BULLETIN *of the*
Malaysian Mathematical
Sciences Society
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **30***(1) (2007), 13–21*

γ-(α,β)-Semi Open Sets and Some New Generalized Separation Axioms

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Abstract. Let (X,τ) be a topological space and α,β,γ: $P(X) \to P(X)$ be operators associated to τ. We introduce the concept of γ-(α,β)-semi open sets and new generalized forms of separations by γ-(α,β)-semi open sets. Also, we analyze the relations with some well known separation axioms.

2000 Mathematics Subject Classification: 54A05, 54A10, 54D10

Key words and phrases: γ-(α,β)-semi open, γ-(α,β)-generalized semi closed set, γ-(α,β)-semi T_i spaces.

1. Introduction

The study of semi open sets was initiated by Levine [4]. Maheshwari [6] introduced and studied a new separation axiom called semi separation axiom. Rosas et al. [8] defined the α-semi T_i spaces for $i = 0, 1/2, 1, 2$. Rosas et al. [9] introduced and studied the (α,β)-semi open sets and observed that these concepts generalize the separation axioms given by Navalagi [7]. In this work, we introduce and study the notion of γ-(α,β)-semi open sets and observe that in the case that γ is the identity operator, the concepts of (α,β)-semi open and $\gamma - (\alpha,\beta)$-semi open are the same. Also, we obtain improved results in comparison with the results obtained by Rosas et al. [8,9].

2. Preliminaries

In this section, we recall some of the basic definitions and some important results.

Definition 2.1. [4] *Let (X,τ) be a topological space. We say that α is an operator associated to τ, if α: $P(X) \to P(X)$ satisfies $U \subseteq \alpha(U)$ for all $U \in \tau$ where $P(X)$ denote the set of parts of X.*

Definition 2.2. [3] *Let (X,τ) be a topological space and α: $P(X) \to P(X)$ be an operator associated to the topology τ. A subset $A \subseteq X$ is said to be α-semi open set if there exists $U \in \tau$ such that $U \subseteq A \subseteq \alpha(U)$.*

Received: August 22, 2005; *Accepted:* January 12, 2006.

Definition 2.3. [9] *Let (X,τ) be a topological space and α,β: $P(X) \to P(X)$ be operators associated to a topology τ on X. A subset $A \subseteq X$ is said to be an (α,β) -semi open set if for each $x \in A$, there exists a β-semi open set V such that $x \in V$ and $\alpha(V) \subseteq A$. The complement of an (α,β)-semi open set is an (α,β)-semi closed set.*

Definition 2.4. [9] *Let (X,τ) be a topological space and α,β: $P(X) \to P(X)$ be operators associated to the topology τ on X. A subset $A \subseteq X$ is said to be an (α,β)-open set if for each $x \in A$, there exist open sets U, V such that $x \in U$, $x \in V$, and $\alpha(U) \cup \beta(V) \subseteq A$.*

The following theorem gives the relationship between (α,β)-open sets and α-open sets.

Theorem 2.1. *Let (X,τ) be a topological space and α,β: $P(X) \to P(X)$ be operators associated to a topology τ on X. Then, A is an (α,β)-open set if and only if A is an α-open set and a β-open set.*

Proof. (Sufficiency). Given $x \in A$ there exist open sets U, V such that $x \in U$, $x \in V$ and $\alpha(U) \cup \beta(V) \subseteq A$. It follows that $\alpha(U) \subseteq A$, and therefore A is an α-open set. In the same way, $\beta(V) \subseteq A$, and therefore A is a β-open set.

(Necessity). If A is an α-open set and β-open set, then for all $x \in A$ there exist open sets U, V such that $x \in U$, $x \in V$, $\alpha(U) \subseteq A$ and $\beta(V) \subseteq A$, which implies that $\alpha(U) \cup \beta(V) \subseteq A$; therefore, A is an (α,β)-open set. ∎

3. γ-(α,β)-semi open sets

In this section, we will introduce a new class of sets that generalizes taking appropriate operators the different classes of sets defined previously.

