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Study of (Λ, α)-Closed Sets and the Related Notions in Topological Spaces

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Abstract. This paper deals with the notions of Λ_α-sets and (Λ, α)-closed sets which are defined by utilizing the notions of α-open and α-closed sets. We also introduce and characterize some new low separation axioms. Moreover, we introduce and study the notions of (Λ, α)-continuity, (Λ, α)-irresoluteness, (Λ, α)-compactness, and (Λ, α)-connectedness.

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1. Preliminaries

In 1965, Njåstad [10] introduced the notions of α-open set and α-closed set in topological spaces. Since the advent of these notions, several research papers with interesting results in different respects came to light. (see [1], [2], [4–9], [11], [12], [14–17]). In this paper, we define and study some new sets, spaces and functions by using the notion of α-open and α-closed sets.

Throughout the present paper, (X, τ) and (Y, σ) (or X and Y) denote topological spaces in which no separation axiom are assumed unless explicitely stated. Let A be a subset of X. The subset A of a topological space (X, τ) is called *α-open* if $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, where $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure and the interior of A respectively. The complement of an α-open set is called *α-closed*. By $\alpha(X, \tau)$ (resp. $\alpha C(X, \tau)$), we denote the family of all α-open (resp. α-closed) sets of X. The intersection of all α-closed sets containing A is called the *α-closure* of A and is denoted by $\mathrm{Cl}_\alpha(A)$. The *α-interior* of A is the union of all α-open sets contained in A and is denoted by $\mathrm{Int}_\alpha(A)$. Recall that a topological space X is called *Alexandroff* if every point has a minimal neighborhood, or equivalently, has a unique minimal base.

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In section 2, we consider the notion of Λ_α-sets. By definition, a subset A of a space (X, τ) is called a Λ_α-set if A is the intersection of all α-open sets containing A. It turns out that the family τ^{Λ_α} of Λ_α-sets of a space (X, τ) is a topology for X. Recently, the present authors have defined and studied the properties of (Λ, θ)-closed sets (see [16]). In this paper, we introduce and investigate the notion of (Λ, α)-closed sets. The definition is as follows: A subset A of a space (X, τ) is called (Λ, α)-closed if $A = T \cap C$, where T is a Λ_α-set and C is a α-closed set. We also investigate some properties related to the separation axiom α-T_1. By introducing the new notion of α-R_0, we prove that in such a space every singleton is (Λ, α)-closed if and only if it is α-closed. In section 3, we define a function $f : (X, \tau) \to (Y, \sigma)$ to be (Λ, α)-continuous if $f^{-1}(\sigma^{\Lambda_\alpha}) \subset \tau^{\Lambda_\alpha}$ and we obtain their characterizations. In section 4, we introduce and investigate several new low separation axioms by utilizing the notions of (Λ, α)-open sets and $D(\Lambda, \alpha)$-sets. In the last section, we present the notions of (Λ, α)-compactness and (Λ, α)-connectedness.

2. (Λ, α)-closed sets

Definition 2.1. *Let A be a subset of a topological space (X, τ). A subset $\Lambda_\alpha(A)$ is defined as follows* $\Lambda_\alpha(A) = \cap\{O \in \alpha(X, \tau) \mid A \subset O\}$.

Lemma 2.1. *For subsets A, B and A_i $(i \in I)$ of a topological space (X, τ), the following hold*

(1) $A \subset \Lambda_\alpha(A)$.
(2) *If* $A \subset B$, *then* $\Lambda_\alpha(A) \subset \Lambda_\alpha(B)$.
(3) $\Lambda_\alpha(\Lambda_\alpha(A)) = \Lambda_\alpha(A)$.
(4) $\Lambda_\alpha(\cap\{A_i \mid i \in I\}) \subset \cap\{\Lambda_\alpha(A_i) \mid i \in I\}$.
(5) $\Lambda_\alpha(\cup\{A_i \mid i \in I\}) = \cup\{\Lambda_\alpha(A_i) \mid i \in I\}$.

Proof. We prove only statements (4) and (5). (4) Suppose that $x \notin \cap\{\Lambda_\alpha(A_i) \mid i \in I\}$. There exists $i_0 \in I$ such that $x \notin \Lambda_\alpha(A_{i_0})$ and there exists an α-open set O such that $x \notin O$ and $A_{i_0} \subset O$. We have $\cap_{i\in I} A_i \subset A_{i_0} \subset O$ and $x \notin O$. Therefore, $x \notin \Lambda_\alpha(\cap\{A_i \mid i \in I\})$. (5) First $A_i \subset \Lambda_\alpha(A_i) \subset \Lambda_\alpha(\cup_{i\in I} A_i)$ and hence $\Lambda_\alpha(A_i) \subset \Lambda_\alpha(\cup_{i\in I} A_i)$. Therefore, we obtain $\cup_{i\in I}\Lambda_\alpha(A_i) \subset \Lambda_\alpha(\cup_{i\in I} A_i)$. Conversely, suppose that $x \notin \cup_{i\in I}\Lambda_\alpha(A_i)$. Then $x \notin \Lambda_\alpha(A_i)$ for each $i \in I$ and hence there exists $V_i \in \alpha(X, \tau)$ such that $A_i \subset V_i$ and $x \notin V_i$ for each $i \in I$. We have $\cup_{i\in I} A_i \subset \cup_{i\in I} V_i$ and $\cup_{i\in I} V_i$ is an α-open set which does not contain x. Therefore, $x \notin \Lambda_\alpha(\cup_{i\in I} A_i)$. This shows that $\Lambda_\alpha(\cup_{i\in I} A_i) \subset \cup_{i\in I}\Lambda_\alpha(A_i)$. ∎

Remark 2.1. In Lemma 2.1(4), the converse is not always true as the following example shows.

Example 2.1. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$. Now put $B = \{b\}$ and $C = \{c\}$. Then $\Lambda_\alpha(B \cap C) = \Lambda_\alpha(\emptyset) = \emptyset$, $\Lambda_\alpha(B) \cap \Lambda_\alpha(C) = X$ and $\Lambda_\alpha(B) \neq B$.

Definition 2.2. *A subset A of a topological space (X, τ) is called a* Λ_α-set *if* $A = \Lambda_\alpha(A)$.

