BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **29**(1) (2006), 169–177

Fuzzy QS-Algebras with Interval-Valued Membership Functions

ARSHAM BORUMAND SAEID
Department of Mathematics, Islamic Azad University
Kerman Branch, Kerman, Iran
arsham@iauk.ac.ir

Abstract. In this note the notion of interval-valued fuzzy QS-algebra (briefly, i-v fuzzy QS-algebra), as well as the i-v level and strong i-v level QS-subalgebra is introduced. Several theorems which determine the relationship between these notions and QS-subalgebras are stated and proved. The images and inverse images of i-v fuzzy QS-subalgebras are defined, and how the homomorphic images and inverse images of an i-v fuzzy QS-subalgebra become i-v fuzzy QS-algebras is studied as well.

2000 Mathematics Subject Classification: 03B52 , 03G25, 06F35, 94D05

Key words and phrases: QS-algebra, fuzzy QS-subalgebra, interval-valued fuzzy set, interval-valued fuzzy QS-subalgebra.

1. Introduction

In 1966, Imai and Iseki [7] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [5], Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They showed that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Neggers, Ahn and Kim introduced the notion of Q-algebras [11], which is a generalization of $BCH/BCI/BCK$-algebras. In [1], Ahn and Kim introduced the notion of QS-algebras which is a generalization of Q-algebras.

The concept of a fuzzy set, was introduced in [12]. In [13], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). This interval-valued fuzzy set is referred to as an i-v fuzzy set. He constructed a method of approximate inference using his i-v fuzzy sets. Biswas [2], defined interval-valued fuzzy subgroups and Hong *et al.* applied the notion of interval-valued fuzzy sets to BCI-algebras [4].

Received: June 1, 2005; *Revised:* October 19, 2005.

In the present paper, we use the notion of interval-valued fuzzy set and introduce the concept of interval-valued fuzzy QS-subalgebras (briefly i-v fuzzy QS-subalgebras) of a QS-algebra, and we study some of their properties. Among other results, we prove that every QS-subalgebra of a QS-algebra X can be realized as an i-v level QS-subalgebra of an i-v fuzzy QS-subalgebra of X. We also obtain some related results which have been mentioned in the abstract.

2. Preliminary notes

Definition 2.1. [1] *A QS-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:*

(I) $x * x = 0$,
(II) $x * 0 = x$,
(III) $(x * y) * z = (x * z) * y$,
(IV) $(x * y) * (x * z) = z * y$,

for all $x, y, z \in X$.

In X we define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$

Example 2.1. [1] Let $\mathcal{Z}$ be the set of all integers and let $n\mathcal{Z}= \{nz \mid z \in \mathcal{Z}\}$. Then $(\mathcal{Z}; -, 0)$ and $(n\mathcal{Z}; -, 0)$ are both QS-algebras, where "-" is the usual subtraction of integers. Also $(\mathcal{R}; -, 0)$ and $(\mathcal{C}; -, 0)$ are QS-algebras where $\mathcal{R}$ is the set of all real numbers, $\mathcal{C}$ is the set of all complex numbers and "-" is the usual subtraction of real (complex) numbers.

Proposition 2.1. [1] *Let X be a QS-algebra. Then for any x, y and z in X, the following relations hold:*

(a) $x \leq y$ *implies* $z * y \leq z * x$,
(b) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$,
(c) $x * y \leq z$ *implies* $x * z \leq y$,
(d) $(x * z) * (y * z) \leq x * y$,
(e) $x \leq y$ *implies* $x * z \leq y * z$,
(f) $0 * (0 * (0 * x)) = 0 * x$.

Definition 2.2. *A non-empty subset S of a QS-algebra X is called a subalgebra of X if $x * y \in S$ for any $x, y \in S$.*

A mapping $f : X \longrightarrow Y$ of QS-algebras is called a QS-homomorphism if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$.

