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On Relative $1\frac{1}{2}$-StarLindelöfness

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Abstract. A subspace Y of a space X is *strongly* $1\frac{1}{2}$*-starLindelöf* in X if for every open cover $\mathcal{U}$ of X, there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $V \cap Y \neq \emptyset$ for each $V \in \mathcal{V}$ and $Y \subseteq St(\bigcup\mathcal{V}, \mathcal{U})$, where $St(\bigcup\mathcal{V}, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap \bigcup\mathcal{V} \neq \emptyset\}$. A subspace Y of a space X is $1\frac{1}{2}$*-starLindelöf* in X if for every open cover $\mathcal{U}$ of X, there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $Y \subseteq St(\cup\mathcal{V}, \mathcal{U})$. In this paper, we give an example to show the difference between relative strongly $1\frac{1}{2}$-starLindelöfness and relative $1\frac{1}{2}$-starLindelöfness.

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1. Introduction

By a space, we mean a topological space. Recall from [1,2,5] that a subspace Y of a space X is Lindelöf in X if for every open cover $\mathcal{U}$ of X, there exists a countable subfamily covering Y. A space X is *starLindelöf* (for example, see [3,6]) if for every open cover $\mathcal{U}$ of X, there exists a countable subset F of X such that $St(F, \mathcal{U}) = X$. A subspace Y of a space X is *starLindelöf*(*strongly starLindelöf*) (see [5,7,8]) in X if for every open cover $\mathcal{U}$ of X, there exists a countable subset F of X (respectively, $F \subseteq Y$) such that $Y \subseteq St(F, \mathcal{U})$, where $St(F, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$. A space X is $1\frac{1}{2}$*-starLindelöf* (by different names, see [3,6]) if for every open cover $\mathcal{U}$ of X, there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $St(\bigcup\mathcal{V}, \mathcal{U}) = X$. It is natural to define the following classes of spaces:

Definition 1.1. A subspace Y of a space X is *strongly* $1\frac{1}{2}$*-starLindelöf* in X if for every open cover $\mathcal{U}$ of X, there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $V \cap Y \neq \emptyset$ for each $V \in \mathcal{V}$ and $Y \subseteq St(\bigcup\mathcal{V}, \mathcal{U})$.

Definition 1.2. A subspace Y of a space X is $1\frac{1}{2}$*-starLindelöf* in X if for every open cover $\mathcal{U}$ of X, there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $Y \subseteq St(\bigcup\mathcal{V}, \mathcal{U})$.

Let X be a space and Y a subspace of X. From the above definitions, it is clear that if Y is starLindelöf in X, then Y is $1\frac{1}{2}$-starLindelöf in X; if Y is strongly starLindelöf in X, then Y is strongly $1\frac{1}{2}$-starLindelöf in X; if Y is strongly $1\frac{1}{2}$-starLindelöf in X, then Y is $1\frac{1}{2}$-starLindelöf in X. But, the converses do not hold.

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The purpose of this paper is to show the difference among these properties in the class of Tychonoff spaces by giving an example.

The cardinality of a set A is denoted by $|A|$. For a cardinal κ, κ^+ denotes the smallest cardinal greater than κ. Let c denote the cardinality of the continuum. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. For ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$, $[\alpha, \beta) = \{\gamma : \alpha \leq \gamma < \beta\}$ and $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$. Other terms and symbols that we do not define will be used as in [4].

2. An example of relatively $1\frac{1}{2}$-starLindelöf spaces

In this section, we clarify the relations among these spaces defined in the first section by giving an example.

Given a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

Example 2.1. There exist a Tychonoff space X and subspaces Y_1, Y_2 of X such that Y_1 is $1\frac{1}{2}$-starLindelöf in X but Y_1 is neither strongly $1\frac{1}{2}$-starLindelöf in X nor starLindelöf in X and Y_2 is strongly $1\frac{1}{2}$-starLindelöf in X, but Y_2 is not strongly starLindelöf in X.

Proof. Let D be the discrete space of cardinality c and let S_1 be the subspace

$$(\beta D \times (c^+ + 1)) \setminus ((\beta D \setminus D) \times \{c^+\})$$

of the product of βD and $c^+ + 1$. Let S_2 be the subspace

$$(\beta D \times (c + 1)) \setminus ((\beta D \setminus D) \times \{c\})$$

of the product of βD and $c + 1$.