Lemma 2.2. *For subsets A and A_i $(i \in I)$ of a topological space (X, τ), the following hold*

(1) $\Lambda_\alpha(A)$ *is a* Λ_α*-set.*

(2) *If A is α-open, then A is a Λ_α-set.*
(3) *If A_i is a Λ_α-set for each $i \in I$, then $\cap_{i\in I} A_i$ is a Λ_α-set.*
(4) *If A_i is a Λ_α-set for each $i \in I$, then $\cup_{i\in I} A_i$ is a Λ_α-set.*

Proof. This follows readily from Lemma 2.1. ∎

Theorem 2.1. *For a topological space (X, τ), we put $\tau^{\Lambda_\alpha} = \{A \mid A$ is a Λ_α-set of $X\}$. Then the pair $(X, \tau^{\Lambda_\alpha})$ is an Alexandroff space.*

Proof. This is an immediate consequence of Lemma 2.2. ∎

Definition 2.3. *Let A be a subset of a topological space (X, τ). A set $\Lambda^*_\alpha(A)$ is defined as follows $\Lambda^*_\alpha(A) = \cup\{B \in \alpha C(X, \tau) \mid B \subset A\}$.*

Definition 2.4. *A subset A of a topological space (X, τ) is called a Λ^*_α-set if $A = \Lambda^*_\alpha(A)$.*

We obtain the following two lemmas which are similar to Lemma 2.1 and Lemma 2.2.

Lemma 2.3. *For subsets A, B and A_i $(i \in I)$ of a topological space (X, τ) the following properties hold*

(1) $\Lambda^*_\alpha(A) \subseteq A$.
(2) *If $A \subseteq B$, then $\Lambda^*_\alpha(A) \subseteq \Lambda^*_\alpha(B)$.*
(3) *If A is α-closed, then $\Lambda^*_\alpha(A) = A$.*
(4) $\Lambda^*_\alpha(\cap\{A_i : i \in I\}) = \cap\{\Lambda^*_\alpha(A_i) : i \in I\}$.
(5) $\cup\{\Lambda^*_\alpha(A_i) : i \in I\} \subseteq \Lambda^*_\alpha(\cup\{A_i : i \in I\})$.
(6) $\Lambda_\alpha(X - A) = X - \Lambda^*_\alpha(A)$ *and* $\Lambda^*_\alpha(X - A) = X - \Lambda_\alpha(A)$.

Lemma 2.4. *For subsets A, B and A_i $(i \in I)$ of a topological space (X, τ) the following properties hold*

(1) *$\Lambda^*_\alpha(A)$ is a Λ^*_α-set.*
(2) *If A is an α-closed, then A is a Λ^*_α-set.*
(3) *If A_i is a Λ^*_α-set for each $i \in I$, then $\cup\{A_i \mid i \in I\}$ and $\cap\{A_i \mid i \in I\}$ are Λ^*_α-sets.*

Remark 2.2. For a topological space (X, τ), we set $\tau^{\Lambda^*_\alpha} = \{A \mid A \;\; is \;\; a \;\; \Lambda^*_\alpha - set \;\; of \;\; X\}$, then the pair $(X, \tau^{\Lambda^*_\alpha})$ is an Alexandroff space.

Definition 2.5. *A subset A of a topological space (X, τ) is called* (Λ, α)-closed *if $A = T \cap C$, where T is a Λ_α-set and C is an α-closed set.*

Theorem 2.2. *Let A be (Λ, α)-closed subset of a topological space (X, τ). Then, we have*

(1) $A = T \cap \mathrm{Cl}_\alpha(A)$, *where T is a Λ_α-set;*
(2) $A = \Lambda_\alpha(A) \cap \mathrm{Cl}_\alpha(A)$.

Proof. (1) Let $A = T \cap C$, where T is a Λ_α-set and C is an α-closed set. Since $A \subset C$, we have $\mathrm{Cl}_\alpha(A) \subset C$ and $A = T \cap C \supset T \cap \mathrm{Cl}_\alpha(A) \supset A$. Therefore, we obtain $A = T \cap \mathrm{Cl}_\alpha(A)$.

(2) Let $A = T \cap \mathrm{Cl}_\alpha(A)$, where T is a Λ_α-set. Since $A \subset T$, we have $\Lambda_\alpha(A) \subset \Lambda_\alpha(T) = T$ and hence $A \subset \Lambda_\alpha(A) \cap \mathrm{Cl}_\alpha(A) \subset T \cap \mathrm{Cl}_\alpha(A) = A$. Therefore, we obtain $A = \Lambda_\alpha(A) \cap \mathrm{Cl}_\alpha(A)$. ∎

Lemma 2.5. *Every Λ_α-set (resp. α-closed set) is (Λ,α)-closed.*

Definition 2.6. *A subset A of a topological space (X,τ) is said to be (Λ,α)-open if the complement of A is (Λ,α)-closed.*

Theorem 2.3. *Let A_i $(i \in I)$ be a subset of a topological space (X,τ).*

(1) *If A_i is (Λ,α)-closed for each $i \in I$, then $\cap\{A_i \mid i \in I\}$ is (Λ,α)-closed.*
(2) *If A_i is (Λ,α)-open for each $i \in I$, then $\cup\{A_i \mid i \in I\}$ is (Λ,α)-open.*

Proof. (1) Suppose that A_i is (Λ,α)-closed for each $i \in I$. Then, for each i, there exist a Λ_α-set T_i and an α-closed set C_i such that $A_i = T_i \cap C_i$. We have $\cap_{i\in I} A_i = \cap_{i\in I}(T_i \cap C_i) = (\cap_{i\in I} T_i) \cap (\cap_{i\in I} C_i)$. By Lemma 2.2, $\cap_{i\in I} T_i$ is a Λ_α-set and $\cap_{i\in I} C_i$ is an α-closed. This shows that $\cap_{i\in I} A_i$ is (Λ,α)-closed.

(2) Let A_i ibe (Λ,α)-open for each $i \in I$. Then $X - A_i$ is (Λ,α)-closed and $X - \cup_{i\in I} A_i = \cap_{i\in I}(X - A_i)$. Therefore, by (1) $\cup_{i\in I} A_i$ is (Λ,α)-open. ∎

Definition 2.7. (*see* [6]) *A subset A of a topological space (X,τ) is called a (α,α)-*generalized-closed *set (briefly (α,α)-g-closed) if $\mathrm{Cl}_\alpha(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X,τ). A subset A is said to be (α,α)-g-open if $X - A$ is (α,α)-g-closed.*

The following two lemmas are obtained easily from the definitions.

