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# About Uniform Boundedness and Convergence of Solutions of Certain Non-Linear Differential Equations of Fifth-Order

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**Abstract.** In this paper, we establish sufficient conditions under which all solutions of equation of the type $x^{(5)} + f(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) + \phi(t, \dot{x}, \ddot{x}, \dddot{x}) + \psi(t, x, \dot{x}, \ddot{x}) + g(t, x, \dot{x}) + e(t)h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})$ are uniformly bounded and tend to zero as $t \to \infty$. Our theorem is stated in a more general form; it extends some related results known in the literature. Also, the relevance of our result is to show that the results established in Abou El-Ela and Sadek [2,3] and Sadek [13] contain some superfluous conditions.

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## 1. Introduction

We consider the fifth order nonlinear differential equation

$$\begin{aligned} x^{(5)} &+ f(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}) + \phi(t, \dot{x}, \ddot{x}, \dddot{x}) + \psi(t, x, \dot{x}, \ddot{x}) + g(t, x, \dot{x}) \\ &+ e(t)h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}), \end{aligned} \tag{1.1}$$

where $f$, $\phi$, $\psi$, $g$, $e$, $h$ and $p$ are continuous functions for the arguments displayed explicitly and the dots denote differentiation with respect to $t$. All solutions considered are assumed real valued.

As we know, the investigation of qualitative behavior of solutions, peculiarly, the discussion of uniform boundedness and convergence of solutions is a very important subject in the theory of ordinary differential equations. Up to now, one of the most effective methods to study the uniform boundedness and convergence of solutions of certain nonlinear ordinary differential equations is still the Lyapunov's direct (or second) method [9]. The major advantage of this method is that the uniform boundedness and convergence of solutions can be obtained without any prior knowledge of solutions. Its chief characteristic is based on the construction of an appropriate scalar function, Lyapunov function. This function and its time derivative along the

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system under consideration must satisfy some fundamental inequalities. However, it is well-known that finding an appropriate Lyapunov function is in general a difficult task. Notwithstanding the difficulty, during the past 40 years or so, by employing the Lyapunov's method, many good and interesting results have been obtained concerning the uniform boundedness and convergence of solutions for various second-, third-, fourth and fifth order certain nonlinear differential equations (See, for example, [1–8,10–26,28] and the references quoted therein). But, in this connection, only a few results exist in the relevant literature on the uniform boundedness and convergence of solutions of certain nonlinear differential equations of the fifth order. The interested reader is advised to look up the references [1–3,21,23,28]. It should be noted that, in 1999 and 2000, Abou-El-Ela and Sadek [2,3], in 2002, Sadek [13], in 2003, Tunc [21] and, recently, Tunç ([23], [24)] discussed the same matter for the differential equations given by

$$\begin{aligned} &x^{(5)} + a(t)f_1(\ddot{x}, \dddot{x})x^{(4)} + b(t)f_2(\ddot{x}, \dddot{x}) + c(t)f_3(\ddot{x}) + d(t)f_4(\dot{x}) \\ &\quad + e(t)f_5(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \\ &x^{(5)} + f(t, \dot{x}, \ddot{x}, \dddot{x})x^{(4)} + \phi(t, \ddot{x}, \dddot{x}) + \psi(t, \ddot{x}) + g(t, \dot{x}) \\ &\quad + e(t)h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}), \\ &x^{(5)} + f_1(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})x^{(4)} + b(t)f_2(\ddot{x}, \dddot{x}) + c(t)f_3(\dot{x}, \ddot{x}) + d(t)f_4(\dot{x}) \\ &\quad + e(t)f_5(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}), \\ &x^{(5)} + f_1(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})x^{(4)} + b(t)f_2(x, \dot{x}, \ddot{x}, \dddot{x}) + c(t)f_3(x, \dot{x}, \ddot{x}) + d(t)f_4(x, \dot{x}) \\ &\quad + e(t)f_5(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)}), \\ &x^{(5)} + f(t, x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})x^{(4)} + \phi(t, \ddot{x}, \dddot{x}) + \psi(t, \dot{x}, \ddot{x}) + g(t, x, \dot{x}) \\ &\quad + e(t)h(x) = p(t, x, \dot{x}, \ddot{x}, x, x^{(4)}), \end{aligned}$$

respectively. The motivation for the present work has been inspired basically by the papers mentioned above and the references listed in that paper.