Since $|D| = c$, we can enumerate D as $\{d_\alpha : \alpha < c\}$. Let

$$\varphi : D \times \{c^+\} \to D \times \{c\}$$

be a map defined by $\varphi(\langle d_\alpha, c^+\rangle) = \langle d_\alpha, c\rangle$ for each $\alpha < c$. Let X be the quotient space obtained from the disjoint topological sum $S_1 \oplus S_2$ by identifying $\langle d_\alpha, c^+\rangle$ of S_1 and $\langle d_\alpha, c\rangle$ of S_2 for each $\alpha < c$. Let $\pi : S_1 \oplus S_2 \to X$ be the quotient map.

Let $Y_1 = \pi(S_2)$ and $Y_2 = \pi(S_1)$. Let us show that Y_1 is $1\frac{1}{2}$-starLindelöf in X. Let $\mathcal{U}$ be an open cover of X. Since c is locally compact and countably compact[4], it follows from [4, Corollary 3.10.14] that $\beta D \times c$ is countably compact, thus $\pi(\beta D \times c)$ is countably compact, hence, $1\frac{1}{2}$-starLindelöf. Hence, there exists a countable subset $\mathcal{V}'$ of $\mathcal{U}$ such that

$$\pi(\beta D \times c) \subseteq St(\bigcup \mathcal{V}', \mathcal{U}).$$

On the other hand, for each $\alpha < c$, there exist $U_\alpha \in \mathcal{U}$ and $\beta_\alpha < c^+$ such that

$$\pi(\langle d_\alpha, c\rangle) \in \pi(\{d_\alpha\} \times (\beta_\alpha, c^+]) \subseteq U_\alpha.$$

If we choose $\alpha_0 < c^+$ with $\alpha_0 > \sup\{\beta_\alpha : \alpha < c\}$. Then, $\pi(\beta D \times \{\alpha_0\})$ is compact. Hence, there exists a finite set $\mathcal{V}''$ of $\mathcal{U}$ such that

$$\pi(\beta D \times \{\alpha_0\}) \subseteq \cup \mathcal{V}''.$$

Since for each $\alpha < c$, $U_\alpha \cap \bigcup \mathcal{V}'' \neq \emptyset$, then

$$\pi(D \times \{c\}) \subseteq St(\bigcup \mathcal{V}'', \mathcal{U}).$$

If we put $\mathcal{V} = \mathcal{V}' \cup \mathcal{V}''$, then $\mathcal{V}$ is a countable subset of $\mathcal{U}$ such that

$$\pi(S_2) = Y_1 \subseteq St(\bigcup \mathcal{V}, \mathcal{U}),$$

this shows that Y_1 is $1\frac{1}{2}$-starLindelöf in X.

Next, we show that Y_1 is not strongly $1\frac{1}{2}$-starLindelöf in X. Let

$$U_\alpha = \pi((\{d_\alpha\} \times [0, c^+]) \cup (\{d_\alpha\} \times (\alpha, c]))$$

and

$$V_\alpha = \pi(\beta D \times [0, \alpha))$$

for each $\alpha < c$. Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < c\} \cup \{V_\alpha : \alpha < c\} \cup \{\pi(\beta D \times [0, c^+))\}$$

of X. For any countable subfamily $\mathcal{V}$ of $\mathcal{U}$, let

$$\alpha_1 = \sup\{\alpha : U_\alpha \in \mathcal{V}\}$$

and

$$\alpha_2 = \sup\{\alpha : V_\alpha \in \mathcal{V}\}.$$

Then, $\alpha_1, \alpha_2 < c$, since $\mathcal{V}$ is countable. If we pick $\alpha_0 > \max\{\alpha_1, \alpha_2\}$, then

$$\pi(\langle d_{\alpha_0}, c\rangle) \notin St(\cup \mathcal{V}, \mathcal{U}).$$

Since U_{α_0} is the only element of $\mathcal{U}$ containing $\pi(\langle d_{\alpha_0}, c\rangle)$ and $U_{\alpha_0} \cap \cup \mathcal{V} = \emptyset$, this shows that Y_1 is not strongly $1\frac{1}{2}$-starLindelöf in X.