Definition 3.1. *Let (X,τ) be a topological space and α,β,γ: $P(X) \to P(X)$ be operators associated to a topology τ on X. A subset $A \subseteq X$ is said to be a γ-(α,β)-semi open set if for each $x \in$, there exists an (α,β)-semi open set V such that $x \in V$ and $\gamma(V) \subseteq A$. The complement of a γ-(α,β)-semi open set is a γ-(α,β)-semi closed set.*

We can observe in the previous definition that:

1. If $\alpha = \beta = id$, then the notion of γ-(α,β)-semi open set is exactly to the notion of γ-open sets [4].
2. If $\alpha = \gamma = id$ and β is a monotone operator, then the notion of γ-(α,β)-semi open set is the notion of β-semi open set [3].
3. If $\alpha = id$ and β is a monotone operator, then the notion of γ-(α,β)-semi open set is $\Leftrightarrow A$ is the notion of (γ,β)-semi open set [9].
4. If $\gamma = id$, then the definition of γ-(α,β)-semi open set is the definition of (α,β)-semi open [9].
5. If $\gamma = \beta = id$, then the notion of γ-(α,β)-semi open set is the notion of α-open set [4].

Observe that for fixed operators α,β,γ the notions of α-semi open sets, (α,β)-semi open sets and γ-(α,β)-semi open sets are not comparable. In the case that γ is an expansive operator on the family of (α,β)-semi open set, that is $U \subseteq \gamma(U)$ for all (α,β)-semi open set U, then:

if A is a γ-(α, β)-semi open set then A is an (α, β)-semi open set.

Moreover if α and γ are expansive operators on the set of β-semi open and β is a monotone operator and if A is a γ-(α,β)-semi open set, then A is a β-semi open set. The following example shows that there exist (α,β)-semi open sets that are not γ-(α,β)-semi open, where γ is an expansive operator.

Example 3.1. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Consider the operators α, β, γ defined as follows

$$\alpha(A) = \begin{cases} A, & \text{if } A = \{a\}, \\ \mathrm{Cl}(A), & \text{otherwise}, \end{cases}$$
$$\beta(A) = id(A) \text{ and } \gamma(A) = \mathrm{Cl}(A).$$

We can see that

$$\begin{aligned} \beta - SO(X, \tau) &= \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}, \\ (\alpha, \beta) - SO(X, \tau) &= \{\emptyset, X, \{a\}\}, \\ \gamma\text{-}(\alpha, \beta)\text{-}SO(X, \tau) &= \{\emptyset, X\} \end{aligned}$$

and the set $\{a\}$ is an (α, β)-semi open that is not a γ-(α, β)-semi open.

Example 3.2. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Consider α, β, γ operators defined as follows

$$\alpha(A) = \mathrm{Cl}(A)$$
$$\beta(A) = \begin{cases} \mathrm{Cl}(A), & \text{if } b \in A, \\ A, & \text{otherwise}, \end{cases}$$
$$\gamma(A) = \begin{cases} A, & \text{if } b \in A, \\ X, & \text{otherwise}. \end{cases}$$

We can see that:

$$\begin{aligned} (\alpha, \beta)\text{-}SO(X, \tau) &= \{\emptyset, X, \{b\}, \{a, c\}\}, \\ \gamma\text{-}(\alpha, \beta)\text{-}SO(X, \tau) &= \{\emptyset, X, \{b\}\}. \end{aligned}$$

Example 3.3. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Consider the operators α, β, γ defined as follows

$$\alpha(A) = \begin{cases} A, & \text{if } b \in A, \\ \mathrm{Cl}(A), & \text{if } b \notin A \text{ and } A \neq \emptyset, \\ \{c\}, & \text{if } A = \emptyset, \end{cases}$$
$$\beta(A) = \mathrm{Cl}(A)$$

and

$$\gamma(A) = \begin{cases} A, & \text{if } c \notin A, \\ \{c\}, & \text{if } c \in A \text{ and } A \neq X, \\ X, & \text{if } A = X. \end{cases}$$

We can see that

$$\begin{aligned} \beta\text{-}SO(X,\tau) &= \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\},\\ (\alpha,\beta)\text{-}SO(X,\tau) &= \{\emptyset, X, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\},\\ \gamma\text{-}(\alpha,\beta)\text{-}SO(X,\tau) &= \{\emptyset, X,\ \{a,c\}, \{b,c\}, \{a,b\}, \{b\}, \{c\}\}. \end{aligned}$$

These example shows that there exists (α,β)-semi open sets that are not γ-(α,β)-semi open and viceversa.

