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Null Generalized Helices in $\mathbb{L}^{m+2}$

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Abstract. In this paper, we obtain the harmonic curvetures of a null generalized helix in $\mathbb{L}^{m+2}$. Later we get the characterization of null general helices in $\mathbb{L}^{m+2}$.

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1. Introduction

In the geometry of null curves, difficulties arise because the arc length vanishes, so that it is not possible to normalize the tangent vector in the usual way. One method of proceeding is to introduce a new parameter called the pseudo-arc which normalize the derivative of the tangent vector. Bonnor defined two curvatures K_2 and K_3 in terms of the pseudo-arc and a third curvature K_1 which takes only two values, 0 whether the null curve is a straight line, or 1 otherwise (see also the paper by Bonnor [4] and Castagnino [5]. Many authors generalize the results of Bonnor in [4], since for a null curve in an n-dimensional Lorentzian space form they introduce a Frenet frame with the minimum number of curvature functions (which called the Cartan frame), and then they study the null helices in those spaces, that is, null curves with constant curvatures [3].

In this paper, we use the Duggal-Bejancu's Frenet equations introduced in Duggal and Bejancu [1] and distinguished Frenet frame $F = \{T, N, W_1, ..., W_m\}$ with respect to distinguished parameter t to define and study null generalized helices in the $(m + 2)$-dimensional Lorentzian space for null curves. We don't consider a Frenet frame with the minimum number of curvature functions which are called the Cartan frame, but we consider all of the curvature functions which are k_i, $1 \leq i \leq 2m - 1$, $(k_{2m} = 0)$. Also since t is a distinguished parameter from Duggal and Bejancu [1], we assume that $h = 0$ [1, 3]. Later, we obtain harmonic curvatures H_i, $1 < i \leq m$, of null helix in $\mathbb{L}^{m+2}$. Thus we show that, for the first five harmonic curvatures H_i, $1 \leq i \leq 5$ we can not get a general formulae. If $i > 5$, then we can get a general

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formulae. Finally, we obtain the following characterization of null generalized helices in $\mathbb{L}^{m+2}$:

The null curve α is a generalized helix if and only if $2H_1 + \sum_{i=2}^{m} H_i^2$ is a constant.

2. Preliminaries

2.1. Symmetric bilinear forms. Let $\mathbb{V}$ be a real vector space. A bilinear form on $\mathbb{V}$ is an r-bilinear function:

$$\langle \ , \ \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$$

and we consider only the symmetric case. A symmetric bilinear form $\langle \ , \ \rangle$ on $\mathbb{V}$ is

(a) positive [negative] definite provided $v \neq 0$ implies $\langle v, v\rangle \ > 0 \ [< 0\ ]$,

(b) positive [negative] semidefinite provided $\langle v, v\rangle \geq 0 \ (\leq 0)$ for all $v \in \mathbb{V}$.

(c) nondegenerate provided $\langle v, w\rangle = 0$ for all $w \in \mathbb{V}$ implies $v \neq 0$.

If $\langle \ , \ \rangle$ is a symmetric bilinear form on $\mathbb{V}$ then for any subspace $\mathbb{W}$ of $\mathbb{V}$ the restriction $\langle \ , \ \rangle |_{\mathbb{W} \times \mathbb{W}}$ denoted merely by $\langle \ , \ \rangle |_{\mathbb{W}}$, is again symmetric and bilinear. If $\langle \ , \ \rangle$ is [semi-] definite, so is $\langle \ , \ \rangle |_{\mathbb{W}}$.

The index q of a symmetric bilinear form $\langle \ , \ \rangle$ on $\mathbb{V}$ is the largest integer that is the dimension of a subspace $\mathbb{W} \subset \mathbb{V}$ on which $\langle \ , \ \rangle |_{\mathbb{W}}$ is negative definite. Thus $0 \leq q \leq \dim \mathbb{V}$, and $q = 0$ if and only if $\langle \ , \ \rangle$ is positive semidefinite [6].

2.2. Scalar product. A scalar product $\langle \ , \rangle$ on a vector space $\mathbb{V}$ is a nondegenerate symmetric bilinear form on $\mathbb{V}$ [6].

Lemma 2.1. [6] *A scalar product space $V \neq 0$ has an orthonormal basis. The matrix of $\langle \ , \ \rangle$ relative to an orthonormal basis $e_1, e_2, ..., e_n$ for $\mathbb{V}$ is diagonal; in fact:*

$$\langle e_i, e_j\rangle = \delta_{ij}\varepsilon_j \ \textit{where} \ \varepsilon_j = \langle e_j, e_j\rangle = \pm 1.$$

Lemma 2.2. [5] *Let $e_1, e_2, ..., e_n$ be an orthonormal basis for $\mathbb{V}$, with $\varepsilon_j = \langle e_j, e_j\rangle$. Then each $v \in \mathbb{V}$ has a unique expression $v = \sum \varepsilon_i \langle v, e_i\rangle e_i$.*

Lemma 2.3. [6] *For any orthonormal basis $e_1, e_2, ..., e_n$ for $\mathbb{V}$ the number of negative signs in the signature $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ is the index q of $\mathbb{V}$.*

Lemma 2.4. *For any orthonormal basis $e_1, e_2, ..., e_n$ for $\mathbb{V}$ the number of for an integer q with $0 \leq q \leq n$, changing the first q plus signs above the minus gives tensor:*

$$\langle v_p, w_p\rangle = -\sum_{i=1}^{q} v^i w^i + \sum_{j=q+1}^{n} v^j w^j$$

of index q.