We now review some fuzzy logic concepts (see [12]). Let X be a set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \longrightarrow [0, 1]$. Let f be a mapping from the set X to the set Y and let B be a fuzzy set in Y with membership function μ_B. The inverse image of B, denoted $f^{-1}(B)$, is the fuzzy set in X with membership function $\mu_{f^{-1}(B)}$ defined by $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ for all $x \in X$. Conversely, let A be a fuzzy set in X with membership function μ_A Then the image of A, denoted by $f(A)$, is the fuzzy set in Y such that:

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} \mu_A(z) & \text{if } f^{-1}(y) = \{x : f(x) = y\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

A fuzzy set A in the QS-algebra X with the membership function μ_A is said to have the sup property if for any subset $T \subseteq X$ there exists $x_0 \in T$ such that

$$\mu_A(x_0) = \sup_{t \in T} \mu_A(t).$$

An interval-valued fuzzy set (briefly, i-v fuzzy set) A defined on X is given by

$$A = \{(x, [\mu_A^L(x), \mu_A^U(x)])\}, \ \forall x \in X.$$

Briefly, it is denoted by $A = [\mu_A^L, \mu_A^U]$ where μ_A^L and μ_A^U are any two fuzzy sets in X such that $\mu_A^L(x) \leq \mu_A^U(x)$ for all $x \in X$.

Let $\overline{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$, for all $x \in X$ and let $D[0,1]$ denote the family of all closed sub-intervals of $[0,1]$. It is clear that if $\mu_A^L(x) = \mu_A^U(x) = c$, where $0 \leq c \leq 1$ then $\overline{\mu}_A(x) = [c, c]$ is in $D[0,1]$. Thus $\overline{\mu}_A(x) \in D[0,1]$, for all $x \in X$. Therefore the i-v fuzzy set A is given by

$$A = \{(x, \overline{\mu}_A(x))\}, \ \forall x \in X$$

where

$$\overline{\mu}_A : X \longrightarrow D[0,1].$$

Now we define the refined minimum (briefly, rmin) and order " $\leq$ " on elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of $D[0,1]$ as:

$$\text{rmin}(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}],$$

$$D_1 \leq D_2 \Longleftrightarrow a_1 \leq a_2 \ \wedge \ b_1 \leq b_2.$$

Similarly we can define $\geq$ and $=$.

Definition 2.3. [3] *Let μ be a fuzzy set in a QS-algebra X. Then μ is called a fuzzy QS-subalgebra (QS-algebra) of X if*

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\},$$

for all $x, y \in X$.

Proposition 2.2. [3] *Let f be a QS-homomorphism from X into Y and G be a fuzzy QS-subalgebra of Y with the membership function μ_G. Then the inverse image $f^{-1}(G)$ of G is a fuzzy QS-subalgebra of X.*

Proposition 2.3. [3] *Let f be a QS-homomorphism from X onto Y and D be a fuzzy QS-subalgebra of X with the* sup *property. Then the image $f(D)$ of D is a fuzzy QS-subalgebra of Y.*

3. Interval-valued fuzzy QS-algebra

From now on X is a QS-algebra, unless otherwise is stated.

Definition 3.1. *An i-v fuzzy set A in X is called an interval-valued fuzzy QS-subalgebras (briefly i-v fuzzy QS-subalgebra) of X if*

$$\overline{\mu}_A(x * y) \geq \text{rmin}\{\overline{\mu}_A(x), \overline{\mu}_A(y)\},$$

for all $x, y \in X$.

Example 3.1. Let $X = \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

Then $(X, *, 0)$ is a QS-algebra, but not a $BCH/BCI/BCK$-algebra.
Define $\overline{\mu}_A$ as:

$$\overline{\mu}_A(x) = \begin{cases} [0.3, 0.9] & \text{if } x \in \{0, 2\} \\ [0.1, 0.6] & \text{otherwise.} \end{cases}$$

It is easy to check that A is an i-v fuzzy QS-subalgebra of X.