Example 3.4. Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Consider the operators α,β,γ defined as follows

$$\alpha(A) = \begin{cases} A, & \text{if } b \in A,\\ \mathrm{Cl}(A), & \text{if } b \notin A \text{ and } A \neq \emptyset,\\ \{c\}, & \text{if } A = \emptyset, \end{cases}$$

$$\beta(A) = \gamma(A) = \mathrm{Cl}(A).$$

We can see that:

$$\begin{aligned} \beta\text{-}SO(X,\tau) &= \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\},\\ (\alpha,\beta)\text{-}SO(X,\tau) &= \{\emptyset, X, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\},\\ \gamma\text{-}(\alpha,\beta)\text{-}SO(X,\tau) &= \{\emptyset, X,\ \{a,c\}, \{b,c\}\}. \end{aligned}$$

Also, Example 3.4 shows that γ-(α,β)-$SO(X,\tau)$ is not a topology.

The following lemmas give information about some fundamental properties of the γ-(α,β)-semi open sets (resp. γ-(α,β)-semi closed sets).

Lemma 3.1. *Let (X,τ) be a topological space and α,β,γ: $P(X) \to P(X)$ be operators associated to a topology τ on X. If $\{A_i : i \in I\}$ is a collection of γ-(α,β)-semi open sets, then $\bigcup_{i\in I} A_i$ is a γ-(α,β)-semi open set.*

Proof. Given $x \in \bigcup_{i\in I} A_i$, then $x \in A_j$ for some $j \in I$. In this case, there exists an (α,β)-semi open set V_j such that $x \in V_j$ and $\gamma(V_j) \subseteq A_j \subseteq \bigcup_{i\in I} A_i$. Therefore, given $x \in \bigcup_{i\in I} A_i$, there exists an (α,β)-semi open set V_j such that $\gamma(V_j) \subseteq \bigcup_{i\in I} A_i$. This implies that $\bigcup_{i\in I} A_i$ is a γ-(α,β-semi open set. ∎

Now using the above lemma and the De Morgan laws, we obtain the following corollary.

Corollary 3.1. *Let (X,τ) be a topological space and α,β,γ:$P(X) \to P(X)$ be operators associated to a topology τ on X. If $\{A_i : i \in I\}$ is a collection of γ-(α,β)-semi closed sets, then $\bigcap_{i\in I} A_i$ is a γ-(α,β)-semi closed set.*

We can observe that it is possible to define in a natural way the γ-(α,β)-semi closure of a set $A \subseteq X$ as the intersection of all γ-(α,β)-semi closed sets F that contain A and the γ-(α,β)-semi interior of a set $A \subseteq X$ as the union of all γ-(α,β)-semi open sets G that are containing in A. They will be denoted by γ-(α,β)-sCl(A) and γ-(α,β)-sInt(A), respectively.

In a topological space (X,τ) for which it has the associated operators α,β,γ: $P(X) \to P(X)$, we have in a natural way some properties that are well known as we can see in the following lemma.

Lemma 3.2. *Let* (X,τ) *be a topological space and* α,β,γ*:* $P(X)\to P(X)$ *be operators associated to a topology* τ *on* X*. Then:*

