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A Note on ψ-Operator

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Abstract. Studing ψ-operator closely, we introduce a new type of sets and consider the interrelation of such sets with some generalized open sets already known in literature.

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1. Introduction

An ideal **I** as we know is a nonempty collection of subsets of X closed with respect to finite union and hereditary. For a subset A of X, $A^* = \{\ x \in X\colon\ U \cap A \notin \mathbf{I}$, for every $U \in \tau\,(x)$ where $\tau\,(x)$ is the collection of all nonempty open sets containing $x\ \}$. A^* is a closed subset for any $A \subset X$ [5]. Now theory of ideals gets a new dimension in case it satisfies $\mathbf{I} \cap \tau = \{\emptyset\}$ [2]. Such ideals have been termed as 'codense ideal' by Dontchev, Ganster and Rose in 1999 who have also defined a set $D \subset X$ as **I**-dense if $D^* = X$ [2]. Eventually **I** is codense if and only if $X = X^*$. With the help of $()^*$-operator, another operator called Ψ-operator is defined as $\Psi\,(A) = X - (X - A)^*$ [3]. In this paper we have used the Ψ-operator to define an interesting generalized open sets and study its properties. A topological space with an ideal **I** is denoted by $(X, \tau, \mathbf{I})$.

2. Set operator Ψ

In this section we discuss a few properties of the set operator Ψ. We first prove:

Theorem 2.1. *Let $(X, \tau, \mathbf{I})$ be a topological space, then $U \subset \Psi\,(U)$ for every open set U of (X, τ).*

Proof. We know that $\Psi\,(U) = X - (X - U)^*$. Now $(X - U)^* \subset \mathrm{cl}\,(X - U) = X - U$, since $X - U$ is closed. Therefore $X - (X - U)^* \supset X - (X - U) = U$ implying $U \subset \Psi\,(U)$. ∎

Now we give an example of a set A which is not open but satisfies $A \subset \Psi\,(A)$.

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Example 2.1. Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a, c\}\}, \mathbf{I} = \{\emptyset, \{c\}\}$. Now $\Psi(\{a\}) = X - \{X - \{a\})^* = X - \{b, c\}^* = X - \{b\} = \{a, c\}$. Therefore $\{a\} \subset \Psi(\{a\})$, but $\{a\}$ is not open.

Corollary 2.1. *Let $(X, \tau, \mathbf{I})$ be a space, then* $\operatorname{int} A \subset \Psi(A)$ *for any subset A of X.*

Proof. We know that $\operatorname{int} A$ is open, then, by Theorem 2.1,

$$\operatorname{int} A \subset \Psi(\operatorname{int} A). \tag{2.1}$$

Again $\operatorname{int} A \subset A$, therefore (see [3])

$$\Psi(\operatorname{int} A) \subset \Psi(A). \tag{2.2}$$

From (2.1) and (2.2), $\operatorname{int} A \subset \Psi(A)$. ∎

Our next result on Ψ-operator seems to be interesting.

Theorem 2.2. *Let $(X, \tau, \mathbf{I})$ be a space, where $\mathbf{I}$ is codense. Then for $A \subset X$, $\Psi(A) \subset A^*$.*

Proof. Suppose $\alpha \in \Psi(A)$ but $\alpha \notin A^*$. Then there exists a nonempty neighborhood U_α of α such that $U_\alpha \cap A \in \mathbf{I}$. Since $\alpha \in \Psi(A)$, therefore $\alpha \in \cup\{M \in \tau : M - A \in \mathbf{I}\}$ [3], which implies that there exists $V \in \tau$ such that $\alpha \in V$ and $V - A \in \mathbf{I}$. Now $U_\alpha \cap V$ is a neighborhood of α. Now $U_\alpha \cap V \cap A \in \mathbf{I}$, by heredity. Again $U_\alpha \cap V - A \in \mathbf{I}$, by heredity. Write $U_\alpha \cap V = (U_\alpha \cap V \cap A) \cup (U_\alpha \cap V - A) \in \mathbf{I}$, by finite additivity. Since $U_\alpha \cap V$ is nonempty open, a contradiction to $\mathbf{I}$ being codense. Therefore $\alpha \in A^*$. This implies that $\Psi(A) \subset A^*$. ∎