Lemma 3.1. *If A is an i-v fuzzy QS-subalgebra of X, then*

$$\overline{\mu}_A(0) \geq \overline{\mu}_A(x)$$

for all $x \in X$.

Proof. For all $x \in X$, we have

$$\begin{aligned} \overline{\mu}_A(0) &= \overline{\mu}_A(x * x) \geq \text{rmin}\{\overline{\mu}_A(x), \overline{\mu}_A(x)\} \\ &= \text{rmin}\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(x), \mu_A^U(x)]\} \\ &= [\mu_A^L(x), \mu_A^U(x)] = \overline{\mu}_A(x). \end{aligned}$$

□

Proposition 3.1. *Let A be an i-v fuzzy QS-subalgebra of X, and let $n \in \mathcal{N}$. Then*

(i) $\overline{\mu}_A(\prod^n x * x) \geq \overline{\mu}_A(x)$, *for any odd number n,*

(ii) $\overline{\mu}_A(\prod^n x * x) = \overline{\mu}_A(0)$, *for any even number n.*

Proof. We proved by induction. Let $x \in X$ and assume that n is odd. Then $n = 2k - 1$ for some positive integer k. The definition and the above lemma imply that $\overline{\mu}_A(x * x) = \overline{\mu}_A(0) \geq \overline{\mu}_A(x)$. Now suppose that $\overline{\mu}_A(\prod^{2k-1} x * x) \geq \overline{\mu}_A(x)$. Then by assumption

$$\begin{aligned} \overline{\mu}_A(\prod^{2(k+1)-1} x * x) &= \overline{\mu}_A(\prod^{2k+1} x * x) \\ &= \overline{\mu}_A(\prod^{2k-1} x * (x * (x * x))) \\ &= \overline{\mu}_A(\prod^{2k-1} x * x) \\ &\geq \overline{\mu}_A(x). \end{aligned}$$

This proves (i), and similarly we can prove (ii). □

Theorem 3.1. *Let A be an i-v fuzzy QS-subalgebra of X. If there exists a sequence $\{x_n\}$ in X, such that*

$$\lim_{n\to\infty} \overline{\mu}_A(x_n) = [1,1].$$

Then $\overline{\mu}_A(0) = [1,1]$.

Proof. By the above lemma we have $\overline{\mu}_A(0) \geq \overline{\mu}_A(x)$, for all $x \in X$, thus $\overline{\mu}_A(0) \geq \overline{\mu}_A(x_n)$, for every positive integer n. Consider the inequality

$$[1,1] \geq \overline{\mu}_A(0) \geq \lim_{n\to\infty} \overline{\mu}_A(x_n) = [1,1].$$

Hence $\overline{\mu}_A(0) = [1,1]$. □

Theorem 3.2. *An i-v fuzzy set $A = [\mu_A^L, \mu_A^U]$ in X is an i-v fuzzy QS-subalgebra of X if and only if μ_A^L and μ_A^U are fuzzy QS-subalgebras of X.*

Proof. Let μ_A^L and μ_A^U be fuzzy QS-subalgebras of X and $x, y \in X$. Observe

$$\begin{aligned} \overline{\mu}_A(x*y) &= [\mu_A^L(x*y), \mu_A^U(x*y)] \\ &\geq [\min\{\mu_A^L(x), \mu_A^L(y)\}, \min\{\mu_A^U(x), \mu_A^U(y)\}] \\ &= \text{rmin}\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\} \\ &= \text{rmin}\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}. \end{aligned}$$

From what was mentioned above we can conclude that A is an i-v fuzzy QS-subalgebra of X.