(a) γ-(α,β)-sInt$(A)\subseteq\gamma-(\alpha,\beta)$-sInt$(B)$ *if* $A\subseteq B$*;*
(b) γ-(α,β)-sCl $(A)\subseteq\gamma$-(α,β)-sCl(B) *if* $A\subseteq B$*;*
(c) A *is a* γ-(α,β)*-semi open set* $\Leftrightarrow A=\gamma$-$\alpha,\beta)$-sInt(A)*;*
(d) B *is a* γ-(α,β)*-semi closed set* $\Leftrightarrow B=\gamma$-(α,β)-sCl(B)*;*
(e) $x\in\gamma$-(α,β)-sInt(A) *if and only if there exists a* γ-(α,β)*-semi-open set* G *such that* $x\in G\subseteq A$*;*
(f) $x\in\gamma$-(α,β)-sCl(B) *if and only if for all* γ-(α,β)*-semi open set* G *such that* $x\in G$*,* $G\cap B\neq\emptyset$*;*
(g) $X\backslash(\gamma(\alpha,\beta)$-sCl $(A)))=\gamma$-(α,β)-sInt $(X\backslash A)$ *and* $X\backslash(\gamma(\alpha,\beta)$-sInt$(A)))=$ γ-(α,β)-sCl$(X\backslash A)$.

Proof. (f) Suppose that $x\notin\gamma$-(α,β)-sCl(B) then there exists a γ-(α,β)-semi closed set F such that $B\subseteq F$ and $x\notin F$, then $x\in X\setminus F$, if we take $G=X\setminus F$, then G is a γ-(α,β) semi open set and $G\cap B=\emptyset$. Reciprocally if there exists a γ-(α,β) semi open set G such that $x\in G$ and $G\cap B=\emptyset$ then $X\setminus G$ is a γ-(α,β) semi closed set containing B and $x\notin B$, this implies that $x\notin$ γ-(α,β)-sCl(B). (e) and (g) follows in a similar form using the definitions of the γ-(α,β)-semi closure and the γ-(α,β)-semi interior. ∎

4. γ-(α,β)-semi T_i spaces

In this section, we introduce the generalized separation axioms using the notions of γ-(α,β)-semi open sets, also we give some characterization of these types of spaces and study the relationships between them and other well known spaces.

Definition 4.1. *Let* (X,τ) be a topological space and $\alpha,\beta,\gamma: P(X)\to P(X)$ be operators associated to a topology τ on X. The space X is said to be:

(i) γ-(α,β)-semi T_0 if for each pair of points $x,y\in X$, $x\neq y$, there is a γ-(α,β)-semi open set containing one of the points, but not the other one.
(ii) γ-(α,β)-semi T_1 if for each pair of distinct points $x,y\in X$ there exist a pair of γ-(α,β)-semi open sets, one of them containing x but not y and the other one containing y but not x.
(iii) γ-(α,β)-semi T_2 if for each pair of distinct points $x,y\in X$ there exist disjoint γ-(α,β)-semi open sets U and V, in X such that $x\in U$ and $y\in V$.

The following theorems characterize the spaces: γ-(α,β)-semi T_0, γ-(α,β)-semi T_1 and γ-(α,β)-semi T_2. We shall observe that there exist some similarities to the usually well known cases. These last ones are more general in the sense that they include the usual cases and others due to the arbitrariness of the considered operators α, β and γ. We can see that the α-T_i spaces [8], with $i=0,1,2$, are the id-(α,id)-semi T_i spaces. If α is a monotone operator, the α-semi T_i spaces [8], with $i=0,1,2$, are the id-(id,α)-semi T_i spaces. If γ is the identity operator, then the (α,β)-semi T_i spaces [9], with $i=0,1,2$, are the id-(α,β)-semi T_i spaces.

Theorem 4.1. *Let* (X,τ) *be a topological space and* α,β,γ*:* $P(X)\to P(X)$ *be operators associated to a topology* τ *on* X*. Then* X *is a* γ-(α,β)*-semi* T_0 *space if* *and only if for any $x, y \in X$ such that $x \neq y$ we have that γ-(α,β)-sCl$(\{x\}) \neq$ γ-(α,β)-sCl $(\{y\})$.*

Proof. (Sufficiency) Suppose that X is a γ-(α,β)-semi T_0 space, then for any pair of distinct points $x, y \in X$ there exists a γ-(α,β)-semi open set U, such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. It follows that γ-(α,β)-sCl $(\{x\}) \neq \gamma$-(α,β)-sCl $(\{y\})$.