Lemma 2.6. *For a subset A of a topological space (X,τ), the following properties hold*

(1) *A is (α,α)-g-closed if and only if $\mathrm{Cl}_\alpha(A) \subset \Lambda_\alpha(A)$.*
(2) *A is α-closed if and only if A is (α,α)-g-closed and (Λ,α)-closed.*

Lemma 2.7. *For a subset A of a topological space (X,τ), the following properties hold*

(1) *A is (α,α)-g-open if and only if $\Lambda^*_\alpha(A) \subset \mathrm{Int}_\alpha(A)$.*
(2) *A is α-open if and only if A is (α,α)-g-open and (Λ,α)-open.*

Theorem 2.4. *Let A be a (Λ,α)-open subset of a topological space (X,τ). Then, we have*

(1) *$A = T \cup C$, where T is a Λ^*_α-set and C is α-open;*
(2) *$A = T \cup \mathrm{Int}_\alpha(A)$, where T is a Λ^*_α-set;*
(3) *$A = \Lambda^*_\alpha(A) \cup \mathrm{Int}_\alpha(A)$.*

Proof. (1) Suppose that A is (Λ,α)-open. Then $X - A$ is (Λ,α)-closed and $X - A = K \cap D$, where K is a Λ_α-set and D is an α-closed set. Hence, we have $A = (X - K) \cup (X - D)$, where $X - K$ is a Λ^*_α-set and $X - D$ is α-open set.

(2) Since A is an (Λ,α)-open we have $A = T \cup C$, where T is an Λ^*_α-set and C is α-open. Also $C \subset A$ and C is α-open, $C \subset \mathrm{Int}_\alpha(A)$ and hence $A = T \cup C \subset T \cup \mathrm{Int}_\alpha(A) \subset A$. Therefore, we obtain $A = T \cup \mathrm{Int}_\alpha(A)$.

(3) Since A is an (Λ,α)-open we have $A = T \cup \mathrm{Int}_\alpha(A)$, where T is a Λ^*_α-set. Also $T \subset A$, we have $\Lambda^*_\alpha(A) \supset \Lambda^*_\alpha(T)$ and hence $A \supset \Lambda^*_\alpha(A) \cup \mathrm{Int}_\alpha(A) \supset \Lambda^*_\alpha(T) \cup \mathrm{Int}_\alpha(A) = T \cup \mathrm{Int}_\alpha(A) = A$. Therefore, we obtain $A = \Lambda^*_\alpha(A) \cup \mathrm{Int}_\alpha(A)$. ∎

Definition 2.8. *A topological space (X,τ) is called an α-R_0 space if for each α-open set U and each $x \in U$, $\mathrm{Cl}_\alpha(\{x\}) \subset U$.*

Definition 2.9. (*see* [6]) *A topological space* (X, τ) *is said to be* α-T_1 *if for any distinct pair of points* x *and* y *in* X*, there is an* α*-open* U *in* X *containing* x *but not* y *and an* α*-open set* V *in* X *containing* y *but not* x*.*

Theorem 2.5. *Let* (X, τ) *be a* α-R_0 *space. A singleton* $\{x\}$ *is* (Λ, α)*-closed if and only if* $\{x\}$ *is* α*-closed.*

Proof. Necessity. Suppose that $\{x\}$ is (Λ, α)-closed. Then, by Theorem 2.2, $\{x\} = \Lambda_\alpha(\{x\}) \cap \mathrm{Cl}_\alpha(\{x\})$. For any α-open set U containing x, $\mathrm{Cl}_\alpha(\{x\}) \subset U$ and hence $\mathrm{Cl}_\alpha(\{x\}) \subset \Lambda_\alpha(\{x\})$. Therefore, we have $\{x\} = \Lambda_\alpha(\{x\}) \cap \mathrm{Cl}_\alpha(\{x\}) \supset \mathrm{Cl}_\alpha(\{x\})$. This shows that $\{x\}$ is α-closed.

Sufficiency. Suppose that $\{x\}$ is α-closed. Since $\{x\} \subset \Lambda_\alpha(\{x\})$, we have $\Lambda_\alpha(\{x\}) \cap \mathrm{Cl}_\alpha(\{x\}) = \Lambda_\alpha(\{x\}) \cap \{x\} = \{x\}$. This shows that $\{x\}$ is (Λ, α)-closed. ∎

Theorem 2.6. *A topological space* (X, τ) *is* α-T_1 *if and only if for each* $x \in X$*, the singleton* $\{x\}$ *is a* Λ_α*-set.*

Proof. Necessity. Suppose that $y \in \Lambda_\alpha(\{x\})$ for some point y distinct from x. Then $y \in \cap\{V_x \mid x \in V_x \;\; and \;\; V_x \;\; is \;\; \alpha - open\}$ and hence $y \in V_x$ for every α-open set V_x containing x. This contradicts that (X, τ) is an α-T_1.

Sufficiency. Suppose that $\{x\}$ is a Λ_α-set for each $x \in X$. Let x and y be any distinct points. Then $y \notin \Lambda_\alpha(\{x\})$ and there exists an α-open set V_x such that $x \in V_x$ and $y \notin V_x$. Similarly, $x \notin \Lambda_\alpha(\{y\})$ and there exists an α-open set V_y such that $y \in V_y$ and $x \notin V_y$. This shows that (X, τ) is α-T_1. ∎

Definition 2.10. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is said to be*

(1) α*-continuous* ([6], [9], [13] *and* [14]) *if the inverse image of each open set in* (Y, σ) *is an* α*-open set in* (X, τ)*.*
(2) α-irresolute [6] *if the inverse image of each* α*-open set in* (Y, σ) *is an* α*-open set in* (X, τ)*.*

Theorem 2.7. *A topological space* (X, τ) *is* α-T_1 *if and only if* $(X, \tau^{\Lambda_\alpha})$ *is the discrete space.*

Proof. Necessity. Suppose that (X, τ) is α-T_1. Let x be any point of X. By Theorem 2.6, $\{x\}$ is a Λ_α-set and $\{x\} \in \tau^{\Lambda_\alpha}$. For any subset A of X, by Lemma 2.2 $A \in \tau^{\Lambda_\alpha}$. This shows that $(X, \tau^{\Lambda_\alpha})$ is discrete.