The resulting semi-Euclidean space $\mathbb{R}_q^n$ reduces to $\mathbb{R}^n$ if $q = 0$. For $n \geq 2$, $\mathbb{R}_1^n$ is called Minkowski n-space; if $n = 4$ it is the simplest example of a relativistic spacetime.

Fix the notation:

$$\varepsilon_i \quad = \quad \begin{cases} -1 & (0 \leq i \leq q-1), \\ 1 & (q \leq i \leq n-1). \end{cases}$$

A Lorentz vector space to be a scalar product space of index 1 and dimension ≥ 2 [5].

2.3. Lorentzian space. Let M be a smooth connected paracompact Hausdorff manifold and let $\pi : TM \to M$ denote the tangent bundle of M. A Lorentzian metric $\langle , \rangle$ for M is a smooth symmetric tensor field of type (0,2) on M such that for each $p \in M$, the tensor $\langle , \rangle_p : T_P M \times T_P M \to \mathbb{R}$ is a nondegenerate inner product of signature $(-, +, \ldots, +)$. In other words, a matrix representation of $\langle \ , \ \rangle$ at p will have one negative eigenvalue and all other eigenvalues will be positive.

A Lorentzian manifold $(M, \langle , \rangle)$ is a manifold M together with a Lorentzian metric $\langle , \rangle$ for M. All noncompact manifolds admit Lorentzian metrics. However, a compact manifold admits a Lorentzian metric if its euler characteristic vanishes [7]. Lorentzian space is the manifold $M = \mathbb{R}^n$ together with the metric

$$ds^2 = -dx_1^2 + \sum_{i=2}^{n} dx_i^2.$$

This space-time is time oriented by the vector field $\partial/\partial x_1$ [7].

Definition 2.1. [6] *A tangent vector $v \in \mathbb{L}^n$ is:*

(i) space-like if $\langle v, v \rangle > 0$ or $v = 0$,
(ii) null if $\langle v, v \rangle = 0$ and $v \neq 0$,
(iii) time-like if $\langle v, v \rangle < 0$.

2.4. Curves. A curve in a Lorentzian space, $\mathbb{L}^n$ is a smooth mapping $\alpha : I \to \mathbb{L}^n$ where I is open interval in the real line $\mathbb{R}$. The interval I has a coordinate system consisting of the identity map u of I. The velocity vector of α at $t \in I$ is $\alpha' = \left.\dfrac{d\alpha(u)}{d(u)}\right|_t$. A curve α is said to be regular if $\alpha'(t)$ does not vanish for all t in I. $\alpha \in \mathbb{L}^n$ is space-like if its velocity vectors α' are space-like for all $t \in I$, similarly for time-like and null. If α is a null curve, we can reparametrize it such that $\langle \alpha'(t), \alpha'(t) \rangle = 0$ and $\alpha'(t) \neq 0$ [6].

Let $(M, \langle \ , \rangle)$ be a proper $(m+2)$-dimensional semi-Riemannian manifold of index q and let us consider α a smooth curve in M locally parametrized by $\alpha : I \subset \mathbb{R} \to M$. The curve α is said to be null or light-like if the tangent vector $\alpha'(t) = T$ at any point is a null vector. That is $\langle T, T \rangle = 0$. The following concepts are taken from Duggal and Bejancu [1].

Let $T\alpha$ denote the tangent bundle of α and define, as in the non-degenerate case, the bundle $T\alpha^{\perp}$ by:

$$T\alpha^{\perp} = \bigcup_{p \in \alpha} T_p\alpha^{\perp}, T_p\alpha^{\perp} = \{\xi_p \in T_pM : \ \langle \xi_p, T_p \rangle = 0, \ T_p \in T_p\alpha\}$$

where T_p is a null vector tangent to α at p. It is well known that $T\alpha^{\perp}$ is of rank m+1. Since T_p is a null vector, it is easily follows that $T\alpha$ is a vector subbundle of $T\alpha^{\perp}$ of rank 1. Then we may consider a complementary vector subbundle $S(T\alpha^{\perp})$ to $T\alpha$ in $T\alpha^{\perp}$ such that:

$$T\alpha^{\perp} = T\alpha \ \perp S(T\alpha^{\perp}),$$

where $\perp$ means orthogonal direct sum. It is known that the subbundle $S(T\alpha^{\perp})$, called the screen vector bundle of α, is non-degenerate and of dimension m. Note that, in contrast with the non-degenerate case, the tangent bundle is contained in the normal bundle, and the screen bundle is not unique. These two properties leads to a much more difficult and also different geometry of null curves with respect to non-degenerate (space-like or time-like) curves.

Since $S(T\alpha^{\perp})$ is non-degenerate, we have the decomposition:

$$TM\mid_{\alpha} = S(T\alpha^{\perp}) \perp S(T\alpha^{\perp})^{\perp},$$

where $S(T\alpha^{\perp})^{\perp}$ is the complementary orthogonal vector bundle to $S(T\alpha^{\perp})$ in $TM(\alpha)$. The following result is well known.