In what follows it will be convenient to use the equivalent differential system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= w \\ \dot{w} &= u \\ \dot{u} &= -f(t, x, y, z, w, u) - \phi(t, y, z, w) - \psi(t, x, y, z) \\ &\quad - g(t, x, y) - e(t)h(x) + p(t, x, y, z, w, u), \end{aligned} \tag{1.2}$$

which is obtained from equation (1.1) by setting $\dot{x} = y$, $\ddot{x} = z$, $x = w$ and $x^{(4)} = u$.

## 2. Assumption and statement of the result

The following assumptions will be accepted on the functions that appeared in equation (1.1).

**Assumptions:**

1. The function $h(x)$is a continuously differentiable function in $\mathbb{R}^1$, and $e(t)$ is a continuously differentiable function in $\mathbb{R}^+ = [0, \infty)$.

2. The function $g(t, x, y)$is continuous in $\mathbb{R}^+ \times \mathbb{R}^2$, and for the function $g(t, x, y)$ there exist non-negative functions $d(t)$, $g_0(x, y)$ and $g_1(x, y)$ which satisfy the inequalities
$$d(t)g_0(x, y) \leq g(t, x, y) \leq d(t)g_1(x, y)$$
for all $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R}^2$. The function $d(t)$ is continuously differentiable for all $t \in \mathbb{R}^+$. Let
$$\tilde{g}(x, y) \equiv \frac{1}{2}\left[g_0(x, y) + g_1(x, y)\right],$$
and $\tilde{g}(x, y)$, $\frac{\partial}{\partial x}\tilde{g}(x, y)$ and $\frac{\partial}{\partial y}\tilde{g}(x, y)$ are continuous for all $(x, y) \in \mathbb{R}^2$.

3. The function $\psi(t, x, y, z)$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^3$. For the function $\psi(t, x, y, z)$ there exist non-negative functions $c(t)$, $\psi_0(x, y, z)$ and $\psi_1(x, y, z)$ which satisfy the inequalities
$$c(t)\psi_0(x, y, z) \leq \psi(t, x, y, z) \leq c(t)\psi_1(x, y, z)$$
for all $(t, x, y, z) \in \mathbb{R}^+ \times \mathbb{R}^3$. The function $c(t)$ is continuously differentiable for all $t \in \mathbb{R}^+$. Let
$$\tilde{\psi}(x, y, z) \equiv \frac{1}{2}\left[\psi_0(x, y, z) + \psi_1(x, y, z)\right],$$
and
$$\tilde{\psi}(x, y, z), \quad \frac{\partial}{\partial x}\tilde{\psi}(x, y, z) \quad \text{and} \quad \frac{\partial}{\partial y}\tilde{\psi}(x, y, z)$$
are continuous for all $(x, y, z) \in \mathbb{R}^3$.
4. The function $\phi(t, y, z, w)$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^3$. For the function $\phi(t, y, z, w)$ there exist non-negative functions $b(t)$, $\phi_0(y, z, w)$ and $\phi_1(y, z, w)$ which satisfy the inequalities
$$b(t)\phi_0(y, z, w) \leq \phi(t, y, z, w) \leq b(t)\phi_1(y, z, w)$$
for all $(t, y, z, w) \in \mathbb{R}^+ \times \mathbb{R}^3$. The function $b(t)$ is continuously differentiable for all $t \in \mathbb{R}^+$. Let
$$\tilde{\phi}(y, z, w) \equiv \frac{1}{2}\left[\phi_0(y, z, w) + \phi_1(y, z, w)\right],$$
and
$$\tilde{\phi}(y, z, w), \quad \frac{\partial}{\partial y}\tilde{\phi}(y, z, w) \quad \text{and} \quad \frac{\partial}{\partial z}\tilde{\phi}(y, z, w)$$
are continuous for all $(y, z, w) \in \mathbb{R}^3$.

**Remark 2.1.** The above assumptions are less restrictive than those established in Sadek [13]. Because the result investigated in Sadek [13] can be proved here without the assumptions
$$a(t)f_0(y, z, w) \leq f(t, y, z, w) \leq a(t)f_1(y, z, w)$$

and

$$\tilde{f}(y,z,w) \equiv \frac{1}{2}\left[f_0(y,z,w) + f_1(y,z,w)\right]$$

constituted there.

## 3. The main result

The following result is established.