Sufficiency. For each $x \in X$, $\{x\} \in \tau^{\Lambda_\alpha}$ and hence $\{x\}$ is Λ_α-set. By Theorem 2.6, (X, τ) is α-T_1. ∎

Theorem 2.8. *If a function* $f : (X, \tau) \to (Y, \sigma)$ *is* α*-irresolute, then* $f : (X, \tau^{\Lambda_\alpha}) \to (Y, \sigma^{\Lambda_\alpha})$ *is continuous.*

Proof. Let V be any Λ_α-set of (Y, σ), i.e. $V \in \sigma^{\Lambda_\alpha}$. Then $V = \Lambda_\alpha(V) = \cap\{W \mid V \subset W \;\; and \;\; W \;\; is \;\; \alpha - open \;\; in \;\; (Y, \sigma)\}$. Since f is α-irresolute, $f^{-1}(W)$ is α-open in (X, τ) for each W. Hence we have $f^{-1}(V) = \cap\{f^{-1}(W) \mid f^{-1}(V) \subset f^{-1}(W) \;\; and \;\; W \;\; is \;\; \alpha - open \;\; in \;\; (Y, \sigma)\} \supset \cap\{U \mid f^{-1}(V) \subset U \;\; and \;\; U \;\; is \;\; \alpha - open \;\; in \;\; (X, \tau)\} = \Lambda_\alpha(f^{-1}(V))$. On the other hand, by the definition $f^{-1}(V) \subset \Lambda_\alpha(f^{-1}(V))$. Therefore, we obtain $f^{-1}(V) = \Lambda_\alpha(f^{-1}(V))$. Hence, $f^{-1}(V) \in \tau^{\Lambda_\alpha}$ and $f : (X, \tau) \to (Y, \sigma)$ is continuous. ∎

3. (Λ,α)-continuous Functions

Definition 3.1. *Let (X,τ) be a topological space and $A \subseteq X$. A point $x \in X$ is called a (Λ,α)-*cluster point *of A if for every (Λ,α)-open set U of X containing x we have $A \cap U \neq \emptyset$. The set of all (Λ,α)-cluster points is called the (Λ,α)-*closure set *(or* Lambda-α-closure*) of A and is denoted by $A^{(\Lambda,\alpha)}$ (or $L\alpha\,\mathrm{Cl}(A)$).*

Lemma 3.1. *Let A and B be subsets of a topological space (X,τ). For the (Λ,α)-closure, the following properties hold*

(1) $A \subset A^{(\Lambda,\alpha)}$ *and* $(A^{(\Lambda,\alpha)})^{(\Lambda,\alpha)} = A^{(\Lambda,\alpha)}$.
(2) $A^{(\Lambda,\alpha)} = \cap\{F \mid A \subset F \ \text{ and } \ F \ \text{ is } \ (\Lambda,\alpha)-closed\}$.
(3) *If* $A \subset B$, *then* $A^{(\Lambda,\alpha)} \subset B^{(\Lambda,\alpha)}$.
(4) A *is* (Λ,α)*-closed if and only if* $A = A^{(\Lambda,\alpha)}$.
(5) $A^{(\Lambda,\alpha)}$ *is* (Λ,α)*-closed.*

Proof. Straightforward. ∎

Definition 3.2. *Let (X,τ) be a topological space, $x \in X$ and $\{x_s, s \in S\}$ be a net of X. We say that the net $\{x_s, s \in S\}$ (Λ,α)-converges to x if for each (Λ,α)-open set U containing x there exists an element $s_0 \in S$ such that $s \geq s_0$ implies $x_s \in U$.*

Lemma 3.2. *Let (X,τ) be a topological space and $A \subseteq X$. A point $x \in A^{(\Lambda,\alpha)}$ if and only if there exists a net $\{x_s, s \in S\}$ of A which (Λ,α)-converges to x.*

Definition 3.3. *Let (X,τ) be a topological space, $\mathcal{F} = \{F_i : i \in I\}$ be a filterbase of X and $x \in X$. We say that the filterbase $\mathcal{F}$ (Λ,α)-converges to x if for each (Λ,α)-open set U containing x there is a member $F_i \in \mathcal{F}$ such that $F_i \subseteq U$.*

Definition 3.4. *A function $f : (X,\tau) \to (Y,\sigma)$ is called (Λ,α)-*continuous *if $f^{-1}(V)$ is a (Λ,α)-open subset of X for every open subset V of Y.*

Theorem 3.1. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following statements are equivalent*

(1) f *is* (Λ,α)*-continuous;*
(2) *For each $x \in X$ and for each open set V of Y containing $f(x)$ there exists a (Λ,α)-open set U of X containing x and $f(U) \subseteq V$;*
(3) *For each $x \in X$ and each filterbase $\mathcal{F}$ which (Λ,α)-converges to x, $f(\mathcal{F})$ converges to $f(x)$.*
(4) *For each $x \in X$ and each net $\{x_s, s \in S\}$ in X which (Λ,α)-converges to x, the net $\{f(x_s), s \in S\}$ of Y converges to $f(x) \in Y$.*

Proof. Obvious. ∎

Definition 3.5. *A function $f : (X,\tau) \to (Y,\sigma)$ is called (Λ,α)-*irresolute *if $f^{-1}(V)$ is a (Λ,α)-open subset of X for every (Λ,α)-open subset V of Y.*

Now we have the following result which its proof is easy and therefore it is left to the reader.

Theorem 3.2. *For a function $f : (X,\tau) \to (Y,\sigma)$, the following statements are equivalent:*

(1) f *is* (Λ,α)*-irresolute;*

(2) *$f^{-1}(B)$ is a (Λ, α)-closed subset of X for every (Λ, α)-closed subset B of Y;*
(3) *For each $x \in X$ and for each (Λ, α)-open set V of Y containing $f(x)$ there exists a (Λ, α)-open set U of X containing x and $f(U) \subseteq V$;*
(4) *$f(A^{(\Lambda,\alpha)}) \subset [f(A)]^{(\Lambda,\alpha)}$ for each subset A of X;*
(5) *$[f^{-1}(B)]^{(\Lambda,\alpha)} \subset f^{-1}(B^{(\Lambda,\alpha)})$ for each subset B of Y;*
(6) *For each $x \in X$ and each filterbase $\mathcal{F}$ which (Λ, α)-converges to x, $f(\mathcal{F})$ (Λ, α)-converges to $f(x)$.*
(7) *For each $x \in X$ and each net $\{x_s, s \in S\}$ in X which (Λ, α)-converges to x, we have that the net $\{f(x_s), s \in S\}$ of Y (Λ, α)-converges to $f(x) \in Y$.*

Proof. Obvious. ∎

Definition 3.6. *A function $f : (X, \tau) \to (Y, \sigma)$ is called* quasi-(Λ, α)-irresolute *if $f^{-1}(V)$ is a (Λ, α)-open subset of X for every α-open subset V of Y.*

Theorem 3.3. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent*

(1) *f is quasi-(Λ, α)-irresolute;*
(2) *For each $x \in X$ and for each α-open set V of Y containing $f(x)$ there exists a (Λ, α)-open set U of X containing x and $f(U) \subseteq V$;*
(3) *For each $x \in X$ and each filterbase $\mathcal{F}$ which (Λ, α)-converges to x $f(\mathcal{F})$ α-converges to $f(x)$ (that is, for each α-open set U containing $f(x)$ there is a member $F_i \in \mathcal{F}$ such that $F_i \subseteq U$)*
(4) *For each $x \in X$ and each net $\{x_s, s \in S\}$ in X which (Λ, α)-converges to x, the net $\{f(x_s), s \in S\}$ of Y α-converges to $f(x) \in Y$ (i.e. for each α-open set U containing $f(x)$ there exists an element $s_0 \in S$ such that $s \geq s_0$ implies $f(x_s) \in U$).*

Proof. Obvious. ∎

Theorem 3.4. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are true*

(1) If the map f is (Λ, α)-irresolute, then the map f is (Λ, α)-continuous and quasi-(Λ, α)-irresolute.