(Necessity) Suppose that $x, y \in X$ and $x \neq y$, imply γ-(α,β)-sCl$(\{x\}) \neq \gamma$-(α,β)-sCl$(\{y\})$. It follows that, given $x \neq y$, there is a point $z \in X$ such that $z \in$ γ-(α,β)-sCl $(\{y\})$ and $z \notin \gamma$-(α,β)-sCl $(\{x\})$ or $z \in \gamma$-(α,β)-sCl$(\{x\})$ and $z \notin \gamma$-(α,β)-sCl $\{y\})$. If $z \in \gamma$-(α,β)-sCl $(\{y\})$ and $z \notin \gamma$-(α,β)-sCl $(\{x\})$, there exist a γ-(α,β)-semi open set V such that $y \in V$ and $V \cap \{x\} = \emptyset$. In case that $z \in \gamma$-(α,β)-sCl$(\{x\})$ and $z \notin \gamma$-(α,β)-sCl$(\{y\})$, there exist a γ-(α,β)-semi open set U such that $x \in U$ and $V \cap \{y\} = \emptyset$. This shows that X is γ-(α,β)-semi T_0. ∎

Theorem 4.2. *Let (X,τ) be a topological space and $\alpha, \beta, \gamma : P(X) \to P(X)$ be operators associated to a topology τ on X. For the topological space (X,τ), the followings conditions are equivalent:*

(a) *X is a γ-(α,β)-semi T_1 space.*
(b) *Each singleton set $\{x\}$, $x \in X$, is a γ-(α,β)-semi closed set.*
(c) *Each subset of X is the intersection of all super sets γ-(α,β)-semi open containing it.*

Proof. (a)⇒(b). Let X be a γ-(α,β)-semi T_1 space. Given $y \in X \setminus x$, then $y \neq x$, by hypothesis there are γ-(α,β)-semi open sets $U, V \subseteq X$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Therefore, $y \in V \subseteq X \setminus x$, because $V \cap \{x\} = \emptyset$. It follows that $X \setminus \{x\}$ is a γ-(α,β)-semi open set and, therefore, $\{x\}$ is a γ-(α,β)-semi closed set.

(b) ⇒ *(c)*. Let us suppose that each $\{x\}, x \in X$, is a γ-(α,β)-semi closed set. Given $A \subseteq X$ and define the set $D(A)$ as follows: $D(A) = \bigcap\{S : A \subseteq S$ and S is a γ-(α,β)-semi open set. We are going to prove that $A = D(A)$. In general, $A \subset D(A)$. Suppose that $x \notin A$. Then $A \subseteq X \setminus \{x\}$ and $X \setminus \{x\}$ is a γ-(α,β)-semi open because $\{x\}$ is γ-(α,β)-semi closed. Therefore, $x \notin D(A)$ and hence $D(A) \subseteq A$. Consequently $A = D(A)$.

(c) ⇒ *(a)*. Let $D(x) = \{S : x \in S$ and S is γ-(α,β)-semi open$\}$. By hypothesis, $\{x\} = \bigcap_{S \in D(x)} S$. Therefore if $y \neq x$ then $y \notin \bigcap_{S \in D(x)} S$ and there is an γ-(α,β)-semi open set S such that $x \in S$ and $y \notin S$, in analogue form, if $x \notin \bigcap_{S' \in D(y)} S'$ and there is a γ-(α,β)-semi open set S' such that $y \in S'$ and $x \notin S'$. It said that X is a γ-(α,β)-semi T_1 space. ∎

From the above definitions, we can see easily the following relations γ-(α,β)-semi $T_2 \Rightarrow \gamma$-(α,β)-semi $T_1 \Rightarrow (\alpha,\beta)$-semi T_0. But the converse need not be true.

In the same way, we can introduce the notions of semi regularity and γ-(α,β)-semi T_3 spaces, using γ-(α,β)-semi open sets.