**Theorem 3.1.** *Assume that the assumptions 1–4 hold and also suppose the existence of arbitrary positive constants $\alpha_1, ..., \alpha_5$ and of sufficiently small positive constants $\varepsilon$, $\varepsilon_0$, $\varepsilon_1, \ldots,$ $\varepsilon_5$ such that the following conditions are satisfied*

(i) *$B$,$C$, $D$,$E$, $b_0$, $c_0$, $d_0$ and $e_0$ are some constants satisfying the inequalities $B \geq b(t) \geq b_0 \geq 1$, $C \geq c(t) \geq c_0 \geq 1$, $D \geq d(t) \geq d_0 \geq 1$, $E \geq e(t) \geq e_0 \geq 1$ for all $t \in \mathbb{R}^+$.*

(ii) *The constants $\alpha_1, \ldots, \alpha_5$ satisfy the inequalities.*

$$\alpha_1 > 0, \quad \alpha_1\alpha_2 - \alpha_3 > 0, \quad (\alpha_1\alpha_2 - \alpha_3)\alpha_3 - (\alpha_1\alpha_4 - \alpha_5)\alpha_1 > 0, \tag{3.1}$$

$$\delta_o := \alpha_1 > 0, (\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3) - (\alpha_1\alpha_4 - \alpha_5)^2 > 0, \quad \alpha_5 > 0, \tag{3.2}$$

$$\Delta_1 := \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \left[\alpha_1 d(t)\frac{\partial}{\partial y}\tilde{g}(x,y) - \alpha_5\right] \geq 2\varepsilon\alpha_2 \tag{3.3}$$

*for all $t \in \mathbb{R}^+$ and all $x$ and $y$,*

$$\Delta_2 := \frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} - \frac{(\alpha_1\alpha_4 - \alpha_5).\gamma.d(t)}{\alpha_4(\alpha_1\alpha_2 - \alpha_5)} - \frac{\varepsilon}{\alpha_1} > 0 \tag{3.4}$$

*for all $t \in \mathbb{R}^+$ and all $x$ and $y$, where*

$$\gamma := \begin{cases} \dfrac{\tilde{g}(x,y)}{y} & (y \neq 0) \\ \dfrac{\partial}{\partial y}\tilde{g}(x,0) & (y = 0). \end{cases} \tag{3.5}$$

(iii) *$\varepsilon_0 \leq \dfrac{f(t,x,y,z,w,u)}{u} - \alpha_1 \leq \varepsilon_1$ for all $t \in \mathbb{R}^+$ and all $x$, $y$, $z$, $w$ and $u \neq 0$.*

(iv) *$\phi(t,y,z,0) = 0$, $0 \leq \dfrac{\phi(t,y,z,w)}{w} - \alpha_2 \leq \varepsilon_2$ for all $t \in \mathbb{R}^+$ and all $x$, $y$, $z$ and $w \neq 0$, $zw\dfrac{\partial}{\partial y}\tilde{\phi}(y,z,w) \leq 0$ and $\dfrac{\partial}{\partial z}\tilde{\phi}(y,z,w) \leq 0$ for all $y$, $z$ and $w$.*

(v) *$\psi(t,x,y,0) = 0$, $0 \leq \dfrac{\psi(t,x,y,z)}{z} - \alpha_3 \leq \varepsilon_3$ for all $t \in \mathbb{R}^+$and all $x$, $y$ and $z \neq 0$, $yz\dfrac{\partial}{\partial x}\tilde{\psi}(x,y,z) \leq 0$ and $\dfrac{\partial}{\partial y}\tilde{\psi}(x,y,z) \leq 0$ for all $x$, $y$ and $z$.*

(vi) *The following inequalities holds*

$$\tilde{g}(x,0) = 0, \frac{g(t,x,y)}{y} \geq E\alpha_4 \text{ for all } t \in \mathbb{R}^+ \text{ and all } x \text{ and } y \neq 0, \tag{3.6}$$

$$\frac{\partial}{\partial y}\tilde{g}(x,y) - \frac{\tilde{g}(x,y)}{y} \leq \frac{\alpha_5\delta_0}{D\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)} \text{ for all } x \text{ and } y \neq 0, \tag{3.7}$$

$$\left|\alpha_4 - \frac{\partial}{\partial y}\tilde{g}(x,y)\right| \leq \varepsilon_4 \ for\ all\ x\ and\ y, \tag{3.8}$$

$$\left[\frac{\partial}{\partial x}\tilde{g}(x,y)\right]^2 \leq \min\left[\frac{\varepsilon^2\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}{16D(\alpha_1\alpha_4-\alpha_5)}, \frac{\varepsilon^2\alpha_2\alpha_4^2(\alpha_1\alpha_2-\alpha_3)}{16D\alpha_1^2(\alpha_1\alpha_4-\alpha_5)}\right] \quad for\ \ all\ x\ and\ y, \tag{3.9}$$

