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On 2-Quasi-Umbilical Pseudosymmetric Hypersurfaces in the Euclidean Space

CIHAN ÖZGÜR
Balikesir University, Faculty of Art and Sciences,
Department of Mathematics, 10145 Balikesir, Turkey
cozgur@balikesir.edu.tr

Abstract. In this paper, we investigate 2-quasi-umbilical pseudosymmetric hypersurfaces in the Euclidean space $\mathbb{E}^{n+1}$. We find the curvature characterization of pseudosymmetric hypersurfaces in the Euclidean space $\mathbb{E}^{n+1}$.

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1. Introduction

Let (M,g) be an n-dimensional, $n \geq 3$, connected Riemannian manifold of class C^{∞}. We denote by ∇, R, C, S and κ the Levi-Civita connection, the Riemann-Christoffel curvature tensor, the Weyl conformal curvature tensor, the Ricci tensor and the scalar curvature of (M,g) respectively. The Ricci operator $\mathcal{S}$ is defined by $g(\mathcal{S}X,Y) = S(X,Y)$, where $X, Y \in \chi(M)$, $\chi(M)$ being Lie algebra of vector fields on M. We next define endomorphisms $X \wedge Y$, $\mathcal{R}(X,Y)$ and $\mathcal{C}(X,Y)$ of $\chi(M)$ by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y, \tag{1.1}$$

$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.2}$$

$$\mathcal{C}(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-2}(X \wedge \mathcal{S}Y + \mathcal{S}X \wedge Y - \frac{\kappa}{n-1} X \wedge Y)Z, \tag{1.3}$$

respectively, where $X, Y, Z \in \chi(M)$.

The Riemannian Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M,g) are defined by

$$R(X,Y,Z,W) = g(\mathcal{R}(X,Y)Z, W), \tag{1.4}$$

$$C(X,Y,Z,W) = g(\mathcal{C}(X,Y)Z, W), \tag{1.5}$$

respectively, where $W \in \chi(M)$.

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For a $(0,k)$-tensor field T, $k \geq 1$, on (M,g) we define the tensors $R.T$ and $Q(g,T)$ by

$$(1.6)\quad \begin{aligned}(R(X,Y)\cdot T)(X_1,\ldots,X_k) &= -T(\mathcal{R}(X,Y)X_1,X_2,\ldots,X_k)\\ &\quad -\cdots - T(X_1,\ldots,X_{k-1},\mathcal{R}(X,Y)X_k),\end{aligned}$$

$$(1.7)\quad \begin{aligned}Q(g,T)(X_1,\ldots,X_k;X,Y) &= -T((X\wedge Y)X_1,X_2,\ldots,X_k)\\ &\quad -\cdots - T(X_1,\ldots,X_{k-1},(X\wedge Y)X_k),\end{aligned}$$

respectively.

If the tensors $R\cdot R$ and $Q(g,R)$ are linearly dependent then M is called *pseudosymmetric*. This is equivalent to

$$(1.8)\quad R\cdot R = L_R Q(g,R)$$

holding on the set $U_R = \{x \mid Q(g,R)\neq 0 \text{ at } x\}$, where L_R is some function on U_R. If $R\cdot R = 0$ then M is called *semisymmetric* (see Deszcz [3]).

If the tensors $R\cdot S$ and $Q(g,S)$ are linearly dependent then M is called *Ricci-pseudosymmetric*. This is equivalent to

$$(1.9)\quad R\cdot S = L_S Q(g,S)$$

holding on the set $U_S = \{x \mid S \neq \frac{\kappa}{n} g \text{ at } x\}$, where L_S is some function on U_S. Every pseudosymmetric manifold is Ricci pseudosymmetric but the converse statement is not true. If $R\cdot S = 0$ then M is called *Ricci-semisymmetric* (see Deszcz [3]).