Conversely, suppose that A is an i-v fuzzy QS-subalgebra of X. For any $x, y \in X$ we have

$$\begin{aligned} [\mu_A^L(x*y), \mu_A^U(x*y)] &= \overline{\mu}_A(x*y) \\ &\geq \text{rmin}\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} \\ &= \text{rmin}\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\} \\ &= [\min\{\mu_A^L(x), \mu_A^L(y)\}, \min\{\mu_A^U(x), \mu_A^U(y)\}. \end{aligned}$$

Therefore $\mu_A^L(x*y) \geq \min\{\mu_A^L(x), \mu_A^L(y)\}$ and $\mu_A^U(x*y) \geq \min\{\mu_A^U(x), \mu_A^U(y)\}$, whence we get that μ_A^L and μ_A^U are fuzzy QS-subalgebras of X. □

Theorem 3.3. *Let A_1 and A_2 be i-v fuzzy QS-subalgebras of X. Then $A_1 \cap A_2$ is an i-v fuzzy QS-subalgebra of X.*

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and A_2, since A_1 and A_2 are i-v fuzzy QS-subalgebras of X, by the above theorem we have:

$$\begin{aligned} \overline{\mu}_{A_1\cap A_2}(x*y) &= [\mu_{A_1\cap A_2}^L(x*y), \mu_{A_1\cap A_2}^U(x*y)] \\ &= [\min(\mu_{A_1}^L(x*y), \mu_{A_2}^L(x*y)), \min(\mu_{A_1}^U(x*y), \mu_{A_2}^U(x*y))] \\ &\geq [\min((\mu_{A_1\cap A_2}^L(x), \mu_{A_1\cap A_2}^L(y)), \min((\mu_{A_1\cap A_2}^U(x), \mu_{A_1\cap A_2}^U(y))] \\ &= \text{rmin}\{\overline{\mu}_{A_1\cap A_2}(x), \overline{\mu}_{A_1\cap A_2}(y)\}. \end{aligned}$$

which proves the theorem. □

Corollary 3.8. Let $\{A_i | i \in \Lambda\}$ be a family of i-v fuzzy QS-subalgebras of X. Then $\bigcap_{i\in\Lambda} A_i$ is also an i-v fuzzy QS-subalgebra of X.

Definition 3.2. *Let A be an i-v fuzzy set in X and $[\delta_1, \delta_2] \in D[0,1]$. Then the i-v level QS-subalgebra $U(A; [\delta_1, \delta_2])$ of A and strong i-v level QS-subalgebra $U(A; >, [\delta_1, \delta_2])$ of X are defined as follows:*

$$U(A; [\delta_1, \delta_2]) := \{x \in X \mid \overline{\mu}_A(x) \geq [\delta_1, \delta_2]\},$$

$$U(A; >, [\delta_1, \delta_2]) := \{x \in X \mid \overline{\mu}_A(x) > [\delta_1, \delta_2]\}.$$

Theorem 3.4. *Let A be an i-v fuzzy QS-subalgebra of X and let B be the closure of the image of μ_A. Then the following conditions are equivalent:*

(i) *A is an i-v fuzzy QS-subalgebra of X.*
(ii) *For all $[\delta_1, \delta_2] \in \Im(\mu_A)$, the nonempty i-v level subset $U(A; [\delta_1, \delta_2])$ of A is a QS-subalgebra of X.*
(iii) *For all $[\delta_1, \delta_2] \in \Im(\mu_A) \setminus B$, the nonempty strong i-v level subset $U(A; >, [\delta_1, \delta_2])$ of A is a QS-subalgebra of X.*
(iv) *For all $[\delta_1, \delta_2] \in D[0,1]$, the nonempty strong i-v level subset $U(A; >, [\delta_1, \delta_2])$ of A is a QS-subalgebra of X.*
(v) *For all $[\delta_1, \delta_2] \in D[0,1]$, the nonempty i-v level subset $U(A; [\delta_1, \delta_2])$ of A is a QS-subalgebra of X.*

Proof. (i) $\Longrightarrow$ (iv). Let A be an i-v fuzzy QS-subalgebra of X, $[\delta_1, \delta_2] \in D[0,1]$ and $x, y \in U(A; <, [\delta_1, \delta_2])$, then we have

$$\overline{\mu}_A(x * y) \geq \text{rmin}\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} > \text{rmin}\{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = [\delta_1, \delta_2],$$

and thus $x * y \in U(A; >, [\delta_1, \delta_2])$. Hence $U(A; >, [\delta_1, \delta_2])$ is a QS-subalgebra of X.