*and*

$$\frac{1}{y}\int_0^y \frac{\partial}{\partial x}\tilde{g}(x,\eta)d\eta \leq -\frac{\varepsilon\alpha_4}{2}\ for\ \ all\ \ x\ \ and\ y \neq\ 0. \tag{3.10}$$

(vii) $h(0) = 0$, $h(x)\operatorname{sgn} x > 0 \quad (x \neq 0)$, $H(x) \equiv \int_0^x h(\xi)d\xi \to \infty$ *as* $|x| \to \infty$, *and* $0 \leq \alpha_5 - h'(x) \leq \varepsilon_5$ *for all* $x$.

(viii) $\int_0^\infty \beta_o(t)dt < \infty, e'(t) \to 0$ *as* $t \to \infty$, *where* $\beta_o(t) := b'_+(t)+c'_+(t)+|d'(t)|+|e'(t)|$, $b'_+(t) := \max\{b'(t),0\}$, $c'_+(t) := \max\{c'(t),0\}$.

(ix) $|B(\phi_1-\phi_0)+C(\psi_1-\psi_0)+D(g_1-g_0)| \leq \Delta(y^2+z^2+w^2+u^2)^{\frac{1}{2}}$.

(x) $|p(t,x,y,z,w,u)| \leq p_1(t)+p_2(t)\left[H(x)+y^2+z^2+w^2+u^2\right]^{\frac{\sigma}{2}}+\Delta(y^2+z^2+w^2+u^2)^{\frac{1}{2}}$,

*where* $\delta$, $\Delta$ *are constants such that* $0 \leq \sigma \leq 1$, $\Delta \geq 0$ *(is sufficiently small), and* $p_1$, $p_2$ *are non-negative continuous functions satisfying*

$$\int_0^\infty p_i(t)dt < \infty \quad (i = 1, 2). \tag{3.11}$$

*Then all solutions* $x(t)$ *of equation (1.1) are uniformly bounded and satisfy*

$$x(t), \dot{x}(t), \ddot{x}(t), \dddot{x}(t), x^{(4)}(t) \to 0$$

*as* $t \to \infty$.

**Remark 3.1.** It should also be noted that the theorem just stated above includes the results [2,3,13,21,23] and improves the result obtained in [2,3,13] because the result stated in here can be proved without the restriction $A \geq a(t) \geq a_0 \geq 1$ in the theorems in [2,3,13].

## 4. Lyapunov function $V_0(t,x,y,z,w,u)$

The proof of the theorem depends on a scalar continuously differentiable Lyapunov function $V_0 = V_0(t,x,y,z,w,u)$.

We define $V_0$ as follows

$$
\begin{aligned}
2V_0 = {} & u^2 + 2\alpha_1 uw + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} uz + 2\delta uy + 2b(t)\int_0^w \tilde{\phi}(y,z,\rho)d\rho \\
& + \left[\alpha_1^2 - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\right] w^2 + 2\left[\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta\right] wz \\
& + 2\alpha_1\delta wy + 2d(t)w\tilde{g}(x,y) + 2e(t)wh(x) + 2\alpha_1 c(t)\int_0^z \tilde{\psi}(x,y,\zeta)d\zeta \\
& + \left[\frac{\alpha_2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_4 - \alpha_1\delta\right] z^2 + 2\delta\alpha_2 yz + 2\alpha_1 d(t) z\tilde{g}(x,y) \\
& - 2\alpha_5 zy + 2\alpha_1 e(t) zh(x) + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} d(t)\int_0^y \tilde{g}(x,\eta)d\eta \\
& + (\delta\alpha_3 - \alpha_1\alpha_5)y^2 + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} e(t)yh(x) + 2\delta e(t)\int_0^x h(\xi)d\xi + k
\end{aligned}
\tag{4.1}
$$

where $\delta$ is a positive constant satisfying

$$
\delta := \frac{\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} + \varepsilon, \tag{4.2}
$$

and $k$ is a positive constant to be determined later in the proof. The following lemmas will be needed in the proof of our main result.