Theorem 2.1. [8] *Let α be a null curve of a semi-Riemannian manifold $(M, \langle\ ,\rangle)$ and consider $S(T\alpha^{\perp})$ a screen vector bundle of α. Then there exist a unique vector bundle E over α, of rank 1, such that on each coordinate neighbourhood $U \subset \alpha$ there is a unique section $N \in \Gamma(E\mid_{\alpha})$ satisfying:*

$$\langle T, N\rangle = 1 \qquad (\alpha^{'}(t) = T)$$

and

$$\langle N, N\rangle \ = \ \langle N, X\rangle = 0, \quad \textit{for all} \ \ X \in \Gamma(S(T\alpha^{\perp}).$$

The above vector bundle E will be denoted by $\mathrm{ntr}(\alpha)$ and it is called the null transversal bundle of α with respect to $S(T\alpha^{\perp})$. The vector field N is called the null transversal vector field of α with respect to $\alpha^{'}(t)$. We define the transversal vector bundle of α, $tr(\alpha)$, as the vector bundle

$$\mathrm{tr}(\alpha) = \mathrm{ntr}(\alpha) \perp S(T\alpha^{\perp}),$$

and then we have

$$TM\mid_{\alpha} = T\alpha \oplus \mathrm{tr}(\alpha) = (T\alpha \oplus \mathrm{ntr}(\alpha)) \perp S(T\alpha^{\perp}),$$

from which the following result easily follows:

Proposition 2.1. [1] *Let α be a null curve of semi-Riemannian manifold $(M, \langle\ ,\rangle)$ of index q. Then any screen vector bundle of semi-Riemannian of index $q-1$. Hence, if M is a Lorentzian manifold, then any screen vector bundle is Riemannian.*

2.5. Harmonic curvatures.

Definition 2.2. *Let α be a null curve in $\mathbb{L}^{m+2}$ and T be the first Frenet vector field of $\alpha(\alpha^{'}(t) = T)$. $X \in \chi(\mathbb{L}^{m+2})$ being a constant unit vector field, if*

$$\langle T,\ X\rangle = constant,$$

then α is called a general helix (inclined curves) in $\mathbb{L}^{m+2}$. The space $Sp\{X\}$ is called slope axis.

Definition 2.3. *Let $\alpha \to \mathbb{L}^{m+2}$ be a general helix. Assume X is a unit and constant vector field of $\mathbb{L}^{m+2}$ and $\{T, N, W_1, ..., W_m\}$ is the Frenet $(m+2)$ - frame at the point of $\alpha(t)$.*

$$H_j : I \to \mathbb{R}$$

$$\langle N,\ X\rangle = H_1\langle T,\ X\rangle, \quad \langle W_1,\ X\rangle = 0,$$
$$\langle W_i,\ X\rangle = H_i\langle T,\ X\rangle, \qquad 2 \leq i \leq m$$

where the value of the H_i function at $\alpha(t)$ is called as the j-th harmonic curvature according to X at the point of $\alpha(t)$ of α.

3. Harmonic Curvature of a Null Generalized Helix in $\mathbb{L}^{m+2}$

Suppose α is a null curve of an $(m+2)$-dimensional Lorentz manifold $(M, \langle,\rangle)$. Denote by ∇ the Levi-Civita connection on M and $\alpha^{'} = T$. In this case, $\{T, N, W_1, ..., W_m\}$ is the Frenet frame of $\alpha \subset \mathbb{L}^{m+2}$, where T and N are null vectors and W_i, $1 \leq i \leq m$, are space-like vectors. Thus the Frenet equations of a null curve in an $(m+2)$-dimensional Lorentz manifold are as follows:

$$\begin{aligned}
\nabla_T T &= hT + k_1 W_1 \\
\nabla_T N &= -hN + k_2 W_1 + k_3 W_2 \\
\nabla_T W_1 &= -k_2 T - k_1 N + k_4 W_2 + k_5 W_3 \\
\nabla_T W_2 &= -k_3 T - k_4 W_1 + k_6 W_3 + k_7 W_4 \\
\nabla_T W_3 &= -k_5 W_1 - k_6 W_2 + k_8 W_4 + k_9 W_5 \\
&\vdots \\
\nabla_T W_{m-2} &= -k_{2m-5} W_{m-4} - k_{2m-4} W_{m-3} + k_{2m-2} W_{m-1} + k_{2m-1} W_m \\
\nabla_T W_{m-1} &= -k_{2m-3} W_{m-3} - k_{2m-2} W_{m-2} + k_{2m} W_m \\
\nabla_T W_m &= -k_{2m-1} W_{m-2} - k_{2m} W_{m-1}
\end{aligned} \tag{3.1}$$

where h and $\{k_i\}$, $1 \leq i \leq 2m$, are smooth functions and $\{W_1, W_2, ..., W_m\}$ is a certain orthonormal basis of $\Gamma(S(T\alpha^{\perp}))$. If $h = 0$, then the parameter t is said to be a distinguished parameter. Moreover, if the last curvature k_{2m} vanishes, then $\{T, N, W_1, W_2, ..., W_m\}$ is called a distinguished Frenet frame [1].