(iv) $\Longrightarrow$ (iii). It is clear.

(iii) $\Longrightarrow$ (ii). If $[\delta_1, \delta_2] \in \Im(\mu_A)$, then $U(A; [\delta_1, \delta_2])$ is nonempty, since

$$U(A; [\delta_1, \delta_2]) = \bigcap_{[\delta_1, \delta_2] > [\alpha_1, \alpha_2]} U(A; >, [\delta_1, \delta_2]),$$

where $[\alpha_1, \alpha_2] \in \Im(\mu_A) \setminus B$. Then by (iii) and Corollary 3.7, $U(A; [\delta_1, \delta_2])$ is a QS-subalgebra of X.

(ii) $\Longrightarrow$ (v). Let $[\delta_1, \delta_2] \in D[0,1]$ and $U(A; [\delta_1, \delta_2])$ be nonempty. Suppose $x, y \in U(A; [\delta_1, \delta_2])$. Let $[\beta_1, \beta_2] = \min\{\mu_A(x), \mu_A(y)\}$, it is clear that

$$[\beta_1, \beta_2] = \min\{\mu_A(x), \mu_A(y)\} \geq \{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = [\delta_1, \delta_2].$$

Thus $x, y \in U(A; [\beta_1, \beta_2])$ and $[\beta_1, \beta_2] \in \Im(\mu_A)$, by (ii) $U(A; [\beta_1, \beta_2])$ is a QS-subalgebra of X, hence $x * y \in U(A; [\beta_1, \beta_2])$. Then we have

$$\overline{\mu}_A(x * y) \geq \text{rmin}\{\mu_A(x), \mu_A(y)\} \geq \{[\beta_1, \beta_2], [\beta_1, \beta_2]\} = [\beta_1, \beta_2] \geq [\delta_1, \delta_2].$$

Therefore $x * y \in U(A; [\delta_1, \delta_2])$. Then $U(A; [\delta_1, \delta_2])$ is a QS-subalgebra of X.

(v) $\Longrightarrow$ (i). Assume that the nonempty set $U(A; [\delta_1, \delta_2])$ is a QS-subalgebra of X, for every $[\delta_1, \delta_2] \in D[0,1]$. In the contrary, let $x_0, y_0 \in X$ be such that

$$\overline{\mu}_A(x_0 * y_0) < \text{rmin}\{\overline{\mu}_A(x_0), \overline{\mu}_A(y_0)\}.$$

Let $\overline{\mu}_A(x_0) = [\gamma_1, \gamma_2]$, $\overline{\mu}_A(y_0) = [\gamma_3, \gamma_4]$ and $\overline{\mu}_A(x_0 * y_0) = [\delta_1, \delta_2]$. Then

$$[\delta_1, \delta_2] < \text{rmin}\{[\gamma_1, \gamma_2], [\gamma_3, \gamma_4]\} = [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}].$$

So $\delta_1 < \min\{\gamma_1, \gamma_3\}$ and $\delta_2 < \min\{\gamma_2, \gamma_4\}$.