(2) If the map f quasi-(Λ, α)-irresolute, then the map f is (Λ, α)-continuous.

(3) If the map f is α-irresolute., then the map f is quasi-(Λ, α)-irresolute.

(4) If the map f is α-continuous., then the map f is (Λ, α)-continuous.

Proof. Obvious. ∎

Example 3.1. Let (X, τ) be a topological space such that $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, X, \{a\}\}$. We have $\alpha(X, \tau) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $\alpha C(X, \tau) = \{\emptyset, X, \{c\}, \{b\}, \{b, c\}\}$. Also, the family of all Λ^*_α-sets is $\{\emptyset, X, \{c\}, \{b\}, \{b, c\}\}$ and the family of all (Λ, α)-open sets is $\{\emptyset, X, \{a, b\}, \{a, c\}, \{b\}, \{c\}, \{a\}, \{b, c\}\}$.

We consider the function $f : X \to X$ defined by $f(c) = a$ and $f(a) = f(b) = c$. We have

(1) f is (Λ, α)-irresolute, quasi-(Λ, α)-irresolute, and (Λ, α)-continuous,
(2) f is not α-irresolute, since if $x = c$ and $\{a\}$ is the α-open neighbourhood of $f(c) = a$ in X, then for every α-open neighbourhood of c in X we have $f(U) \not\subseteq \{a\}$, and

(3) f is not α-continuous and the proof is similar to that of (2).

Example 3.2. Let (X,τ) be a topological space such that $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, X, \{a,b\}, \{b\}\}$. We have $\alpha(X,\tau) = \{\emptyset, X, \{b\}, \{a,b\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}$ and $\alpha C(X,\tau) = \{\emptyset, X, \{c,d\}, \{a,c,d\}, \{a,d\}, \{a,c\}, \{d\}, \{c\}\}$. Also, the family of all Λ^*_α-sets is $\{\emptyset, X, \{c,d\}, \{a,c,d\}, \{a,d\}, \{a,c\}, \{d\}, \{c\}\}$, and the family of all (Λ,α)-open sets is $\{\emptyset, X, \{a,b\}, \{b\}, \{c\}, \{d\}, \{b,c,d\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{a,d\}, \{a,c\}, \{c,d\}\}$. We consider the function $f : X \to X$ defined as follows: $f(a) = d$, $f(b) = c$, $f(c) = d$ and $f(d) = a$. The following hold

(1) f is not (Λ,α)-irresolute at the point a since if $\{d\}$ is the (Λ,α)-open neighbourhood of $f(a) = d$ in X, then $f(U) \not\subseteq \{d\}$ for every (Λ,α)-open neighbourhood of a in X, and

(2) f is (Λ,α)-continuous.

4. $D(\Lambda,\alpha)$-sets and associated separation axioms

Definition 4.1. *A subset A of a topological space X is called a $D(\Lambda,\alpha)$-set if there are two (Λ,α)-open sets U, V in X such that $U \neq X$ and $A = U - V$.*

Observe that a (Λ,α)-open set $A \neq X$ is $D(\Lambda,\alpha)$-set since $A = U$ and $V = \emptyset$.

Definition 4.2. *A topological space (X,τ) is (Λ,α)-D_0 (resp. (Λ,α)-D_1) if for $x, y \in X$ such that $x \neq y$ there exists a $D(\Lambda,\alpha)$-set of X containing x but not y or (resp. and) a $D(\Lambda,\alpha)$-set containing y but not x.*

Definition 4.3. *A topological space (X,τ) is (Λ,α)-D_2 if for $x, y \in X$ such that $x \neq y$ there exist disjoint $D(\Lambda,\alpha)$-set G_1 and G_2 such that $x \in G_1$ and $y \in G_2$.*

Definition 4.4. *A topological space (X,τ) is called (Λ,α)-T_0 if for any distinct pair of points in X, there is a (Λ,α)-open set containing one of the points but not the other.*

Definition 4.5. *A topological space (X,τ) is called (Λ,α)-T_1 if for any distinct pair of points x and y in X, there is a (Λ,α)-open U in X containing x but not y and a (Λ,α)-open set V in X containing y but not x.*

Definition 4.6. *A topological space (X,τ) is called (Λ,α)-T_2 if for any distinct pair of points x and y in X, there exist (Λ,α)-open sets U and V in X containing x and y, respectively, such that $U \cap V = \emptyset$.*

Remark 4.1. (i) If (X,τ) is (Λ,α)-T_i, then (X,τ) is (Λ,α)-T_{i-1} , $i = 1, 2$. (ii) If (X,τ) is (Λ,α)-T_i , then it is (Λ,α)-D_i, $i = 1, 2$. (iii) If (X,τ) is (Λ,α)-D_i, then (X,τ) is (Λ,α)-D_{i-1},$i = 1, 2$.

Theorem 4.1. *For a topological space (X,τ) the following statements are true*

(1) *(X,τ) is (Λ,α)-D_0 if and only if it is (Λ,α)-T_0.*

(2) *(X,τ) is (Λ,α)-D_1 if and only if it is (Λ,α)-D_2.*

Proof. The sufficiency is stated in Remark 4.1(ii). To prove necessity, let (X,τ) be (Λ,α)-D_0. Then for each distinct pair $x, y \in X$, at least one of x, y, say x, belongs to a $D(\Lambda,\alpha)$-set G but $y \notin G$. Suppose $G = U_1 \setminus U_2$ where $U_1 \neq X$ and U_1,U_2

are (Λ, α)-open sets of (X, τ). Then $x \in U_1$, and for $y \notin G$ we have two cases: (a) $y \notin U_1$; (b) $y \in U_1$ and $y \in U_2$.

In case (a), U_1 contains x but does not contain y; in case (b), U_2 contains y but does not contain x. Hence X is (Λ, α)-T_0.

(2) Sufficiency. Remark 4.1(iii).