**Lemma 4.1.** *Assume that the conditions (i)–(vii) of the theorem hold. Then there exist positive constants $D_7$ and $D_8$ such that*

$$
\begin{aligned}
& D_7\left[H(x) + y^2 + z^2 + w^2 + u^2 + k\right] \\
& \quad \le V_0 \le D_8\left[H(x) + y^2 + z^2 + w^2 + u^2 + k\right].
\end{aligned}
\tag{4.3}
$$

*Proof.* The proof of this lemma follows the lines indicated in Tunç [21], except for some minor modifications. Therefore, we omit the detailed proof of the lemma. ∎

**Lemma 4.2.** *Assume that all the conditions of the theorem are satisfied. Then there exist positive constants $D_i$ $(i = 11, 12, 13)$ such that*

$$
\begin{aligned}
\dot{V}_0 \le {} & -D_{13}(y^2 + z^2 + w^2 + u^2) + 2D_{12}(y^2 + z^2 + w^2 + u^2)^{\frac{1}{2}}[p_1(t) + p_2(t)] \\
& + 2D_{12}p_2(t)\lfloor H(x) + y^2 + z^2 + w^2 + u^2\rfloor + D_{11}\gamma_0 V_0.
\end{aligned}
\tag{4.4}
$$

*Proof.* Along any solution $(x, y, z, w, u)$ of system (1.2) from (4.1) and (1.2) for $y, z, w \neq 0$ we have

$$
\begin{aligned}
\dot{V}_0 = & - u^2\left[\frac{f(t,x,y,z,w,u)}{u} - \alpha_1\right] \\
& - w^2\left[\alpha_1\frac{\phi(t,y,z,w)}{w} - \left\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta\right\}\right] \\
& - z^2\left[\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}\frac{\psi(t,x,y,z)}{z} - \left\{\delta\alpha_2 + \alpha_1 d(t)\frac{\partial}{\partial y}\tilde{g}(x,y) - \alpha_5\right\}\right] \\
& - y^2\left[\delta\frac{g(t,x,y)}{y} - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} e(t)h'(x)\right]
\end{aligned}
$$

$$
\begin{aligned}
& - \alpha_1 uw \left[ \frac{f(t,x,y,z,w,u)}{u} - \alpha_1 \right] - uz \left[ \frac{\psi(t,x,y,z)}{z} - \alpha_3 \right] \\
& - wz \left[ \alpha_4 - d(t) \frac{\partial}{\partial y} \tilde{g}(x,y) \right] + b(t) z \int_0^w \frac{\partial}{\partial y} \tilde{\phi}(y,z,\rho) d\rho \\
& + b(t) w \int_0^w \frac{\partial}{\partial z} \tilde{\phi}(y,z,\rho) d\rho - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} uz \left[ \frac{f(t,x,y,z,w,u)}{u} - \alpha_1 \right] \\
& - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} wz \left[ \frac{\phi(t,y,z,w)}{w} - \alpha_2 \right] - \delta uy \left[ \frac{f(t,x,y,z,w,u)}{u} - \alpha_1 \right] \\
(4.5) \qquad & - wye(t) \left[ \alpha_5 - h'(x) \right] - \delta wy \left[ \frac{\phi(t,y,z,w)}{w} - \alpha_2 \right] - \alpha_1 e(t) zy \left[ \alpha_5 - h'(x) \right] \\
& - \delta zy \left[ \frac{\psi(t,x,y,z)}{z} - \alpha_3 \right] - \alpha_5 wy \left[ 1 - e(t) \right] - \alpha_1\alpha_5 zy \left[ 1 - e(t) \right] \\
& + d(t) wy \frac{\partial}{\partial x} \tilde{g}(x,y) + \alpha_1 c(t) y \int_0^z \frac{\partial}{\partial x} \tilde{\psi}(x,y,\zeta) d\zeta \\
& + \alpha_1 c(t) z \int_0^z \frac{\partial}{\partial y} \tilde{\psi}(x,y,\zeta) d\zeta + \alpha_1 d(t) zy \frac{\partial}{\partial x} \tilde{g}(x,y) \\
& + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} d(t) y \int_0^y \frac{\partial}{\partial x} \tilde{g}(x,\eta) d\eta + \frac{1}{2} \left[ c(t)(\psi_1 - \psi_0) \right] \alpha_1 w \\
& + \frac{1}{2} \left[ d(t)(g_1 - g_0) \right] \times \left[ u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z \right] \\
& + \frac{1}{2} \left[ b(t)(\phi_1 - \phi_0) \right] u \\
& + \left[ u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y \right] .p(t,x,y,z,w,u) \frac{\partial V_0}{\partial t}.
\end{aligned}
$$