Definition 4.2. *Let (X,τ) be a topological space and $\alpha, \beta, \gamma : P(X) \to P(X)$ be operators associated to a topology τ on X. X is said to be a γ-(α,β)-semi regular* *space if whenever A is a γ-(α,β)-semi closed set in X and $x \notin A$, there are disjoint γ-(α,β)-semi open sets U and V with $x \in U$ and $A \subseteq V$.*

The following proposition characterize the γ-(α,β)-semi regular spaces.

Theorem 4.3. *Let (X,τ) be a topological space and $\alpha,\beta,\gamma : P(X) \to P(X)$ be operators associated to a topology τ on X. The following are equivalent:*

(a) *X is γ-(α,β)-semi regular.*

(b) *If U is a γ-(α,β)-semi open set and $x \in U$, there is a γ-(α,β)-semi open set V such that $x \in V$ and γ-(α,β)-$\mathrm{sCl}(V) \subseteq U$.*

Proof. (a) $\Rightarrow$ (b). Suppose X is γ-(α,β)-semi regular, U is γ-(α,β)-semi open set and $x \in U$. Then $X \setminus U$ is a γ-(α,β)-semi closed set in X not containing x, so disjoint γ-(α,β)-semi open sets V and W can be found with $x \in V$ and $X \setminus U \subseteq W$. Then $X \setminus W$ γ-(α,β)-semi closed set contained in U and contain V, so γ-(α,β)-$\mathrm{sCl}(V) \subseteq U$.

(b) $\Rightarrow$ (a). Let A be a γ-(α,β)-semi closed set and $x \notin A$, then $x \in X \setminus A$. Since $X \setminus A$ is γ-(α,β)-semi open then by hypothesis there exist V γ-(α,β)-semi open set such that $x \in V \subseteq \gamma$-(α,β)-$\mathrm{sCl}(V) \subseteq X \setminus A$. It follows that V and $X \setminus \gamma$-(α,β)-sCl (V) separate x and A. ∎

Definition 4.3. *Let (X,τ) be a topological space and $\alpha,\beta,\gamma : P(X) \to P(X)$ be operators associated to a topology τ on X. X is said to be a γ-(α,β)-semi T_3 space, if X is γ-(α,β)-semi regular and γ-(α,β)-semi T_1.*

Clearly every γ-(α,β)-semi T_3 space is γ-(α,β)-semi T_2.

5. γ-(α,β)-generalized semi closed sets and γ-(α,β)-semi $T_{1/2}$ spaces

We recall that if $A \subseteq X$ and α,β,γ: $P(X) \to P(X)$ are associated operators to a topology τ on X, then the γ-(α,β)-$\mathrm{sCl}(A)$ is a γ-(α,β)-semi closed set. In consequence, we can introduce the notions of γ-(α,β)-semi $T_{1/2}$ spaces in a natural way, using the concept of γ-(α,β)-generalized semi closed sets. Also we can study the relations with other spaces that we have studied before.

Definition 5.1. *Let (X,τ) be a topological space and α,β,γ: $P(X) \to P(X)$ be operators associated to a topology τ on X. $A \subseteq X$ is said to be a γ-(α,β)-generalized semi closed set if the γ-(α,β)-$\mathrm{sCl}(A) \subseteq S$ for all γ-(α,β)-semi open set S such that $A \subseteq S$.*

Definition 5.2. *Let (X,τ) be a topological space and $\alpha,\beta,\gamma : P(X) \to P(X)$ be operators associated to a topology τ on X. X is said to be a γ-(α,β)-semi $T_{1/2}$ space if all γ-(α,β)-generalized semi closed set is a γ-(α,β)-semi closed set.*

Observe that when β is a monotone operator and $\alpha = id = \gamma$, then the γ-(α,β)-generalized semi closed sets are the β-generalized semi closed sets, therefore the γ-(α,β)-semi $T_{1/2}$ spaces are the β-semi $T_{1/2}$ spaces.