Necessity. Let X be (Λ, α)-D_1. Then for each distinct pair $x, y \in X$, we have $D(\Lambda, \alpha)$-sets G_1, G_2 such that $x \in G_1$, $y \notin G_1$; $y \in G_2$, $x \notin G_2$. Let $G_1 = U_1 \backslash U_2$, $G_2 = U_3 \backslash U_4$. By $x \notin G_2$, we have either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. Now we consider these two cases as follows

(1) $x \notin U_3$. From $y \notin G_1$, we obtain the following two subcases

(a) $y \notin U_1$. From $x \in U_1 \backslash U_2$ we have $x \in U_1 \backslash (U_2 \cup U_3)$ and from $y \in U_3 \backslash U_4$ we have $y \in U_3 \backslash (U_1 \cup U_4)$. It is easy to see that $(U_1 \backslash (U_2 \cup U_3)) \cap (U_3 \backslash (U_1 \cup U_4) = \emptyset$.

(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 \backslash U_2$, $y \in U_2$. $(U_1 \backslash U_2) \cap U_2 = \emptyset$.

(2) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 \backslash U_4$, $x \in U_4$. $(U_3 \backslash U_4) \cap U_4 = \emptyset$. Therefore, the space X is (Λ, α)-D_2. ∎

Theorem 4.2. *If (X, τ) is (Λ, α)-D_1, then it is (Λ, α)-T_0.*

Proof. It follows from Remark 4.1 and Theorem 4.1. ∎

Example 4.1. Let (X, τ) be a topological space such that $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$ (see Example 3.1). Clearly, the singletons $\{a\}$, $\{b\}$ and $\{c\}$ are $D(\Lambda, \alpha)$-sets. Now we have

(1) (X, τ) is not T_i, $i = 0, 1, 2$,
(2) (X, τ) is not α-T_i, $i = 1, 2$, but is α-T_0,
(3) (X, τ) is (Λ, α)-T_i, $i = 0, 1, 2$,
(4) (X, τ) is (Λ, α)-D_i, $i = 0, 1, 2$, and
(5) (X, τ) is not α-R_0 and R_0.

Example 4.2. Let (X, τ) be a topological space such that $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a, b\}, \{b\}\}$ (see Example 3.2). The singletons $\{a\}, \{b\}, \{c\}$ and $\{d\}$ are $D(\Lambda, \alpha)$-sets We have

(1) (X, τ) is not T_i, $i = 0, 1, 2$,
(2) (X, τ) is not α-T_i, $i = 1, 2$, but is α-T_0,
(3) (X, τ) is (Λ, α)-T_i , $i = 0, 1, 2$, and
(4) (X, τ) is (Λ, α)-D_i, $i = 0, 1, 2$.

Example 4.3. Let (X, τ) be a topological space such that $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a, b\}\}$. We have $\alpha(X, \tau) = \{\emptyset, X, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $\alpha C(X, \tau) = \{\emptyset, X, \{c, d\}, \{d\}, \{c\}\}$. The family of Λ^*_α-sets is $\{\emptyset, X, \{c, d\}, \{d\}, \{c\}\}$ and the family of (Λ, α)-open sets is $\{\emptyset, X, \{a, b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b\}, \{c, d\}, \{d\}, \{c\}\}$. So, we have

(1) (X, τ) is not T_i, $i = 0, 1, 2$,
(2) (X, τ) is not (α)-T_i, $i = 0, 1, 2$,
(3) (X, τ) is not (Λ, α)-D_i, $= 0, 1, 2$,
(4) (X, τ) is not $(\alpha$-R_0 and $\alpha)$-R_1 and

(5) (X,τ) is sober (α)-R_0 (i.e., $\cap_{x\in X}\operatorname{Cl}_\alpha(\{x\})=\emptyset$).

Theorem 4.3. *A topological space (X,τ) is (Λ,α)-T_0 if and only if for each pair of distinct points x, y of X, $\{x\}^{(\Lambda,\alpha)}\neq\{y\}^{(\Lambda,\alpha)}$.*

Proof. (Sufficiency). Suppose that $x,y\in X$, $x\neq y$ and $\{x\}^{(\Lambda,\alpha)}\neq\{y\}^{(\Lambda,\alpha)}$. Let z be a point of X such that $z\in\{x\}^{(\Lambda,\alpha)}$ but $z\notin\{y\}^{(\Lambda,\alpha)}$. We claim that $x\notin\{y\}^{(\Lambda,\alpha)}$. For, if $x\in\{y\}^{(\Lambda,\alpha)}$ then $\{x\}^{(\Lambda,\alpha)}\subset\{y\}^{(\Lambda,\alpha)}$. This contradicts the fact that $z\notin\{y\}^{(\Lambda,\alpha)}$. Consequently x belongs to the (Λ,α)-open set $[\{y\}^{(\Lambda,\alpha)}]^c$ to which y does not belong.

(Necessity). Let (X,τ) be a (Λ,α)-T_0 space and x, y be any two distinct points of X. There exists a (Λ,α)-open set G containing x or y, say x but not y. Then G^c is a (Λ,α)-closed set which does not contain x but contains y. Since $\{y\}^{(\Lambda,\alpha)}$ is the smallest (Λ,α)-closed set containing y (Lemma 3.1(2)), $\{y\}^{(\Lambda,\alpha)}\subset G^c$, and so $x\notin\{y\}^{(\Lambda,\alpha)}$. Therefore $\{x\}^{(\Lambda,\alpha)}\neq\{y\}^{(\Lambda,\alpha)}$. ∎

Theorem 4.4. *A topological space (X,τ) is (Λ,α)-T_1 if and only if the singletons are (Λ,α)-closed sets.*

Proof. Suppose (X,τ) is (Λ,α)-T_1 and x be any point of X. Let $y\in\{x\}^c$. Then $x\neq y$ and so there exists a (Λ,α)-open set U_y such that $y\in U_y$ but x $\notin U_y$. Consequently $y\in U_y\subset\{x\}^c$ i.e., $\{x\}^c=\bigcup\{U_y/y\in\{x\}^c\}$ which is (Λ,α)-open.

To prove the converse, suppose $\{p\}$ is (Λ,α)-closed for every $p\in X$. Let $x,y\in X$ with $x\neq y$. Now $x\neq y$ implies $y\in\{x\}^c$. Hence $\{x\}^c$ is a (Λ,α)-open set containing y but not containing x. Similarly $\{y\}^c$ is a (Λ,α)-open set containing x but not containing y. This means that X is a (Λ,α)-T_1 space. ∎

We say that a subset A of a topological space X is said to be a (Λ,α)*-neighborhood* of a point $x\in X$ if there exists a (Λ,α)-open set U such that $x\in U\subset A$.