If the tensors $R\cdot C$ and $Q(g,C)$ are linearly dependent then M is called *Weyl-pseudosymmetric*. This is equivalent to

$$(1.10)\quad R\cdot C = L_C Q(g,C)$$

holding on the set $U_C = \{x \mid C\neq 0 \text{ at } x\}$. Every pseudosymmetric manifold is Weyl pseudosymmetric but the converse statement is not true. If $R\cdot C = 0$ then M is called *Weyl-semisymmetric* (see Deszcz [3]).

The manifold M is a *manifold with pseudosymmetric Weyl tensor* if and only if

$$(1.11)\quad C\cdot C = L_C Q(g,C)$$

holds on the set U_C, where L_C is some function on U_C (see Deszcz, Verstraelen, and Yaprak, [4]). The tensor $C\cdot C$ is defined in the same way as the tensor $R\cdot R$.

2. 2-quasi umbilical hypersurfaces

Let M^n be an $n\geq 3$ dimensional connected hypersurface immersed isometrically in the Euclidean space $\mathbb{E}^{n+1}$. We denote by $\widetilde{\nabla}$ and ∇ the Levi-Civita connections corresponding to $\mathbb{E}^{n+1}$ and M^n, respectively. Let ξ be a local unit normal vector field on M^n in $\mathbb{E}^{n+1}$. We can present the Gauss formula and the Weingarten formula of M^n in $\mathbb{E}^{n+1}$ of the form: $\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$ and $\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X\xi$, respectively, where X,Y are vector fields tangent to M^n and D is the normal connection of M^n (see Chen [2]).

For the plane section $e_i\wedge e_j$ of the tangent bundle TM^n spanned by the vectors e_i and e_j $(i\neq j)$ the scalar curvature of M^n is defined by $\kappa = \sum_{i,j=1}^{n} K(e_i\wedge e_j)$ where K denotes the sectional curvature of M^n. We denote by shortly $K_{rs} = K(e_r\wedge e_s)$.

Hypersurface M^n with three distinct principal curvatures, their multiplicities are 1, 1 and $n-2$, is said to be *2-quasi umbilical.* So the shape operator of a 2-quasi-umbilical hypersurface is of the form

$$A_\xi = \begin{bmatrix} a & 0 & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & 0 & \cdots & 0 \\ 0 & 0 & c & 0 & \cdots & 0 \\ 0 & 0 & 0 & c & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c \end{bmatrix}. \tag{2.1}$$

2-quasi-umbilical hypersurfaces are the extended class of quasi-umbilical hypersurfaces. It is well-known that a hypersurface M^n which has a principal curvature with multiplicity $\geq n-1$ is said to be *quasi-umbilical.* The well-known result of E. Cartan gives us "A hypersurface M^n, $n \geq 4$, isometrically in the Euclidean space $\mathbb{E}^{n+1}$, is conformally flat if and only if it is quasi umbilical." In Özgür [8], the present author studied conformally flat submanifolds with flat normal connection.

By (2.1) for a 2-quasi-umbilical hypersurface, one can get easily the following corollaries:

Corollary 2.1. *Let M^n be a 2-quasi-umbilical hypersurface of $\mathbb{E}^{n+1}$, $n \geq 4$, then*

$$\begin{aligned} K_{12} &= ab, & & \\ K_{1j} &= ac, & &(j > 2) \\ K_{2j} &= bc, & &(j > 2) \\ K_{ij} &= c^2, & &(i, j > 2). \end{aligned} \tag{2.2}$$

where $i, j > 2$. Furthermore, $\mathcal{R}(e_i, e_j)e_k = 0$ if i, j and k are mutually different.

Theorem 2.1. [5]. *Any 2-quasi-umbilical hypersurface M^n, $dim M^n \geq 4$, immersed isometrically in a semi-Riemannian conformally flat manifold N is a manifold with pseudosymmetric Weyl tensor.*

On the other hand, it is known that in a hypersurface M^n of a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, if M^n is a Ricci-pseudosymmetric manifold with pseudosymmetric Weyl tensor then it is a pseudosymmetric manifold (see Deszcz, Verstraelen, and Yaprak [4]). Moreover from Arslan, Deszcz, and Yaprak [1], we know that, in a hypersurface M^n of a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, the Weyl pseudosymmetry and the pseudosymmetry conditions are equivalent. So using the previous facts and Theorem 2.1 one can obtain the following corollary.