Corollary 2.2. *Let $(X, \tau, \mathbf{I})$ be a topological space, where $\mathbf{I}$ is codense. Then for $A \subset X$, $\Psi(A) \subset \operatorname{cl} A$.*

Proof. This follows from Theorem 2.2 and the fact that $A^* \subset \operatorname{cl} A$ for any $A \subset X$. ∎

We shall now prove Theorem 2.3. Some of the results in the theorem have been proved by Hamlett and Jankovic [3]. However using Theorem 2.2 and Corollary 2.2, the proofs have become much simpler.

Theorem 2.3. *Let $(X, \tau, \mathbf{I})$ be a topological space and $\mathbf{I}$ be codense. Then*

(i) for any $A \subset X$, $\Psi(A) \subset \operatorname{int} \operatorname{cl} A$.
(ii) for any closed subset A, $\Psi(A) \subset A$.
(iii) for any $A \subset X$, $\operatorname{int} \operatorname{cl} A = \Psi(\operatorname{int} \operatorname{cl} A)$.
(iv) for any regular open subset A, $A = \Psi(A)$.
(v) for any $U \in \tau$, $\Psi(U) \subseteq \operatorname{int} \operatorname{cl} U \subseteq U^*$.
(vi) for $J \in \mathbf{I}$, $\Psi(J) = \emptyset$.

Proof.

(i) From Corollary 2.2 $\Psi(A) \subset \operatorname{cl} A$. Since $\Psi(A)$ is open, then $\Psi(A) \subset \operatorname{int} \operatorname{cl} A$.
(ii) Proof is obvious.
(iii) Now for any set A, $\Psi(\operatorname{int} \operatorname{cl} A) \subset \operatorname{cl} \operatorname{int} \operatorname{cl} A$, by Corollary 2.2. Since $\Psi(\operatorname{int} \operatorname{cl} A)$ is open, $\Psi(\operatorname{int} \operatorname{cl} A) \subset \operatorname{int} \operatorname{cl} \operatorname{int} \operatorname{cl} A$ implying

$$\Psi(\operatorname{int} \operatorname{cl} A) \subset \operatorname{int} \operatorname{cl} A \tag{2.3}$$

Since $\operatorname{int}\operatorname{cl} A$ is open, therefore by Theorem 2.1

$$\operatorname{int}\operatorname{cl} A \subset \Psi(\operatorname{int}\operatorname{cl} A). \tag{2.4}$$

From (2.3) and (2.4), $\operatorname{int}\operatorname{cl} A = \Psi(\operatorname{int}\operatorname{cl} A)$

(iv) If A is regular open, therefore $A = \operatorname{int}\operatorname{cl} A$. Now from (iii), $A = \Psi(A)$.

(v) By Corollary 2.2, $\Psi(U) \subset \operatorname{cl} U$. Since $\Psi(A)$ is open, therefore

$$\Psi(U) \subset \operatorname{int}\operatorname{cl} U \tag{2.5}$$

Here $\mathbf{I}$ is codense and U is open, therefore $U^* = \operatorname{cl} U$ implies that

$$\operatorname{int}\operatorname{cl} U \subset U^* \tag{2.6}$$

From (2.5) and (2.6), $\Psi(U) \subset \operatorname{int}\operatorname{cl} U \subset U^*$.

(vi) Proof is follows from Theorem 2.2. ∎

We now prove Theorem 2.4.

Theorem 2.4. *Let $(X, \tau, \mathbf{I})$ be a topological space. Then for each $x \in X$, $X - \{x\}$ is $\mathbf{I}$ -dense if and only if $\Psi(\{x\}) = \emptyset$.*

Proof. Proof follows from the definition of **I**-dense set, since $\Psi(\{x\}) = \emptyset$ if and only if $(X - \{x\})^* = X$. ∎

3. Ψ - C set

In this section, using Ψ-operator, we discuss a new class of sets which happens to contain the class of all open sets.