Definition 4.7. *A point $x\in X$ which has X as the (Λ,α)-neighborhood is called $D(\Lambda,\alpha)$-neat point.*

Theorem 4.5. *For a (Λ,α)-T_0 topological space (X,τ) the following are equivalent*
(1) (X,τ) is (Λ,α)-D_1;
(2) (X,τ) has no $D(\Lambda,\alpha)$-neat point.

Proof. (1) ⇒ (2). Since (X,τ) is (Λ,α)-D_1, so each point x of X is contained in a $D(\Lambda,\alpha)$-set $O=U-V$ and thus in U. By definition $U\neq X$. This implies that x is not a $D(\Lambda,\alpha)$-neat point.

(2) ⇒ (1). If X is (Λ,α)-T_0, then for each distinct pair of points $x,y\in X$, at least one of them, x(say) has a (Λ,α)-neighborhood U containing x and not y. Thus U which is different from X is a $D(\Lambda,\alpha)$-set. If X has no $D(\Lambda,\alpha)$-neat point, then y is not a $D(\Lambda,\alpha)$-neat point. This means that there exists a (Λ,α)-neighborhood V of y such that $V\neq X$. Thus $y\in(V-U)$ but not x and $V-U$ is a $D(\Lambda,\alpha)$-set. Hence X is (Λ,α)-D_1. ∎

Remark 4.2. It should be noted that a (Λ,α)-T_0 topological space (X,τ) is not (Λ,α)-D_1 if and only if there is a unique $D(\Lambda,\alpha)$-neat point in X. It is unique because if x and y are both $D(\Lambda,\alpha)$-neat point in X, then at least one of them say x has a (Λ,α)-neighborhood U containing x but not y. But this is a contradiction since $U\neq X$.

Definition 4.8. *A topological space (X, τ) is called a (Λ, α)-symmetric if for x and y in X, $x \in y^{(\Lambda,\alpha)}$ (or $x \in L\alpha\,\mathrm{Cl}(\{y\})$) implies $y \in x^{(\Lambda,\alpha)}$ (or $y \in L\alpha\,\mathrm{Cl}(\{x\})$).*

Theorem 4.6. *A topological space (X, τ) is (Λ, α)-symmetric if and only if for $x \in X$, $x^{(\Lambda,\alpha)} \subseteq E$ whenever $x \in E$ and E is (Λ, α)-open in (X, τ).*

Proof. Assume that $x \in y^{(\Lambda,\alpha)}$ but $y \notin x^{(\Lambda,\alpha)}$. This means that the complement of $x^{(\Lambda,\alpha)}$ contains y. Therefore the set $\{y\}$ is a subset of the complement of $x^{(\Lambda,\alpha)}$. This implies that $y^{(\Lambda,\alpha)}$ is a subset of the complement of $x^{(\Lambda,\alpha)}$. Now the complement of $x^{(\Lambda,\alpha)}$ contains x which is a contradiction.

Conversely, suppose that $\{x\} \subset E$ and E is (Λ, α)-open in (X, τ) but $x^{(\Lambda,\alpha)}$ is not a subset of E. This means that $x^{(\Lambda,\alpha)}$ and the complement of E are not disjoint. Let y belongs to their intersection. Now we have $x \in y^{(\Lambda,\alpha)}$ which is a subset of the complement of E and $x \notin E$. But this is a contradiction. ∎

Theorem 4.7. *For a (Λ, α)-symmetric topological space (X, τ), the following are equivalent*

(1) (X, τ) is (Λ, α)-T_0;
(2) (X, τ) is (Λ, α)-D_0;
(3) (X, τ) is (Λ, α)-D_1.

Proof. (1) $\Leftrightarrow$ (2) : Theorem 4.1.
(3) $\Rightarrow$ (2) : Remark 4.1.
(1) $\Rightarrow$ (3) : Let $x \neq y$ and by (1), we may assume that $x \in E \subset \{y\}^c$ for some E (Λ, α)-open in (X, τ). Then $x \notin y^{(\Lambda,\alpha)}$ and hence $y \notin x^{(\Lambda,\alpha)}$. Thus there exists a (Λ, α)-open set F such that $y \in F \subset \{x\}^c$. Since every (Λ, α)-open set is a $D(\Lambda, \alpha)$-set, we have that (X, τ) is a (Λ, α)-D_1 space. ∎

Theorem 4.8. *If $f : (X, \tau) \to (Y, \sigma)$ is a (Λ, α)-irresolute surjective function and E is a $D(\Lambda, \alpha)$-set in Y, then the inverse image of E is a $D(\Lambda, \alpha)$-set in X.*

Proof. Let E be a $D(\Lambda, \alpha)$-set in Y. Then there are (Λ, α)-open sets U_1 and U_2 in Y such that $S = U_1 - U_2$ and $U_1 \neq Y$. By the (Λ, α)- irresoluteness of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are (Λ, α)-open in X. Since $U_1 \neq Y$, we have $f^{-1}(U_1) \neq X$. Hence $f^{-1}(E) = f^{-1}(U_1) - f^{-1}(U_2)$ is a $D(\Lambda, \alpha)$-set. ∎

Theorem 4.9. *If (Y, σ) is (Λ, α)-D_1 and $f : (X, \tau) \to (Y, \sigma)$ is (Λ, α)-irresolute and bijective, then (X, τ) is (Λ, α)-D_1.*

Proof. Suppose that Y is a (Λ, α)-D_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is (Λ, α)-D_1, there exist $D(\Lambda, \alpha)$-sets G_x and G_y of Y containing $f(x)$ and $f(y)$ respectively, such that $f(y) \notin G_x$ and $f(x) \notin G_y$. By Theorem 4.8, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are $D(\Lambda, \alpha)$-sets in X containing x and y respectively. This implies that X is a (Λ, α)-D_1 space. ∎

Theorem 4.10. *A topological space (X, τ) is (Λ, α)-D_1 if and only if for each pair of distinct points $x, y \in X$, there exists a (Λ, α)-irresolute surjective function $f : (X, \tau) \to (Y, \sigma)$, where Y is a (Λ, α)-D_1 space such that $f(x)$ and $f(y)$ are distinct.*

Proof. (Necessity). For every pair of distinct points of X, it suffices to take the identity function on X. (Sufficiency). Let x and y be any pair of distinct points in

X. By hypothesis, there exists a (Λ, α)-irresolute, surjective function f of a space X onto a (Λ, α)-D_1 space Y such that $f(x) \neq f(y)$. Therefore, there exist disjoint $D(\Lambda, \alpha)$-sets G_x and G_y in Y such that $f(x) \in G_x$ and $f(y) \in G_y$. Since f is (Λ, α)-irresolute and surjective, by Theorem 4.8, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint $D(\Lambda, \alpha)$-sets in X containing x and y, respectively. Hence by Theorem 4.1, X is (Λ, α)-D_1 space. ∎