Theorem 3.1. *Assume that $\alpha \subset \mathbb{L}^4$ is a null generalized helix given by distinguished Frenet frame $\{T, N, W_1, W_2\}$ and curvature functions k_1, k_2, k_3, k_4. Let X be a unit and constant vector field (time-like or space-like) of $\mathbb{L}^4$. By accepting the slope axis of α curve is $Sp\{X\}$ and harmonic curvatures are H_1, H_2, the following equations are obtained*

$$\langle N,\, X\rangle = H_1\langle T,\, X\rangle,\ \langle W_1,\, X\rangle = 0,\ \langle W_2,\, X\rangle = H_2\langle T,\, X\rangle,$$

where $H_1 = -k_2/k_1$ and $H_2 = H_1^{'}/k_3$.

Proof. From equations (3.1), it is easy to see that the Frenet equations of a null curve α in a 4-dimensional Lorentzian manifold can be written down as follows:

$$\begin{aligned}
\nabla_T T &= hT + k_1 W_1 \\
\nabla_T N &= -hN + k_2 W_1 + k_3 W_2 \\
\nabla_T W_1 &= -k_2 T - k_1 N + k_4 W_2 \\
\nabla_T W_2 &= -k_3 T - k_4 W_1
\end{aligned} \tag{3.2}$$

where $T = \alpha^{'}$. If $h = 0$, then the parameter t is said to be a distinguished parameter. Moreover, if the last curvature k_4 vanishes, $\{T, N, W_1, W_2\}$ is then called a distinguished Frenet frame [1,3].

These equalities are present from Theorem 2.1:

$$\langle T, T\rangle = 0,\ \langle N, N\rangle = 0, \langle T, N\rangle = 1,\ \langle T, W_i\rangle = 0, \langle N, W_i\rangle = 0,$$
$$\langle W_i\ , W_j\rangle = \delta_{ij}\ ,\ \text{for } i,\ j = 1, 2.$$

By using the Definition 2.2, which is given by:

$$\langle T,\, X\rangle = \text{constant}, \tag{3.3}$$

we obtain

$$\langle \nabla_T T,\, X\rangle = 0. \tag{3.4}$$

Using the first equation from (3.2), the result is

$$\langle W_1,\, X\rangle = 0. \tag{3.5}$$

Taking the inner product of the third equation in (3.2) by X

$$\langle \nabla_T W_1,\, X\rangle = -k_2\langle T,\, X\rangle - k_1\langle N, X\rangle,$$

is obtained, where since $\langle W_1, X\rangle = 0$, it is written as $\langle \nabla_T W_1,\, X\rangle = 0$. Thus it is

$$\langle N, X\rangle = \frac{-k_2}{k_1}\langle T,\, X\rangle,$$

where $H_1 = -k_2/k_1$ is called the 1-st harmonic curvature function. As a consequence

$$\langle N, X\rangle = H_1\langle T,\, X\rangle \tag{3.6}$$

is found. By taking the derivative of equation (3.6) with respect to T, we deduce the following equation

$$\langle \nabla_T N,\, X\rangle = H_1^{'}\langle T,\, X\rangle. \tag{3.7}$$

Taking the inner product of the second Frenet equation in (3.2) by X and using the equations (3.5) and (3.7)

$$\langle W_2,\, X\rangle = \frac{H_1^{'}}{k_3}\langle T,\, X\rangle$$

is obtained, where $H_2 = H_1^{'}/k_3$ is called 2-nd harmonic curvature function. Thus the deduction is

$$\langle W_2,\, X\rangle = H_2\langle T,\, X\rangle. \tag{3.8}$$

∎

Theorem 3.2. *Let $\alpha \subset \mathbb{L}^6$ be a null generalized helix given by distinguished Frenet frame $\{T, N, W_1, W_2, W_3, W_4\}$ and curvature functions k_i, $1 \leq i \leq 8$. If the slope axis of α is $Sp\{X\}$ and harmonic curvature functions are H_i, $i = 1, 2, 3, 4$, the following equations are existent*

$$\langle W_3,\, X\rangle = H_3\langle T,\, X\rangle,\ \langle W_4,\, X\rangle = H_4\langle T,\, X\rangle,$$

where $H_3 = -k_4/k_5H_2$ and $H_4 = \{H_2' + k_3 - k_6H_3\}/k_7$.

Proof. The Frenet equations of a null curve in a 6-dimensional Lorentzian manifold write down as follows

$$\begin{aligned}
\nabla_T T &= hT + k_1W_1\\
\nabla_T N &= -hN + k_2W_1 + k_3W_2\\
\nabla_T W_1 &= -k_2T - k_1N + k_4W_2 + k_5W_3\\
\nabla_T W_2 &= -k_3T - k_4W_1 + k_6W_3 + k_7W_4\\
\nabla_T W_3 &= -k_5W_1 - k_6W_2 + k_8W_4\\
\nabla_T W_4 &= -k_7W_2 - k_8W_3
\end{aligned} \tag{3.9}$$

where $T = \alpha^{'}$. If the last curvature k_8 vanishes, then $\{T, N, W_1, W_2, W_3, W_4\}$ is called a distinguished Frenet frame [1,3].

