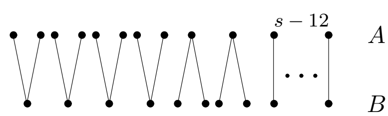
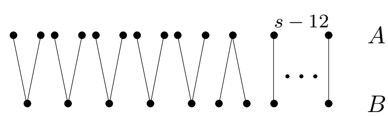
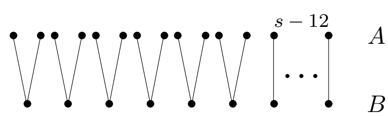
Name of Graph, $G_{6,i}$	Graphs $G'_{6,i}$ ($G'_{6,i} = K_{p,q} - G_{6,i}$) $\lvert A\rvert = p,\ \lvert B\rvert = q$	$\alpha''(G_{6,i}, 4)$	Conditions on s
$G_{6,41}$		$5(2^{p-3} - 2^{q-3}) - 3$	$11 \le s \le q-1$
$G_{6,42}$		$6(2^{p-3} - 2^{q-3})$	$12 \le s \le q-1$
$G_{6,43}$		$5(2^{p-3} - 2^{q-3})$	$12 \le s \le q-1$
$G_{6,44}$		$4(2^{p-3} - 2^{q-3})$	$12 \le s \le q-1$
$G_{6,45}$		$3(2^{p-3} - 2^{q-3})$	$12 \le s \le q-1$
$G_{6,46}$		$2(2^{p-3} - 2^{q-3})$	$12 \le s \le q-1$
$G_{6,47}$		$2^{p-3} - 2^{q-3}$	$12 \le s \le q-1$
$G_{6,48}$		0	$12 \le s \le q-1$

TABLE 1: $\mathbf{t} = \mathbf{6}$ (7 of 8)

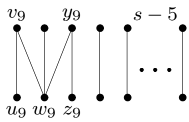
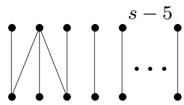
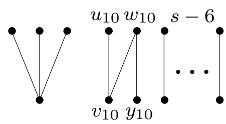
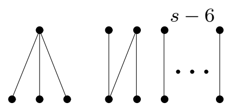
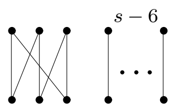
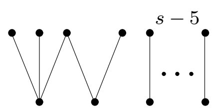
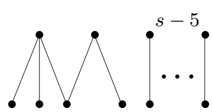
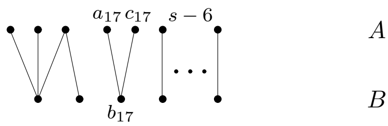
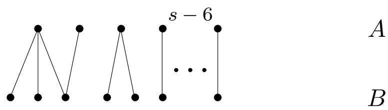
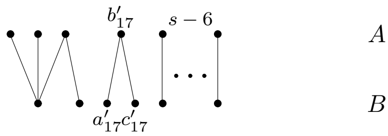
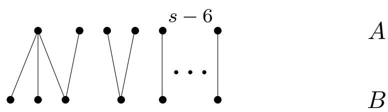
Name of Graph, $G_{6,i}$	Graphs $G'_{6,i}$ ($G'_{6,i} = K_{p,q} - G_{6,i}$) $\lvert A\rvert = p,\ \lvert B\rvert = q$	$\alpha''(G_{6,i}, 4)$	Conditions on s
$G_{6,49}$	 A B	$3 \cdot 2^{p-4} - 2^{q-2} - 28$	$5 \leq s \leq q-1$
$G_{6,50}$	 A B	$4 \cdot 2^{p-3} - 9 \cdot 2^{q-4} - 28$	$5 \leq s \leq q-1$
$G_{6,51}$	 A B	$2^{p-4} - 2^{q-3} - 12$	$6 \leq s \leq q-1$
$G_{6,52}$	 A B	$5 \cdot 2^{p-3} - 11 \cdot 2^{q-4} - 12$	$6 \leq s \leq q-1$
$G_{6,53}$	 A B	$3 \cdot 2^{p-3} - 3 \cdot 2^{q-3} - 24$	$6 \leq s \leq q-1$
$G_{6,54}$	 A B	$2^{p-4} - 2^{q-3} - 24$	$5 \leq s \leq q-1$
$G_{6,55}$	 A B	$5 \cdot 2^{p-3} - 11 \cdot 2^{q-4} - 24$	$5 \leq s \leq q-1$

TABLE 1: $\mathbf{t} = \mathbf{6}$ (8 of 8)

Name of Graph, $G_{6,i}$	Graphs $G'_{6,i}$ ($G'_{6,i} = K_{p,q} - G_{6,i}$) $\lvert A \rvert = p,\ \lvert B \rvert = q$	$\alpha''(G_{6,i}, 4)$	Conditions on s
$G_{6,56}$		$2^{p-4} - 2^{q-3} - 18$	$6 \leq s \leq q-1$
$G_{6,57}$		$5 \cdot 2^{p-3} - 11 \cdot 2^{q-4} - 18$	$6 \leq s \leq q-1$
$G_{6,58}$		$3 \cdot 2^{p-4} - 2^{q-2} - 18$	$6 \leq s \leq q-1$
$G_{6,59}$		$2^{p-1} - 9 \cdot 2^{q-4} - 18$	$6 \leq s \leq q-1$