Corollary 2.2. *In the class of 2-quasi-umbilical hypersurfaces of the Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$, the conditions of the pseudosymmetry, Ricci pseudosymmetry and Weyl pseudosymmetry are equivalent.*

In Özgür and Arslan [9], the present author and K. Arslan studied pseudosymmetry type hypersurfaces in the Euclidean space satisfying Chen's equality. It is known that, a hypersurface satisfying Chen's equality is a special 2-quasi-umbilical hypersurface.

In this study, our aim is to generalize the study Özgür and Arslan [9] and to find the characterization of 2-quasi-umbilical hypersurfaces satisfying pseudosymmetry curvature condition. Since pseudosymmetry, Ricci-pseudosymmetry and Weyl-pseudosymmetry curvature conditions for a 2-quasi-umbilical hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, are equivalent, it is sufficient to investigate only pseudosymmetry curvature condition.

Firstly we have:

Lemma 2.1. *Let M^n be a 2-quasi-umbilical hypersurface of $\mathbb{E}^{n+1}$, $n \geq 4$. Then*

$$(\mathcal{R}(e_1, e_3) \cdot \mathcal{R})(e_2, e_3)e_1 = abc[c - a]e_2, \tag{2.3}$$

$$(\mathcal{R}(e_2, e_3) \cdot \mathcal{R})(e_1, e_3)e_2 = abc[c - b]e_1. \tag{2.4}$$

Proof. Using (1.6) we have

$$\begin{aligned} &(\mathcal{R}(e_1, e_3) \cdot \mathcal{R})(e_2, e_3)e_1 \\ &= \mathcal{R}(e_1, e_3)(\mathcal{R}(e_2, e_3)e_1) - \mathcal{R}(\mathcal{R}(e_1, e_3)e_2, e_3)e_1 \\ &\quad - \mathcal{R}(e_2, \mathcal{R}(e_1, e_3)e_3)e_1 - \mathcal{R}(e_2, e_3)(\mathcal{R}(e_1, e_3)e_1) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &(\mathcal{R}(e_2, e_3) \cdot \mathcal{R})(e_1, e_3)e_2 \\ &= \mathcal{R}(e_2, e_3)(\mathcal{R}(e_1, e_3)e_2) - \mathcal{R}(\mathcal{R}(e_2, e_3)e_1, e_3)e_2 \\ &\quad - \mathcal{R}(e_1, \mathcal{R}(e_2, e_3)e_3)e_2 - \mathcal{R}(e_1, e_3)(\mathcal{R}(e_2, e_3)e_2). \end{aligned} \tag{2.6}$$

Since $\mathcal{R}(e_i, e_j)e_k = (A_\xi e_i \wedge A_\xi e_j)e_k$, using (2.2), one can get

$$\begin{aligned} \mathcal{R}(e_1, e_3)e_1 &= -K_{13}e_3, & \mathcal{R}(e_1, e_3)e_3 &= K_{13}e_1 \\ \mathcal{R}(e_2, e_1)e_1 &= K_{12}e_2, & \mathcal{R}(e_2, e_1)e_2 &= -K_{12}e_1 \\ \mathcal{R}(e_2, e_3)e_2 &= -K_{23}e_3, & \mathcal{R}(e_2, e_3)e_3 &= K_{23}e_2. \end{aligned} \tag{2.7}$$

So substituting (2.7) and (2.2) into (2.5) and (2.6), respectively we obtain (2.3) and (2.4). ∎