The condition (iii) of the theorem shows that

$$\left[ \frac{f(t,x,y,z,w,u)}{u} - \alpha_1 \right] \geq \varepsilon_0.$$

Clearly, (iv) and (4.2) yield

$$
\begin{aligned}
& \alpha_1 \frac{\phi(t,y,z,w)}{w} - \left\{ \alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta \right\} \\
& \geq \alpha_1 \left[ \frac{\phi(t,y,z,w)}{w} - \alpha_2 \right] + \left[ \alpha_1\alpha_2 - \alpha_3 + \delta - \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \right] \geq \varepsilon
\end{aligned}
$$

for $w \neq 0$. In view of (v), (4.2) and (3.3) (for $z \neq 0$), we see that

$$
\begin{aligned}
& \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \frac{\psi(t,x,y,z)}{z} - \left[ \delta\alpha_2 + \alpha_1 d(t) \frac{\partial}{\partial y} \tilde{g}(x,y) - \alpha_5 \right] \\
& \geq \frac{\alpha_3\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \left[ \frac{\alpha_2\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} + \alpha_2\varepsilon + \alpha_1 d(t) \frac{\partial}{\partial y} \tilde{g}(x,y) - \alpha_5 \right] \\
& = \frac{(\alpha_1\alpha_2 - \alpha_3)(\alpha_3\alpha_4 - \alpha_2\alpha_5)}{\alpha_1\alpha_4 - \alpha_5} - \left[ \alpha_1 d(t) \frac{\partial}{\partial y} \tilde{g}(x,y) - \alpha_5 \right] - \varepsilon\alpha_2 \\
& > \varepsilon\alpha_2.
\end{aligned}
$$

Next, observe that (i), (vi), (vii) of the theorem and (4.2) imply

$$\begin{aligned}\delta\frac{g(t,x,y)}{y} &- \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}e(t)h'(x)\\ &\geq \delta E\alpha_4 - \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}Eh'(x)\\ &= \varepsilon E\alpha_4 + \frac{\alpha_4 E(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\left[\alpha_5 - h'(x)\right]\\ &\geq \varepsilon\alpha_4 E\end{aligned}$$

for $y \neq 0$. In view of the above discussions, it follows for the first four terms in (4.5) that

$$\begin{aligned}&-u^2\left[\frac{f(t,x,y,z,w,u)}{u}-\alpha_1\right]\\ &\quad - w^2\left[\alpha_1\frac{\phi(t,y,z,w)}{w} - \left\{\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5} - \delta\right\}\right]\\ &\quad - z^2\left[\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\frac{\psi(t,x,y,z)}{z} - \left\{\delta\alpha_2 + \alpha_1 d(t)\frac{\partial}{\partial y}\tilde{g}(x,y) - \alpha_5\right\}\right]\\ &\quad - y^2\left[\delta\frac{g(t,x,y)}{y} - \frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}e(t)h'(x)\right]\\ &\geq -(\varepsilon_0 u^2 + \varepsilon w^2 + \varepsilon\alpha_2 z^2 + \varepsilon\alpha_4 E y^2).\end{aligned} \tag{4.6}$$

Now consider the terms

$$\begin{aligned}&\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}d(t)y\int_0^y \frac{\partial}{\partial x}\tilde{g}(x,\eta)d\eta\\ &\quad + d(t)\left[wy\frac{\partial}{\partial x}\tilde{g}(x,y) + \alpha_1 zy\frac{\partial}{\partial x}\tilde{g}(x,y)\right]\end{aligned}$$

which are contained in (4.5). As it is shown in Tunc [23], the estimate there gives

$$\begin{aligned}&-\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}d(t)y^2\left[\frac{1}{y}\int_0^y \frac{\partial}{\partial x}\tilde{g}(x,\eta)d\eta\right]\\ &\quad - d(t)\left[wy\frac{\partial}{\partial x}\tilde{g}(x,y) + \alpha_1 zy\frac{\partial}{\partial x}\tilde{g}(x,y)\right]\\ &\geq -\left(\frac{\varepsilon}{16}\right)w^2 - \left(\frac{\varepsilon\alpha_2}{16}\right)z^2.\end{aligned} \tag{4.7}$$