Definition 3.1. *Let $(X, \tau, \mathbf{I})$ be a topological space and $A \subset X$, A is said to be a Ψ-C set if $A \subset \operatorname{cl}\Psi(A)$. The collection of all Ψ-C sets in $(X, \tau, \mathbf{I})$ is denoted by $\Psi(X, \tau)$.*

Theorem 3.1. *Let $(X, \tau, \mathbf{I})$ be a topological space. If $A \in \tau$ then $A \in \Psi(X, \tau)$.*

Proof. The proof follows from Theorem 2.1. From Theorem 3.1 it follows that $\tau \subset \Psi(X, \tau)$ holds in a topological space $(X, \tau, \mathbf{I})$. ∎

Now we give an example which shows that the reverse inclusion is not true.

Example 3.1. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, X, \{c, d\}\}, \mathbf{I} = \{\emptyset, \{c\}\}$ denoting $C(\tau)$ the closed sets in (X, τ). Therefore $C(\tau) = \{\emptyset, X, \{a, b\}\}$. Now $\Psi(\{a, d\}) = X - \{b, c\}^* = X - \{a, b\} = \{c, d\}$. Thus $\operatorname{cl}\Psi(\{a, d\}) = X$. Therefore $\{a, d\} \subset \operatorname{cl}\Psi(\{a, d)\}$, but $\{a, d\}$ is not open in τ.

We give an example which shows that any closed set in $(X, \tau, \mathbf{I})$ may not be a Ψ-C set.

Example 3.2. Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}, \mathbf{I} = \{\emptyset, \{a\}\}$. $C(\tau) = \{\emptyset, X, \{a, c\}, \{c\}, \{a\}\}$. Now $\Psi(\{a\}) = X - \{b, c\}^* = X - \{a, b, c\} = \emptyset$. Therefore $\{a\}$ is closed in (X, τ) but $\{a\} \not\subset \operatorname{cl}\Psi(\{a\})$.

Now we prove that the arbitrary union of Ψ-C sets is a Ψ-C .

Theorem 3.2. *Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of nonempty Ψ-C sets in a topological space $(X, \tau, \mathbf{I})$ then $\bigcup_\alpha A_\alpha \in \Psi(X, \tau)$.*

Proof. For each α,

$$A_\alpha \subset \mathrm{cl}\,\Psi\left(A_\alpha\right) \subset \mathrm{cl}\,\Psi\left(\bigcup_{\alpha\in\Delta} A_\alpha\right).$$

This implies that

$$\bigcup_{\alpha} A_\alpha \subset \mathrm{cl}\,\Psi\left(\bigcup_{\alpha} A_\alpha\right).$$

Thus $\bigcup_{\alpha\in\Delta} A_\alpha \in \Psi\left(X,\tau\right)$. ∎

Following example shows that intersection of two Ψ-C sets in $(X,\tau,\mathbf{I})$ may not be a Ψ-C set.

Example 3.3. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, X, \{a\}, \{b,c\}, \{a,b,c\}\}$, $\mathbf{I} = \{\emptyset, \{c\}\}$. $C\left(\tau\right) = \{\emptyset, X, \{b,c,d\}, \{a,d\}, \{d\}\}$. Now $\Psi\left(\{a,d\}\right) = X - \{b,c\}^* = X - \{b,c,d\} = \{a\}$. Therefore $\mathrm{cl}\,\Psi\left(\{a,d\}\right) = \{a,d\}$, implies that $\{a,d\} \subset \mathrm{cl}\,\Psi\left(\{a,d\}\right)$. Again $\Psi\left(\{b,c,d\}\right) = X - \{a\}^* = X - \{a,d\} = \{b,c\}$, implies that $\mathrm{cl}\,\Psi\left(\{b,c,d\}\right) = \{b,c,d\}$. Therefore $\{b,c,d\} \subset \mathrm{cl}\,\Psi\left(\{b,c,d\}\right)$. Now $\{b,c,d\} \cap \{a,d\} = \{d\}$ and $\Psi\left(\{d\}\right) = X - \{a,b,c\}^* = X - \{a,b,c,d\} = \emptyset$. Therefore $\{d\} \not\subset \mathrm{cl}\,\Psi\left(\{d\}\right)$.