Then the following equalities are existent from Theorem 2.1

$$\langle T,T\rangle = 0, \langle N,N\rangle = 0, \langle T,N\rangle = 1, \langle T,W_i\rangle = 0,$$
$$\langle N,W_i\rangle = 0, \langle W_i,W_j\rangle = \delta_{ij}, \text{ for } i,j = 1,2,3,4$$

Taking the inner product of the third Frenet equation in (3.9) by X:

$$\langle \nabla_T W_1,\, X\rangle = -k_2\langle T,\, X\rangle - k_1\langle N, X\rangle + k_4\langle W_2,\, X\rangle + k_5\langle W_3,\, X\rangle$$

is found. By using the equations (3.5), (3.6), (3.8) and Definition 2.3, we obtain

$$\langle W_3,\, X\rangle = \frac{-k_4}{k_5} H_2\langle T,\, X\rangle.$$

Thus $H_3 = -k_4/k_5 H_2$ is called the 3-rd harmonic curvature function. That is

$$\langle W_3,\, X\rangle = H_3\langle T,\, X\rangle. \tag{3.10}$$

If we take the inner product of the 4-th equation in (3.9) by X, we get

$$\langle \nabla_T W_2,\, X\rangle = -k_3\langle T,\, X\rangle - k_4\langle W_1, X\rangle m + \ k_6\langle W_3,\, X\rangle + \ k_7\ \langle W_4,\, X\rangle, \tag{3.11}$$

where we know that $\langle W_1, X\rangle = 0$ and $\langle W_3, X\rangle = H_3\langle T,\, X\rangle$. Also if we take the derivative of (3.8) with respect to T, we have $\langle \nabla_T W_2,\, X\rangle = H_2^{'}\langle T,\, X\rangle$. Using these values in equation (3.11), we get

$$\langle W_4,\, X\rangle = \frac{1}{k_7}\left\{H_2^{'} + k_3 - k_6 H_3\right\}\langle T,\, X\rangle,$$

where $H_4 = \{H_2' + k_3 - k_6 H_3\}/k_7$ is called the 4-th harmonic curvature function. Thus we obtain

$$\langle W_4,\, X\rangle = H_4\langle T,\, X\rangle, \tag{3.12}$$

or

$$\langle \nabla_T W_4,\, X\rangle = H_4^{'}\langle T,\, X\rangle. \tag{3.13}$$

∎

Theorem 3.3. *Let $\alpha \subset \mathbb{L}^7$ be a null generalized helix given by distinguished Frenet frame $\{T, N, W_1, W_2, W_3, W_4, W_5\}$ and consider $Sp\{X\}$ is slope axis of α. For the curvature functions are k_i, $1 \le i \le 10$, and harmonic curvature functions are H_i, $1 \le i \le 5$, the following conclusion is existent:*

$$\langle W_5,\, X\rangle = H_5\langle T,\, X\rangle,$$

where $H_5 = \{H_3' + k_6 H_2 - k_8 H_4\}/k_9$ is the 5th harmonic curvature function.

Proof. Recall that the Frenet equations of a null curve α in a 7-dimensional Lorentzian manifold can be written down as follows

$$\begin{aligned}
\nabla_T T &= hT + k_1 W_1\\
\nabla_T N &= -hN + k_2 W_1 + k_3 W_2\\
\nabla_T W_1 &= -k_2 T - k_1 N + k_4 W_2 + k_5 W_3\\
\nabla_T W_2 &= -k_3 T - k_4 W_1 + k_6 W_3 + k_7 W_4\\
\nabla_T W_3 &= -k_5 W_1 - k_6 W_2 + k_8 W_4 + k_9 W_5
\end{aligned} \tag{3.14}$$

$$\nabla_T W_4 = -k_7 W_2 - k_8 W_3 + k_{10} W_5$$
$$\nabla_T W_5 = -k_9 W_3 - k_{10} W_4$$

where $T = \alpha^{'}$. If the last curvature k_{10} vanishes, then $\{T, N, W_1, W_2, W_3, W_4, W_5\}$ is called a distinguished Frenet frame [1,3]. Thus we have from Theorem 2.1 that

$$\langle T,T\rangle = 0,\ \langle N,N\rangle = 0,\ \langle T,N\rangle = 1,\ \langle T,W_i\rangle = 0,\ \langle N,W_i\rangle = 0,$$
$$\langle W_i, W_j\rangle = \delta_{ij},\ \text{for} i,\ j = 1,2,3,4,5.$$

Taking the inner product of the 5th equation in (3.14) by X, we obtain

$$\begin{aligned}\langle \nabla_T W_3,\ X\rangle = \ & -k_5\langle W_1,\ X\rangle - k_6\langle W_2,\ X\rangle \\ & +k_8\langle W_4,\ X\rangle + k_9\langle W_5,\ X\rangle.\end{aligned} \tag{3.15}$$