It also follows form the conditions (iv) and (v) of the theorem that

$$\begin{aligned}z\int_0^w \frac{\partial}{\partial y}\tilde{\phi}(y,z,\rho)d\rho \leq 0, &\quad w\int_0^w \frac{\partial}{\partial z}\tilde{\phi}(y,z,\rho)d\rho \leq 0,\\ y\int_0^z \frac{\partial}{\partial x}\tilde{\psi}(x,y,\zeta)d\zeta \leq 0, &\quad z\int_0^z \frac{\partial}{\partial y}\tilde{\psi}(x,y,\zeta)d\zeta \leq 0.\end{aligned} \tag{4.8}$$

Making use of the estimates (4.6)–(4.8) in relation (4.5) we obtain

$$
\begin{aligned}
\dot{V}_0 = &- \varepsilon_0 u^2 - \left(\frac{15\varepsilon}{16}\right) w^2 - \left(\frac{15\varepsilon\alpha_2}{16}\right) z^2 - (\varepsilon\alpha_4 E) y^2 \\
&- \alpha_1 uw \left[\frac{f(t,x,y,z,w,u)}{u} - \alpha_1\right] - uz\left[\frac{\psi(t,x,y,z)}{z} - \alpha_3\right] \\
&- wz\left[\alpha_4 - d(t)\frac{\partial}{\partial y}\tilde{g}(x,y)\right] \\
&- \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} uz \left[\frac{f(t,x,y,z,w,u)}{u} - \alpha_1\right] \\
&- \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} wz \left[\frac{\phi(t,y,z,w)}{w} - \alpha_2\right] - \delta uy\left[\frac{f(t,x,y,z,w,u)}{u} - \alpha_1\right] \\
&- wye(t)\left[\alpha_5 - h'(x)\right] - \delta wy\left[\frac{\phi(t,y,z,w)}{w} - \alpha_2\right] - \alpha_1 e(t) zy\left[\alpha_5 - h'(x)\right] \\
&- \delta zy\left[\frac{\psi(t,x,y,z)}{z} - \alpha_3\right] - \alpha_5 wy\left[1 - e(t)\right] - \alpha_1\alpha_5 zy\left[1 - e(t)\right] \\
&+ \frac{1}{2}\left[b(t)(\phi_1 - \phi_0)\right] u + \frac{1}{2}\left[c(t)(\psi_1 - \psi_0)\right]\alpha_1 w \\
&+ \frac{1}{2}\left[d(t)(g_1 - g_0)\right] \times \left[u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z\right] \\
&+ \left[u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y\right] p(t,x,y,z,w,u) + \frac{\partial V_0}{\partial t}.
\end{aligned}
\tag{4.9}
$$

Now let $R(t,x,y,z,w,u)$ denote the sum of the remaining terms, except the first four terms in (4.9). It can be easily seen from the conditions (i), (iii)–(vii) of the theorem that the absolute value of each coefficient of $uw$, $uz$, $uy$, $wz$, $wy$ or $zy$ in the $R(t,x,y,z,w,u)$ that can't exceed $D_9\varepsilon_i$ $(i = 1,2,3,4,5)$, where $D_9$ is a positive constant. Thus, using the inequalities

$$
|uw| \le \frac{1}{2}(u^2 + w^2), \quad |uz| \le \frac{1}{2}(u^2 + z^2), \quad |uy| \le \frac{1}{2}(u^2 + y^2),
$$

$$
|wz| \le \frac{1}{2}(w^2 + z^2), \quad |wy| \le \frac{1}{2}(w^2 + y^2), \quad |zy| \le \frac{1}{2}(z^2 + y^2)
$$

and Cauchy's inequality we obtain that

$$
\begin{aligned}
|R(t,x,y,z,w,u)| \le\ & D_9(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5)(y^2 + z^2 + w^2 + u^2) \\
&+ \frac{1}{2}\left|b(t)(\phi_1 - \phi_0)u\right| + \frac{1}{2}\left|c(t)(\psi_1 - \psi_0)\alpha_1 w\right| \\
&+ \frac{1}{2}\left|d(t)(g_1 - g_0)\left[u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z\right]\right| \\
&+ \left|u + \alpha_1 w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z + \delta y\right| . \left|p(t,x,y,z,w,u)\right| \\
\le\ & D_9(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5)(y^2 + z^2 + w^2 + u^2) \\
&+ \left[B(\phi_1 - \phi_0) + C(\psi_1 - \psi_0) + D(g_1 - g_0)\right] \times
\end{aligned}
\tag{4.10}
$$