Consider

$$[\lambda_1, \lambda_2] = \frac{1}{2}\overline{\mu}_A(x_0 * y_0) + \text{rmin}\{\overline{\mu}_A(x_0), \overline{\mu}_A(y_0)\}.$$

We find that

$$\begin{aligned}[\lambda_1, \lambda_2] &= \frac{1}{2}([\delta_1, \delta_2] + [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}]) \\ &= [\frac{1}{2}(\delta_1 + \min\{\gamma_1, \gamma_3\}), \frac{1}{2}(\delta_2 + \min\{\gamma_2, \gamma_4\})].\end{aligned}$$

Therefore

$$\min\{\gamma_1, \gamma_3\} > \lambda_1 = \frac{1}{2}(\delta_1 + \min\{\gamma_1, \gamma_3\}) > \delta_1,$$

$$\min\{\gamma_2, \gamma_4\} > \lambda_2 = \frac{1}{2}(\delta_2 + \min\{\gamma_2, \gamma_4\}) > \delta_2.$$

Hence

$$[\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] > [\lambda_1, \lambda_2] > [\delta_1, \delta_2] = \overline{\mu}_A(x_0 * y_0),$$

so that $x_0 * y_0 \notin U(A; [\delta_1, \delta_2])$ which is a contradiction, since

$$\overline{\mu}_A(x_0) = [\gamma_1, \gamma_2] \geq [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] > [\lambda_1, \lambda_2],$$

$$\overline{\mu}_A(y_0) = [\gamma_3, \gamma_4] \geq [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] > [\lambda_1, \lambda_2],$$

imply that $x_0, y_0 \in U(A; [\delta_1, \delta_2])$. Thus $\overline{\mu}_A(x * y) \geq \text{rmin}\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$ for all $x, y \in X$, which completes the proof. □

Theorem 3.5. *Each QS-subalgebra of X is an i-v level QS-subalgebra of an i-v fuzzy QS-subalgebra of X.*

Proof. Let Y be a QS-subalgebra of X, and A be an i-v fuzzy set on X defined by

$$\overline{\mu}_A(x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in Y \\ [0, 0] & \text{otherwise} \end{cases}$$

where $\alpha_1, \alpha_2 \in [0, 1]$ with $\alpha_1 < \alpha_2$. It is clear that $U(A; [\alpha_1, \alpha_2]) = Y$. Let $x, y \in X$. If $x, y \in Y$, then $x * y \in Y$ and therefore

$$\overline{\mu}_A(x * y) = [\alpha_1, \alpha_2] = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = \text{rmin}\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}.$$

If $x, y \notin Y$, then $\overline{\mu}_A(x) = [0, 0] = \overline{\mu}_A(y)$ and so

$$\overline{\mu}_A(x * y) \geq [0, 0] = \text{rmin}\{[0, 0], [0, 0]\} = \text{rmin}\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}.$$

If $x \in Y$ and $y \notin Y$, then $\overline{\mu}_A(x) = [\alpha_1, \alpha_2]$ and $\overline{\mu}_A(y) = [0, 0]$. Thus

$$\overline{\mu}_A(x * y) \geq [0, 0] = \text{rmin}\{[\alpha_1, \alpha_2], [0, 0]\} = \text{rmin}\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}.$$

Similarly, if $y \in Y$ and $x \notin Y$, then $\overline{\mu}_A(x * y) \geq \text{rmin}\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$. Therefore A is an i-v fuzzy QS-subalgebra of X. □

Theorem 3.6. *Let Y be a subset of X and A be the i-v fuzzy set on X which is given in the proof of Theorem 3.5 If A is an i-v fuzzy QS-subalgebra of X, then Y is a QS-subalgebra of X.*

Proof. Let A be an i-v fuzzy QS-subalgebra of X, and $x, y \in Y$. Then $\overline{\mu}_A(x) = [\alpha_1, \alpha_2] = \overline{\mu}_A(y)$, thus

$$\overline{\mu}_A(x * y) \geq \text{rmin}\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]$$

which implies that $x * y \in Y$. □

Theorem 3.7. *If A is an i-v fuzzy QS-subalgebra of X, then the set*

$$X_{\overline{\mu}_A} := \{x \in X \mid \overline{\mu}_A(x) = \overline{\mu}_A(0)\},$$

is a QS-subalgebra of X.