In addition, taking the derivative of equation (3.10) with respect to T, we set

$$\langle \nabla_T W_3,\ X\rangle = H_3^{'}\langle T,\ X\rangle.$$

On the other hand, the equations (3.5),(3.8), and (3.12) leads to

$$\langle W_5,\ X\rangle = \frac{1}{k_9}\left\{H_3^{'} + k_6 H_2 - k_8 H_4\right\}\langle T,\ X\rangle.$$

Thus $H_5 = \dfrac{1}{k_9}\left\{H_3^{'} + k_6 H_2 - k_8 H_4\right\}$ is called the 5th harmonic curvature function.Thus

$$\langle W_5,\ X\rangle = H_5\langle T,\ X\rangle. \tag{3.16}$$ ∎

Theorem 3.4. *Assume that $\alpha \subset \mathbb{L}^{m+2}$ is a null generalized helix given by distinguished Frenet frame $\{T, N, W_1, W_2, ..., W_m\}$ and curvature functions k_i, $1 \leq i \leq 2m$. Let X be a unit and constant vector field (time-like or space-like) of $\mathbb{L}^{m+2}$. If the curve α has slope axis $Sp\{X\}$ and harmonic curvatures are H_j , $6 \leq j \leq m$, then*

$$\langle W_j, X\rangle = H_j\langle T,\ X\rangle, \quad 6 \leq j \leq m,$$

where H_j is the jth harmonic curvature function and

$$H_j = \frac{1}{k_{2j-1}}\{H_{j-2}^{'} + k_{2j-5}H_{j-4} + k_{2j-4}H_{j-3} - k_{2j-2}H_{j-1}\}.$$

Proof. Recall that $\langle T,\ X\rangle$ = constant for is a null helix α in $\mathbb{L}^{m+2}$. We use the induction method. For the case $j = 6$, we must show that $\langle W_6,\ X\rangle = H_6\langle T,\ X\rangle$. From Theorem 3.1, Theorem 3.2, Theorem 3.3, we obtain

$$\langle W_1,\ X\rangle = 0,\ \langle N,\ X\rangle = H_1\langle T,\ X\rangle$$
$$\langle W_i,\ X\rangle = H_i\langle T,\ X\rangle,\ \ 2 \leq i \leq 5.$$

Considering the inner product of the 6-th equation in (3.1) by X, we have

$$\begin{aligned}\langle \nabla_T W_4,\ X\rangle = \ & -k_7\langle W_2,\ X\rangle - k_8\langle W_3,\ X\rangle \\ & +k_{10}\langle W_5,\ X\rangle + k_{11}\langle W_6,\ X\rangle.\end{aligned}$$

By using the values of (3.8), (3.10) and (3.13) in the equation above, we set

$$\langle W_6,\ X\rangle = \frac{1}{k_{11}}\{H_4^{'} + k_7 H_2 + k_8 H_3 - k_{10}H_5\}\langle T,\ X\rangle.$$

Since

$$H_6 = \frac{1}{k_{11}}\{H_4^{'} + k_7 H_2 + k_8 H_3 - k_{10} H_5\} \tag{3.17}$$

that

$$\langle W_6,\, X\rangle = H_6 \langle T,\, X\rangle. \tag{3.18}$$

Indeed, since $X \in \{T, N, W_1, W_2, ..., W_6\}$, it is clear that

$$\begin{aligned} X = &\langle N,\, X\rangle T + \langle T,\, X\rangle N + \langle W_2,\, X\rangle W_2 + \langle W_3,\, X\rangle W_3 \\ &+ \langle W_4, X\rangle W_4 + \langle W_5,\, X\rangle W_5 + \langle W_6,\, X\rangle W_6. \end{aligned} \tag{3.19}$$

If we take the derivative of (3.19) with respect to T, then we obtain:

$$\begin{aligned} \nabla_T X = &\langle \nabla_T N,\, X\rangle T + \langle N,\, X\rangle \nabla_T T + \langle \nabla_T T,\, X\rangle N + \langle T,\, X\rangle \nabla_T N \\ &+ \langle \nabla_T W_2,\, X\rangle W_2 + \langle W_2,\, X\rangle \nabla_T W_2 + \langle \nabla_T W_3,\, X\rangle W_3 \\ &+ \langle W_3,\, X\rangle \nabla_T W_3 + \langle \nabla_T W_4,\, X\rangle W_4 + \langle W_4,\, X\rangle \nabla_T W_4 \\ &+ \langle \nabla_T W_5,\, X\rangle W_5 + \langle W_5,\, X\rangle \nabla_T W_5 \\ &+ \langle \nabla_T W_6,\, X\rangle W_6 + \langle W_6,\, X\rangle \nabla_T W_6. \end{aligned}$$

Using (3.1) and the value of $\langle W_6,\, X\rangle$ in (3.18) we obtain $\nabla_T X = 0$. This means that X is a constant vector field.