Theorem 5.1. *Let (X,τ) be a topological space and $\alpha,\beta,\gamma : P(X) \to P(X)$ be operators associated to a topology τ on X. X is a γ-(α,β)-semi $T_{1/2}$ space if and only if for each $x \in X$, $\{x\}$ is a γ-(α,β)-semi open set or a γ-(α,β)-semi closed set.*

Proof. (Sufficiency) Suppose that X is a γ-(α,β)-semi $T_{1/2}$ space and $x \in X$, then $\{x\}$ can be a γ-(α,β)-semi closed set or not. In the first case, the proof follows. In the second case, take $A = X \setminus \{x\}$, then A is not a γ-(α,β)-semi closed set, but X is the only γ-(α,β)-semi open set that contain A, then A is a γ-(α,β)-generalized semi closed set, therefore γ-(α,β)-semi closed, this implies that $\{x\}$ is a γ-(α,β)-semi open set.
(Necessity) Let A be a γ-(α,β)-generalized semi closed set and $x \in \gamma$-(α,β)-sCl(A). If $\{x\}$ is a γ-(α,β)-semi open set, then $\{x\} \cap A \neq \emptyset$, therefore $x \in A$. If $\{x\}$ is a γ-(α,β)-semi closed set and $x \notin A$, then $X \setminus \{x\}$ is γ-(α,β)-semi open and $A \subseteq X \setminus \{x\}$. Since A is γ-(α,β)-generalized semi closed, then γ-(α,β)-sCl$(A) \subseteq X \setminus \{x\}$ and $x \notin \gamma$-(α,β)-sCl(A). This is contrary to $x \in \gamma$-(α,β)-sCl(A). Hence $x \in A$ and A is γ-(α,β)-semi closed. ∎

From the above theorem, the following corollary is obtained.

Corollary 5.1. *Let (X,τ) be a topological space and $\alpha,\beta,\gamma : P(X) \to P(X)$ be operators associated to a topology τ on X. X is a γ-(α,β)-semi $T_{1/2}$ space if and only if each subset of X, is the intersection of all γ-(α,β)-semi open sets and γ-(α,β)-semi closed sets containing it.*

Theorem 5.2. *Let (X,τ) be a topological space and $\alpha,\beta,\gamma : P(X) \to P(X)$ be operators associated to a topology τ on X. Every γ-(α,β)-semi $T_{1/2}$ space is a γ-(α,β)-semi T_0 space.*

Proof. Let x, y be any pair of distinct points of X. By Theorem 5.1, the singleton $\{x\}$ is a γ-(α,β)-semi open or γ-(α,β)-semi closed. If $\{x\}$ is a γ-(α,β)-semi open, $x \in \{x\}$ and $y \notin \{x\}$. If $\{x\}$ is a γ-(α,β)-semi closed, then $X \setminus \{x\}$ is a γ-(α,β)-semi open, $y \in X \setminus \{x\}$ and $x \notin X \setminus \{x\}$. Therefore, (X,τ) is γ-(α,β)-semi T_0. ∎

The following example shows that there exist γ-(α,β)-semi T_0 spaces that are not γ-(α,β)-semi $T_{1/2}$ spaces.

Example 5.1. If we take $X, \tau, \alpha, \beta, \gamma$ as in Example 3.4, we obtain that

$$\gamma\text{-}(\alpha,\beta)\text{-}SO(X,\tau) = \{\emptyset, X, \{a,c\}, \{b,c\}\}$$

and the γ-(α,β) semi closed set is $\{\emptyset, X, \{a\}, \{b\}\}$. Using Theorem 4.1, X is a γ-(α,β)-semi T_0 space and by Theorem 5.1, X is not a γ-(α,β)-semi $T_{1/2}$ space.

Theorem 5.3. *Let (X,τ) be a topological space and $\alpha,\beta,\gamma : P(X) \to P(X)$ be operators associated to a topology τ on X. Then γ-(α,β)-semi T_1 implies γ-(α,β)-semi $T_{1/2}$.*

Proof. By Theorem 4.2, for each $x \in X$, the singleton $\{x\}$ is γ-(α,β)-semi closed. Therefore, by Theorem 5.1, (X,τ) is γ-(α,β)-semi $T_{1/2}$. ∎

The following example shows that the existence of a γ-(α,β)-semi $T_{1/2}$ space that is not a γ-(α,β)-semi T_1 space.