Lemma 2.2. *Let M be a 2-quasi-umbilical hypersurface of $\mathbb{E}^{n+1}$, $n \geq 4$. Then*

$$Q(g, \mathcal{R})(e_2, e_3, e_1; e_1, e_3) = b[c - a]e_2, \tag{2.8}$$

$$Q(g, \mathcal{R})(e_1, e_3, e_2; e_2, e_3) = a[c - b]e_1. \tag{2.9}$$

Proof. Using the relation (1.7) we obtain

$$\begin{aligned} &Q(g, \mathcal{R})(e_2, e_3, e_1; e_1, e_3) \\ &= (e_1 \wedge e_3)\mathcal{R}(e_2, e_3)e_1 - \mathcal{R}((e_1 \wedge e_3)e_2, e_3)e_1 \\ &\quad - \mathcal{R}(e_2, (e_1 \wedge e_3)e_3)e_1 - \mathcal{R}(e_2, e_3)((e_1 \wedge e_3)e_1) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &Q(g, \mathcal{R})(e_2, e_3, e_2; e_2, e_3) \\ &= (e_2 \wedge e_3)\mathcal{R}(e_2, e_3)e_2 - \mathcal{R}((e_2 \wedge e_3)e_2, e_3)e_2 \\ &\quad - \mathcal{R}(e_2, (e_2 \wedge e_3)e_3)e_2 - \mathcal{R}(e_2, e_3)((e_2 \wedge e_3)e_2). \end{aligned} \tag{2.11}$$

So substituting (2.7) and (2.2) into (2.10) and (2.11, respectively we obtain (2.8) and (2.9). ∎

Using Lemma 2.1 and Lemma 2.2 we have the following theorem:

Theorem 2.2. *Let M^n be a 2-quasi-umbilical hypersurface of $\mathbb{E}^{n+1}$, $n \geq 4$. Then M^n is proper pseudosymmetric if and only if $a = b$ and $L_R = ac$ holds on M^n.*

Proof. Let M^n be a pseudosymmetric hypersurface in $\mathbb{E}^{n+1}$. Then by definition one can write

$$(\mathcal{R}(e_1, e_3) \cdot \mathcal{R})(e_2, e_3)e_1 = L_R Q(g, \mathcal{R})(e_2, e_3, e_1; e_1, e_3) \tag{2.12}$$

and

$$(\mathcal{R}(e_2, e_3) \cdot \mathcal{R})(e_1, e_3)e_2 = L_R Q(g, \mathcal{R})(e_2, e_3, e_2; e_2, e_3). \tag{2.13}$$

Since M^n is 2-quasi-umbilical then by Lemma 2.1 and Lemma 2.2 the equations (2.12) and (2.13) turn into respectively

$$b(c - a)(L_R - ac) = 0 \tag{2.14}$$

and

$$a(c - b)(L_R - bc) = 0, \tag{2.15}$$

respectively. Extracting the equations (2.14) and (2.15) we get

$$cL_R(b - a) = 0. \tag{2.16}$$

Since M^n is proper pseudosymmetric, it is not semisymmetric. Then the equation (2.16) gives us $b = a$. So the principal curvatures of M^n must be of the form $(a, a, c, ..., c)$, which gives us

$$(\mathcal{R}(e_1, e_3) \cdot \mathcal{R})(e_2, e_3)e_1 = a^2c[c - a]e_2, \tag{2.17}$$

$$Q(g, \mathcal{R})(e_2, e_3, e_1; e_1, e_3) = a[c - a]e_2, \tag{2.18}$$

$$(\mathcal{R}(e_2, e_3) \cdot \mathcal{R})(e_1, e_3)e_2 = a^2c[c - a]e_1, \tag{2.19}$$

and

$$Q(g, \mathcal{R})(e_1, e_3, e_2; e_2, e_3) = a[c - a]e_1. \tag{2.20}$$

So from (2.17)–(2.18) and (2.19)–(2.20) we obtain $L_R = ac$. This completes the proof of the theorem. ∎

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