Conversely, since X is a time-like or space-like vector field, $\langle X,\, X\rangle = \pm 1$. Thus, from (3.19),

$$\begin{aligned} \pm 1 = 2\langle N,\, X\rangle \langle T,\, X\rangle + \langle W_2,\, X\rangle^2 + \langle W_3,\, X\rangle^2 \\ + \langle W_4,\, X\rangle^2 + \langle W_5,\, X\rangle^2 + \langle W_6,\, X\rangle^2 \end{aligned}$$

or by using Theorem 3.1, Theorem 3.2, Theorem 2.3 and equation (3.18), we set

$$1 = 2H_1 + \sum_{i=2}^{6} H_i^2. \tag{3.20}$$

By taking the derivative of (3.20),

$$2H_1^{'} + 2H_2 H_2^{'} + 2H_3 H_3^{'} + 2H_4 H_4^{'} + 2H_5 H_5^{'} + 2H_6 H_6^{'} = 0. \tag{3.21}$$

Additionally, from (3.1), we may derive the following results

$$\begin{aligned} \langle \nabla_T W_2,\, X\rangle &= -k_3 \langle T,\, X\rangle + k_6 \langle W_3,\, X\rangle + k_7 \langle W_4,\, X\rangle, \\ \langle \nabla_T W_3,\, X\rangle &= -k_6 \langle W_2,\, X\rangle + k_8 \langle W_4,\, X\rangle + k_{10} \langle W_5,\, X\rangle, \\ \langle \nabla_T W_5,\, X\rangle &= -k_9 \langle W_3,\, X\rangle - k_{10} \langle W_4,\, X\rangle, \\ \langle \nabla_T W_6,\, X\rangle &= -k_{11} \langle W_4,\, X\rangle. \end{aligned} \tag{3.22}$$

Using the equation $\langle W_i,\, X\rangle = H_i \langle T,\, X\rangle,\ 2 \le i \le 5$ and equation (3.18) it is clear that

$$\langle \nabla_T W_i,\, X\rangle = H_i^{'} \langle T,\, X\rangle, \quad 2 \le i \le 6.$$

Thus from (3.22) , we have

$$
\begin{aligned}
H_2' &= -k_3 + k_6 H_3 + k_7 H_4, \\
H_3' &= -k_6 H_2 + k_8 H_4 + k_9 H_5, \\
H_5' &= -k_9 H_3 - k_{10} H_4, \\
H_6' &= -k_{11} H_4.
\end{aligned} \tag{3.23}
$$

If we put the equalities at (3.23) in equation (3.21) , we obtain

$$
\begin{aligned}
0 \;=\; & H_2\{k_6 H_3 + k_7 H_4\} + H_3\{-k_6 H_2 + k_8 H_4 + k_9 H_5\} \\
& + H_4 H_4' + H_5\{-k_9 H_3 - k_{10} H_4\} + H_6\{-k_{11} H_4\}
\end{aligned}
$$

or

$$
H_6 = \frac{1}{k_{11}}\{H_4' - k_{10} H_5 + k_8 H_3 + k_7 H_2\}. \tag{3.24}
$$

Thus the theorem holds for the case $j = 6$. Assuming that the theorem holds for the case $j < p$, let us show that the theorem also holds for the case $j = p$. This assume that

$$
\langle W_{p-1},\, X\rangle = H_{p-1}\langle T,\, X\rangle \tag{3.25}
$$

and

$$
\langle W_{p-2},\, X\rangle = H_{p-2}\langle T,\, X\rangle. \tag{3.26}
$$

If we take the derivative of (3.26) with respect to T, then, we obtain

$$
\langle \nabla_T W_{p-2},\, X\rangle = H_{p-2}'\langle T,\, X\rangle. \tag{3.27}
$$

If the value of $\nabla_T W_{p-2}$ in (3.1) supersede in (3.27), the inference is

$$
\begin{aligned}
H_{p-2}'\langle T,\, X\rangle = \;& -k_{2p-5}\langle W_{p-4}, X\rangle - k_{2p-4}\langle W_{p-3}, X\rangle \\
& + k_{2p-2}\langle W_{p-1}, X\rangle + k_{2p-1}\langle W_p, X\rangle.
\end{aligned}
$$

Since $\langle W_{p-4}, X\rangle = H_{p-4}\langle T,\, X\rangle$, $\langle W_{p-3}, X\rangle = H_{p-3}\langle T,\, X\rangle$, $\langle W_{p-1}, X\rangle = H_{p-1}\langle T,\, X\rangle$, we have

$$
\{k_{2p-5} H_{p-4} + k_{2p-4} H_{p-3} - k_{2p-2} H_{p-1} + H_{p-2}'\}\langle T,\, X\rangle = k_{2p-1}\langle W_p, X\rangle
$$

or

$$
\langle W_p, X\rangle = \frac{1}{k_{2p-1}}\{k_{2p-5} H_{p-4} + k_{2p-4} H_{p-3} - k_{2p-2} H_{p-1} + H_{p-2}'\}\langle T,\, X\rangle,
$$

where

$$
H_p = \frac{1}{k_{2p-1}}\{k_{2p-5} H_{p-4} + k_{2p-4} H_{p-3} - k_{2p-2} H_{p-1} + H_{p-2}'\}.
$$