Proof. Let $x, y \in X_{\overline{\mu}_A}$. Then $\overline{\mu}_A(x) = \overline{\mu}_A(0) = \overline{\mu}_A(y)$, and so

$$\overline{\mu}_A(x * y) \geq \text{rmin}\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} = \text{rmin}\{\overline{\mu}_A(0), \overline{\mu}_A(0)\} = \overline{\mu}_A(0).$$

By Lemma 3.1, we get that $\overline{\mu}_A(x * y) = \overline{\mu}_A(0)$ which means that $x * y \in X_{\overline{\mu}_A}$. □

Theorem 3.8. *Let N be an i-v fuzzy subset of X defined by*

$$\overline{\mu}_N(x) = \begin{cases} [\alpha_1, \alpha_2] & \textit{if } x \in N \\ [\beta_1, \beta_2] & \textit{otherwise} \end{cases}$$

for all $[\alpha_1, \alpha_2], [\beta_1, \beta_2] \in D[0, 1]$ with $[\alpha_1, \alpha_2] \geq [\beta_1, \beta_2]$. Then N is an i-v fuzzy QS-subalgebra if and only if N is a QS-subalgebra of X. Moreover, in this case $X_{\overline{\mu}_N} = N$.

Proof. Let N be an i-v fuzzy QS-subalgebra. Let $x, y \in X$ be such that $x, y \in N$. Then

$$\overline{\mu}_N(x * y) \geq \text{rmin}\{\overline{\mu}_N(x), \overline{\mu}_N(y)\} = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2],$$

and so $x * y \in N$.

Conversely, suppose that N is a QS-subalgebra of X, and let $x, y \in X$.

(i) If $x, y \in N$ then $x * y \in N$, thus

$$\overline{\mu}_N(x * y) = [\alpha_1, \alpha_2] = \text{rmin}\{\overline{\mu}_N(x), \overline{\mu}_N(y)\}.$$

(ii) If $x \notin N$ or $y \notin N$, then

$$\overline{\mu}_N(x * y) \geq [\beta_1, \beta_2] = \text{rmin}b\{\overline{\mu}_N(x), \overline{\mu}_N(y)\}.$$

This shows that N is an i-v fuzzy QS-subalgebra.

Moreover, we have

$$X_{\overline{\mu}_N} := \{x \in X \mid \overline{\mu}_N(x) = \overline{\mu}_N(0)\} = \{x \in X \mid \overline{\mu}_N(x) = [\alpha_1, \alpha_2]\} = N.$$

□

Definition 3.3. [2] *Let f be a mapping from the set X into a set Y. Let B be an i-v fuzzy set in Y. Then the inverse image of B, denoted by $f^{-1}[B]$, is the i-v fuzzy set in X with the membership function given by $\overline{\mu}_{f^{-1}[B]}(x) = \overline{\mu}_B(f(x))$, for all $x \in X$.*

Lemma 3.2. [2] *Let f be a mapping from the set X into the set Y. Let $m = [m^L, m^U]$ and $n = [n^L, n^U]$ be i-v fuzzy sets in X and Y respectively. Then*

(i) $f^{-1}(n) = [f^{-1}(n^L), f^{-1}(n^U)]$,

(ii) $f(m) = [f(m^L), f(m^U)]$.