5. (Λ, α)-compactness and (Λ, α)-connectedness

Definition 5.1. *A topological space (X, τ) is said to be*

(1) (Λ, α)-compact *if every cover of X by (Λ, α)-open sets of (X, τ) has a finite subcover,*

(2) α-compact ([5] *and* [11]) *if every α-open cover of X has a finite subcover.*

Theorem 5.1. *A topological space (X, τ) is (Λ, α)-compact if and only if for every family $\{A_i : i \in I\}$ of (Λ, α)-closed sets in X satisfying $\cap\{A_i : i \in I\} = \emptyset$, there is a finite subfamily $A_{i_1},...,A_{i_n}$ with $\cap\{A_{i_k} : k = 1, ..., n\} = \emptyset$.*

Proof. Obvious. ∎

Theorem 5.2. *For a topological space (X, τ), the following properties hold*

(1) If $(X, \tau^{\Lambda_\alpha})$ is compact, then (X, τ) is α-compact.

(2) If (X, τ) is (Λ, α)-compact, then (X, τ) is α-compact.

(3) If (X, τ) is (Λ, α)-compact, then $(X, \tau^{\Lambda^*_\alpha})$ is compact.

Proof. (1) Let $\{V_\alpha \mid \alpha \in \nabla\}$ be any α-open cover of X. By Lemma 2.2, every α-open V_α is a Λ_α-set for each $\alpha \in \nabla$. Moreover, by the compactness of $(X, \tau^{\Lambda_\alpha})$ there exists a finite subset ∇_0 of ∇ such that $X = \cup\{V_\alpha \mid \alpha \in \nabla_0\}$. This shows that (X, τ) is α-compact.

(2) Let $\{F_\alpha \mid \alpha \in \nabla\}$ be a family of α-closed sets of (X, τ) such that $\cap\{F_\alpha \mid \alpha \in \nabla\} = \emptyset$. Every α-closed is (Λ, α)-closed for each $\alpha \in \nabla$. By Theorem 5.1, there exists a finite subset ∇_0 of ∇ such that $\cap\{F_\alpha \mid \alpha \in \nabla_0\} = \emptyset$. It follows from [[5], Theorem 3.1] that (X, τ) is α-compact.

(3) Let $\{V_\alpha \mid \alpha \in \nabla\}$ be a cover of X by Λ^*_α-sets of (X, τ). Since $V_\alpha = V_\alpha \cup \emptyset$ and the empty set is α-open, by Lemma 2.2 each V_α is (Λ, α)-open in (X, τ). Since (X, τ) is (Λ, α)-compact, there exists a finite subset ∇_0 of ∇ such that $X = \cup\{V_\alpha \mid \alpha \in \nabla_0\}$. This shows that $(X, \tau^{\Lambda^*_\alpha})$ is compact. ∎

Corollary 5.1. *If (X, τ) is (Λ, α)-compact, then (X, τ) is compact.*

Proof. Every α-compact is compact [4]. ∎

The following example shows that the converse of Corollary 5.1 does not hold.

Example 5.1. Let I be an infinite space and let (X, τ) be a topological space such that $X = \{a\} \cup \{a_i : i \in I\}$ and $\tau = \{\emptyset, X, \{a\}\}$. Clearly, the space (X, τ) is compact but is not (Λ, α)-compact.

Theorem 5.3. *If $f : (X, \tau) \to (Y, \sigma)$ is a (Λ, α)-irresolute surjection and (X, τ) is a (Λ, α)-compact space, then (Y, σ) is (Λ, α)-compact.*

Proof. Let $\{V_\alpha \mid \alpha \in \nabla\}$ be any cover of Y by (Λ, α)-open sets of (Y, σ). Since f is (Λ, α)-irresolute, by Theorem 3.1 $\{f^{-1}(V_\alpha) \mid \alpha \in \nabla\}$ is a cover of X by (Λ, α)-open sets of (X, τ). Thus, there exists a finite subset ∇_0 of ∇ such that $X = \cup\{f^{-1}(V_\alpha) \mid \alpha \in \nabla_0\}$. Since f is surjective, we obtain $Y = f(X) = \cup\{V_\alpha \mid \alpha \in \nabla_0\}$. This shows that (Y, σ) is (Λ, α)-compact. ∎

Definition 5.2. *A topological space (X, τ) is called* (Λ, α)-connected *(resp. α-connected) if X cannot be written as a disjoint union of two non-empty (Λ, α)-open sets (resp. α-open sets).*

The proof of the following theorem is straightforward and therefore is omitted.

Theorem 5.4. *Every (Λ, α)-connected space is α-connected space.*

The following example shows that connectedness does not imply (Λ, α)-connected.

Example 5.2. Let (X, τ) be a topological space such that $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$ (see Example 3.1).
We have

(1) (X, τ) is connected and α-connected, and
(2) (X, τ) is not (Λ, α)-connected.

Theorem 5.5. *For a topological space (X, τ), the following statements are equivalent*

(1) *(X, τ) is (Λ, α)-connected;*
(2) *The only subsets of X, which are both (Λ, α)-open and (Λ, α)-closed are the empty set $\emptyset$ and X*

Proof. Straightforward. ∎

Theorem 5.6. *If a topological space (X, τ) is (Λ, α)-connected, then $(X, \tau^{\Lambda_\alpha})$ is connected.*

Proof. Suppose that $(X, \tau^{\Lambda_\alpha})$ is not connected. There exist nonempty Λ_α-sets G, H of (X, τ) such that $G \cap H = \emptyset$ and $G \cup H = X$. By Lemma 2.5, G and H are (Λ, α)-closed sets. This shows that (X, τ) is not (Λ, α)-connected. ∎

Theorem 5.7. *If $f : (X, \tau) \to (Y, \sigma)$ is a (Λ, α)-irresolute surjection and (X, τ) is (Λ, α)-connected, then (Y, σ) is (Λ, α)-connected.*

Proof. Suppose that (Y, σ) is not (Λ, α)-connected. There exist nonempty (Λ, α)-open sets G, H of Y such that $G \cap H = \emptyset$ and $G \cup H = Y$. Then we have $f^{-1}(G) \cap f^{-1}(H) = \emptyset$ and $f^{-1}(G) \cup f^{-1}(H) = X$. Moreover, $f^{-1}(G)$ and $f^{-1}(H)$ are nonempty (Λ, α)-open sets of (X, τ). This shows that (X, τ) is not (Λ, α)-connected. Therefore, (Y, σ) is (Λ, α)-connected. ∎

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