Thus $\langle W_p, X\rangle = H_p\langle T,\, X\rangle$. ∎

Definition 3.1. *Assume that $\alpha \subset \mathbb{L}^{m+2}$ is a null generalized helix given by distinguished Frenet frame $T, N, W_1, W_2, ..., W_m$ and curvature functions k_i, $1 \le i \le 2m-1$, then higher ordered harmonic curvatures of α are as follows*

$$H_i = \begin{cases} -\dfrac{k_2}{k_1}, & i = 1 \\ \dfrac{H_1^{'}}{k_3}, & i = 2 \\ -\dfrac{k_4}{k_5}H_2, & i = 3 \\ \dfrac{1}{k_7}\{H_2^{'} + k_3 - k_6H_3\}, & i = 4 \\ \dfrac{1}{k_9}\{H_3^{'} + k_6H_2 - k_8H_4\}, & i = 5 \\ \dfrac{1}{k_{2m-1}}\{H_{m-2}^{'} + k_{2m-5}H_{m-4} \\ \quad + k_{2m-4}H_{m-3} - k_{2m-2}H_{m-1}\}, & 5 < i \leq m \end{cases} \tag{3.28}$$

4. The Characterization of Null Generalized Helices in $\mathbb{L}^{m+2}$

Theorem 4.1. *Suppose that $\alpha \subset \mathbb{L}^{m+2}$ is a null curve given by distinguished Frenet frame $\{T, N, W_1, W_2, ..., W_m\}$. Let $\{H_1, H_2, ..., H_m\}$ be harmonic curvatures of α. Then a null curve α is a general helix in $\mathbb{L}^{m+2}$ if and only if $2H_1 + \sum_{i=2}^{m} H_i^2$ is a constant.*

Proof. ($\Rightarrow$) Let $\alpha \subset \mathbb{L}^{m+2}$ be a general null helix, then, for α whose slope axis $Sp\{X\}$ we know that $\langle T, X\rangle$ = constant. Also from Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4 we know that

$$\langle W_j, X\rangle = H_j\langle T, X\rangle, \ \ 2 \leq j \leq m.$$

In addition since $X \in \{T, N, W_1, W_2, ..., W_m\}$, it can be written as:

$$X = \lambda_1 T + \lambda_2 N + \lambda_3 W_1 + ... + \lambda_{m+2} W_m$$

or

$$X = \langle N, X\rangle T + \langle T, X\rangle N + \langle W_2, X\rangle W_2 + ... + \langle W_m, X\rangle W_m.$$

Now, assume that X is a time-like (or space-like) vector field, then since $\langle X, X\rangle = \pm 1$, it is clear that:

$$\pm 1 = \langle X, X\rangle = 2\langle N, X\rangle + \langle W_2, X\rangle^2 + ... + \langle W_m, X\rangle^2$$

or

$$\pm 1 = 2\langle T, X\rangle\langle N, X\rangle + \sum_{i=2}^{m}\langle W_i, X\rangle^2$$

or, since $\langle N, X\rangle = H_1\langle T, X\rangle$,

$$\pm 1 = 2H_1\langle T, X\rangle^2 + \sum_{i=2}^{m}\langle W_i, X\rangle^2$$

which, in view of $\langle W_i, X\rangle = H_i\langle T, X\rangle$, $2 \leq j \leq m$, leads to

$$\pm 1 = \langle 2H_1 + \sum_{i=2}^{m} H_i^2\rangle \ \langle T, X\rangle^2.$$

Thus

$$2H_1 + \sum_{i=2}^{m} H_i{}^2 = \pm \frac{1}{\langle T, X\rangle^2}.$$

Since $\langle T, X\rangle = \text{constant}$, $2H_1 + \sum_{i=2}^{m} H_i{}^2$ is a constant.

($\Leftarrow$) We assume that $2H_1 + \sum\limits_{i=2}^{m} H_i{}^2$ is a constant. We show that $\alpha \subset \mathbb{L}^{m+2}$ be a general null helix. In order to show this, we must indicate that the following vector field X is a constant vector field in $\mathbb{L}^{m+2}$.

$$X = \langle N, X\rangle T + \langle T, X\rangle N + \sum_{i=2}^{m} H_i \langle T, X\rangle.$$

If we take the derivative of X in respect to T, then using Theorem 2, Theorem 3, Theorem 4 and Theorem 5, we obtain:

$$\begin{aligned}\nabla_T X = \;& \langle \nabla_T N, X\rangle T + \langle N, X\rangle \nabla_T T + \langle T, X\rangle \nabla_T N \\ & + \sum_{i=2}^{m} \{H_i' W_i + H_i \nabla_T W_i\} \langle T, X\rangle\end{aligned}$$

or

$$\begin{aligned}\nabla_T X = \;& \Big\{ k_3 H_2 T + k_1 H_1 W_1 + k_2 W_1 + k_3 W_2 + H_2' W_2 \\ & + H_2(-k_3 T - k_4 W_1 + k_6 W_3 + k_7 W_4) + H_3' W_3 \\ & + H_3(-k_5 W_1 - k_6 W_2 + k_8 W_4 + k_9 W_5) \\ & + \sum_{i=4}^{m} [(k_{2i+3} H_{i+2} - k_{2i-1} H_{i-2} - k_{2i} H_{i-1}) W_i \\ & + \; H_i(-k_{2i-1} W_{i-2} - k_{2i} W_{i-1})] \Big\} \langle T, X\rangle\end{aligned}$$

and calculating this, we obtain $\nabla_T X = 0$. Thus X is a constant vector field. ∎

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