Example 5.2. *Let* $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$. Defined α,β,γ as follows:

$$\alpha(A) = A,$$

$$\beta(A) = \begin{cases} A, & \text{if } A = \{a\} \text{ or } \{b\}, \\ X, & \text{otherwise,} \end{cases}$$

$$\gamma(A) = \begin{cases} A, & \text{if } A = \{a\} \text{ or } \{b\}, \\ \emptyset, & \text{if } A = \emptyset, \\ X, & \text{otherwise.} \end{cases}$$

We obtain that

$$\beta\text{-}SO(X,\tau) = \{\emptyset, X, \{a,b\}, \{a,c\}, \{b,c\}, \{a\}, \{b\}, \{c\}\}$$
$$\alpha,\beta)\text{-}SO(X,\tau) = \{\emptyset, X,\ \{a,b\}, \{a,c\}, \{b,c\}, \{a\}, \{b\}, \{c\}\}$$
$$\gamma\text{-}(\alpha,\beta)\text{-}SO(X,\tau) = \{\emptyset, X,\ \{a\}, \{b\}, \{a,b\}\}$$

and the γ-(α,β)-semi closed set is

$$\{\emptyset, X,\ \{b,c\}, \{a,c\}, \{c\}\}.$$

By Theorem 5.1, X is a γ-(α,β)-semi $T_{1/2}$ space and by Theorem 4.2, X is not a γ-(α,β)-semi T_1 space.

Example 5.3. Let $X = \{a,b,c\}$, $\tau = P(X)$. Defined α, β, γ as follows:

$$\alpha(A) = \begin{cases} A, & \text{if } A = \{a,b\} \text{ or } \{a,c\} \text{ or } \{b,c\} \\ X, & \text{otherwise} \end{cases}$$

$$\beta(A) = A \text{ and } \gamma(A) = \alpha(A) \quad \forall A \subseteq X.$$

We obtain

$$\gamma\text{-}(\alpha,\beta)\text{-}SO(X,\tau) = \{\emptyset, X,\ \{a,b\}, \{a,c\}, \{b,c\}\}$$

and the γ-(α,β)-semi closed set is $\{\emptyset, X,\ \{a\}, \{b\}, \{c\}\}$. By Theorem 4.2, X is a γ-(α,β)-semi T_1 space and X is not a γ-(α,β)-semi T_2space.

Acknowledgement. The authors are very grateful to the referee for his careful work.Research Partially Supported by Consejo de Investigación UDO.

References

[1] P. Bhattacharya and B. K. Lahiri, Semigeneralized closed sets in topology, *Indian J. Math.* **29**(3)(1987), 375–382.

[2] N. Biswas, On characterizations of semicontinuos functions, *Atti. Accad. Naz. Lincei. Rend. CL. Sci. Fis. Mat. Natur.* **48**(8)(1970), 399–402.

[3] C. Carpintero, E. Rosas and J. Vielma, Operadores asociados a una topologi a Γ sobre un conjunto X y nociones conexas, *Divulgaciones matemáticas* **6**(2),(1998), 139–148.

[4] N. Levine, Semiopen sets and semicontinuity in topological spaces, *Amer. Math. Monthly* **70**(1963), 36–41.

[5] N. Levine, Generalized closed sets in topology, *Rend. Circ. Mat. Palermo* **19**(2)(1970), 89–96.

[6] S. N. Maheshwari and R. Prasad, Some new separation axioms, *Ann. Soc. Sci. Bruxelles, Ser. I.* **89**(1975), 395–402.

[7] G. B. Navalagi, *Semi generalized separation axioms in topology,* Preprint (2001).

[8] E. Rosas, J. Vielma, C. Carpintero and M. Salas, Espacios α-semi T_i, para $i = 0, 1/2, 1, 2$, *Pro-Mathematica* **27**(2000), 37–48.

[9] E. Rosas, C. Carpintero and J. Sanabria, (α,β)-semi open sets and some new generalized separation axioms, Sci. Math. Jpn. **62**(3) (2005), 397–403.