We show that Y_1 is not starLindelöf in X. For each $\alpha < c$, let

$$U_\alpha = \pi((\{d_\alpha\} \times [0, c^+]) \cup (\{d_\alpha\} \times [0, c])).$$

Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < c\} \cup \{\pi(\beta D \times [0, c))\} \cup \{\pi(\beta D \times [0, c^+))\}.$$

Let F be a countable subset of X. Then, there exists a $\alpha_1 < c$ such that $F \cap U_\alpha = \emptyset$ for every $\alpha > \alpha_1$ by the definition of U_α and countability of F. Pick $\alpha_0 > \alpha_1$. Then,

$$\pi(\langle d_{\alpha_0}, c\rangle) \notin St(F, \mathcal{U}),$$

since U_{α_0} is the only element of $\mathcal{U}$ containing $\pi(\langle d_{\alpha_0}, c\rangle)$ and $U_{\alpha_0} \cap F = \emptyset$, this shows that Y_1 is not starLindelöF in X.

Next, we show that Y_2 is strongly $1\frac{1}{2}$-starLindelöf in X, let $\mathcal{U}$ be an open cover of X. Since c^+ is locally compact and countably compact, it follows from [4, Corollary 3.10.14] that $\beta D \times c^+$ is countably compact. Thus, $\pi(\beta D \times c^+)$ is countably compact, hence, $1\frac{1}{2}$-starLindelöf. Hence, there exists a countable subset $\mathcal{V}' \subseteq \mathcal{U}$ such that

$$\pi(\beta D \times c^+) \subseteq St(\bigcup \mathcal{V}', \mathcal{U}).$$

On the other hand, for each $\alpha < c$, there exist $U_\alpha \in \mathcal{U}$ and $\beta_\alpha < c^+$ such that

$$\pi(\langle d_\alpha, c\rangle) \in \pi(\{d_\alpha\} \times (\beta_\alpha, c^+]) \subseteq U_\alpha.$$

If we choose $\alpha_0 < c^+$ with $\alpha_0 > \sup\{\beta_\alpha : \alpha < c\}$. Then, $\pi(\beta D \times \{\alpha_0\})$ is compact. Hence, there exists a finite subset $\mathcal{V}'' \subseteq \mathcal{U}$ such that

$$\pi(\beta D \times \{\alpha_0\}) \subseteq \cup \mathcal{V}''.$$

Since for each $\alpha < c$, $U_\alpha \cap \bigcup \mathcal{V}'' \neq \emptyset$, then

$$\pi(D \times \{c^+\}) \subseteq St(\bigcup \mathcal{V}'', \mathcal{U}).$$

If we put $\mathcal{V} = \mathcal{V}' \cup \mathcal{V}''$, then $\mathcal{V}$ is a countable subset of $\mathcal{U}$ such that $V \cap Y_2 \neq \emptyset$ for each $V \in \mathcal{V}$ and $\pi(S_1) = Y_2 \subseteq St(\bigcup \mathcal{V}, \mathcal{U})$. This shows that Y_2 is strongly $1\frac{1}{2}$-starLindelöf in X.

Let us show that Y_2 is not strongly starLindelöf in X. Let $U_\alpha = \pi((\{d_\alpha\} \times [0, c^+]) \cup (\{d_\alpha\} \times [0, c]))$ for each $\alpha < c$. Let us consider the open cover

$$\mathcal{U} = \{U_\alpha : \alpha < c\} \cup \{\pi(\beta D \times [0, c))\} \cup \{\pi(\beta D \times [0, c^+))\}.$$

Let F be a countable subset of Y_2. Then, there exists a $\alpha_1 < c$ such that $F \cap U_\alpha = \emptyset$ for every $\alpha > \alpha_1$ by the definition of U_α and countability of F. Pick $\alpha_0 > \alpha_1$. Then,

$$\pi(\langle d_{\alpha_0}, c\rangle) \notin St(F, \mathcal{U}).$$

Since U_{α_0} is the only element of $\mathcal{U}$ containing $\pi(\langle d_{\alpha_0}, c\rangle)$ and $U_{\alpha_0} \cap F = \emptyset$, this shows that Y_2 is not strongly starLindelöf in X, which completes the proof. □

Remark 2.1. From the proof of Example 2.1, it is not difficult to see that X is $1\frac{1}{2}$-starLindelöf and Y_2 is $1\frac{1}{2}$-starLindelöf (in itself), but Y_2 is not starLindelöf in X.

Remark 2.2. The authors do not know if there exist a normal space X and a subspace Y of X such that Y is $1\frac{1}{2}$-starLindelöf in X, but Y is not strongly $1\frac{1}{2}$-starLindelöf in X.

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