Proposition 3.2. *Let f be a QS-homomorphism from X into Y and G be an i-v fuzzy QS-subalgebra of Y with the membership function μ_G. Then the inverse image $f^{-1}[G]$ of G is an i-v fuzzy QS-subalgebra of X.*

Proof. Since $B = [\mu_B^L, \mu_B^U]$ is an i-v fuzzy QS-subalgebra of Y, by Theorem 3.2, we get that μ_B^L and μ_B^U are fuzzy QS-subalgebras of Y. By Proposition 2.2, $f^{-1}[\mu_B^L]$ and $f^{-1}[\mu_B^U]$ are fuzzy QS-subalgebras of X. By the above lemma and Theorem 3.2, we can conclude that $f^{-1}(B) = [f^{-1}(\mu_B^L), f^{-1}(\mu_B^U)]$ is an i-v fuzzy QS-subalgebra of X. □

Definition 3.4. [2] *Let f be a mapping from the set X into a set Y, and A be an i-v fuzzy set in X with membership function μ_A. Then the image of A, denoted by $f[A]$, is the i-v fuzzy set in Y with membership function defined by:*

$$\overline{\mu}_{f[A]}(y) = \begin{cases} \mathrm{rsup}_{z \in f^{-1}(y)} \overline{\mu}_A(z) & \text{if } f^{-1}(y) \neq \emptyset, \forall y \in Y, \\ [0,0] & \text{otherwise} \end{cases}$$

in which $f^{-1}(y) = \{x \mid f(x) = y\}$.

Theorem 3.9. *Let f be a QS-homomorphism from X onto Y. If A is an i-v fuzzy QS-subalgebra of X, then the image $f[A]$ of A is an i-v fuzzy QS-subalgebra of Y.*

Proof. Assume that A is an i-v fuzzy QS-subalgebra of X, then $A = [\mu_A^L, \mu_A^U]$ is an i-v fuzzy QS-subalgebra of X if and only if μ_B^L and μ_B^U are fuzzy QS-subalgebras of X. By Proposition 2.3, $f[\mu_A^L]$ and $f[\mu_A^U]$ are fuzzy QS-subalgebras of Y. By Lemma 3.2, and Theorem 3.2, we can conclude that $f[A] = [f[\mu_A^L], f[\mu_A^U]]$ is an i-v fuzzy QS-subalgebra of Y. □

Acknowledgements. The author wishes to express his gratitude to the referee for his/her suggestions to improve the readability of this paper.

References

[1] S. S. Ahn and H. S. Kim, On QS-algebras, *J. Chungcheong Math. Soc.* **12**(1999), 1–7.
[2] R. Biswas, Rosenfeld's fuzzy subgroups with interval valued membership function, *Fuzzy Sets and Systems* **63**(1) (1994), 87–90.
[3] A. Borumand Saeid, On fuzzy QS-subalgebras, *preprint.*
[4] S. M. Hong, Y. B. Jun, S. J. Kim and G. I. Kim, Fuzzy BCI-subalgebras with interval-valued membership functions, *Int. J. Math. Math. Sci.* **25**(2)(2001), 135–143.
[5] Q. P. Hu and X. Li, On BCH-algebras, *Math. Sem. Notes Kobe Univ.* **11**(2)(1983), part 2, 313–320.
[6] Q. P. Hu and X. Li, On proper BCH-algebras, *Math. Japon.* **30**(4)(1985), 659–661.
[7] Y. Imai and K. Iseki, On axiom systems of propositional calculi, *XIV. Proc. Japan Acad.* **42**(1966), 19–22.
[8] K. Iseki and S. Tanaka, An introduction to the theory of BCK-algebras, *Math. Japon.* **23**(1) (1978/79), 1–26.
[9] K. Iseki, On BCI-algebras, *Math. Sem. Notes Kobe Univ.* **8**(1)(1980), 125–130.
[10] Y. B. Jun and E. H. Roh, On the $BCI-G$ part of BCI-algebras, *Math. Japon.* **38** (4)(1993), 697–702.
[11] J. Neggers, S. S. Ahn and H. S. Kim, On Q-algebras, *Int. J. Math. Math. Sci.* **27**(12)(2001), 749–757.
[12] L. A. Zadeh, Fuzzy Sets, *Information and Control* **8**(1965), 338–353.
[13] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning. I, *Information Sci.* **8**(1975), 199–249.