$$D_{12}\sqrt{u^2+w^2+z^2+y^2}$$
$$+2D_{12}\sqrt{u^2+w^2+z^2+y^2}\left|p(t,x,y,z,w,u)\right|,$$

where

$$D_{12}=\max\left\{1,\alpha_1,\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_4},\delta\right\}.$$

The result (4.10) combined with (4.9) clearly shows that

$$\begin{aligned}\dot{V}_0 \le{}& -\varepsilon_0 u^2-\left(\frac{15\varepsilon}{16}\right)w^2-\left(\frac{15\varepsilon\alpha_2}{16}\right)z^2-(\varepsilon\alpha_4 E)y^2\\ &+D_9(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)(y^2+z^2+w^2+u^2)\\ &+D_{12}\left[B(\phi_1-\phi_0)+C(\psi_1-\psi_0)+D(g_1-g_0)\right]\sqrt{u^2+w^2+z^2+y^2}\\ &+2D_{12}\sqrt{u^2+w^2+z^2+y^2}\left|p(t,x,y,z,w,u)\right|+\frac{\partial V_0}{\partial t}\\ \le{}& -\frac{1}{2}\min\left\{\varepsilon_0,\frac{15\varepsilon}{16},\frac{15\varepsilon\alpha_2}{16},\varepsilon\alpha_4 E\right\}(y^2+z^2+w^2+u^2)\\ &+D_{12}\left[B(\phi_1-\phi_0)+C(\psi_1-\psi_0)+D(g_1-g_0)\right]\sqrt{u^2+w^2+z^2+y^2}\\ &+2D_{12}\sqrt{u^2+w^2+z^2+y^2}\left|p(t,x,y,z,w,u)\right|+\frac{\partial V_0}{\partial t}\end{aligned}$$

provided that

$$D_9(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5)\le\frac{1}{2}\min\left\{\varepsilon_0,\frac{15\varepsilon}{16},\frac{15\varepsilon\alpha_2}{16},\varepsilon\alpha_4 E\right\}. \tag{4.11}$$

Now we assume that $D_9$ and $\varepsilon_1$, $\varepsilon_2$,..., $\varepsilon_5$ are so small that (4.11) holds. The case $y,z,w=0$ is evident. From (4.1) we have

$$\begin{aligned}\frac{\partial V_0}{\partial t}={}& e'(t)\left[wh(x)+\alpha_1 zh(x)+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}yh(x)+\delta\int_0^x h(\xi)d\xi\right]\\ &+d'(t)\left[w\tilde{g}(x,y)+\alpha_1 z\tilde{g}(x,y)+\frac{\alpha_4(\alpha_1\alpha_2-\alpha_3)}{\alpha_1\alpha_4-\alpha_5}\int_0^y\tilde{g}(x,\eta)d\eta\right]\\ &+\alpha_1 c'(t)\int_0^z\tilde{\psi}(x,y,\zeta)d\zeta+b'(t)\int_0^w\tilde{\phi}(y,z,\rho)d\rho.\end{aligned}$$

Under the conditions of the theorem and the use of (4.3), we have

$$\begin{aligned}\frac{\partial V_0}{\partial t}\le{}& D_{10}\left[b'_+(t)+c'_+(t)+|d'(t)|+|e'(t)|\right]\times\\ &\left[H(x)+y^2+z^2+w^2+u^2\right]\le D_{11}\beta_0 V_0,\end{aligned}$$

where $D_{10}$ is a positive constant and $D_{11}=D_{10}D_7^{-1}$. The remainder of the proof proceeds just as in the proof of Lemma 2 in Tunc [24], and hence it is omitted. ∎

## 5. Completion of the proof of theorem

By considering the results of Lemma 4.1, Lemma 4.2 just verified above and Theorem 10.2 and Theorem 14.2 in Yoshizawa [29], if ones follow the lines indicated in Tunç

[24], except for some minor modifications it can be easily made the completion of the proof. Therefore, we omit the details.

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