Recall that a subset $A \subset X$ is semi-open set if $A \subset \mathrm{cl}\,\mathrm{int}\,A$. The collection of all semi-open sets in a topological space (X,τ) is denoted by $SO\ (X,\tau)$.

Now we give the relation between $SO\ (X,\tau)$ and $\Psi\left(X,\tau\right)$ in (X,τ).

Theorem 3.3. *Let $(X,\tau,\mathbf{I})$ be a topological space, then $SO\ (X,\tau) \subset \Psi\left(X,\tau\right)$.*

Proof. Let $A \in SO\left(X,\tau\right)$, therefore $A \subset \mathrm{cl}\,\mathrm{int}\,A$. We know that $\mathrm{int}\,A \subset \Psi\left(A\right)$ by Corallary 2.1. Therefore $\mathrm{cl}\,\mathrm{int}\,A \subset \mathrm{cl}\,\Psi\left(A\right)$. Thus $A \subset \mathrm{cl}\,\mathrm{int}\,A \subset \mathrm{cl}\,\Psi\left(A\right)$. Hence the theorem. ∎

That the reverse inclusion of the above theorem fails to hold follows from Example 3.1 where $\{a,d\} \in \Psi(X,\tau)$ where as $\{a,d\}$ is not a semi-open set.

Now we recall the definition of a semi-preopen set.

Definition 3.2. [1] *A subset A of X is said to be a semi-preopen set if $A \subset \mathrm{cl}\,\mathrm{int}\,\mathrm{cl}\,A$. The collection of all semi-preopen sets in (X,τ) is denoted by $SPO\ (X,\tau)$.*

Theorem 3.3 and Example 3.4 show that if $\mathbf{I}$ is codense $\Psi\left(X,\tau\right)$ in general is a larger class than the class of semi-open sets in (X,τ). However we shall show that the class of semi-preopen sets forms even a larger class than the class of Ψ-C sets.

Theorem 3.4. *Let A be a Ψ-C set in a topological space $(X,\tau,\mathbf{I})$, where $\mathbf{I}$ is codense. Then $A \in SPO\left(X,\tau\right)$.*

Proof. Proof follows directly from Theorem 2.3(i), since $\Psi\left(A\right) \subset \mathrm{cl}\,A$ implies $A \subset \mathrm{cl}\,\mathrm{int}\,\mathrm{cl}\,A$. ∎

By the above theorem we get $\Psi\left(X,\tau\right) \subset SPO\left(X,\tau\right)$. However the inequality in the other direction fails to hold.

Example 3.4. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a,b\}\}$, $\mathbf{I} = \{\emptyset, \{a\}\}$ and $C\left(\tau\right) = \{\emptyset, X, \{c\}\}$. Now $\Psi\left(\{a\}\right) = X - \{b,c\}^* = X - \{a,b,c\} = \emptyset$. Therefore $\{a\} \not\subset \mathrm{cl}\,\Psi\left(\{a\}\right)$, i.e., $\{a\}$ is not a Ψ-C set. But $\{a\} \subset \mathrm{cl}\,\mathrm{int}\,\mathrm{cl}\,\{a\}$, therefore $\{a\}$ is a semi-preopen set.

Corollary 3.1. *$SO\ (X, \tau) \subset \Psi\,(X, \tau) \subset SPO\,(X, \tau)$, when $\mathbf{I}$ is a codense ideal.*

Proof. The proof follows from Theorem 3.3 and Theorem 3.4. ∎

Recall that Njastad in 1965 defined a set $A \subset X$ to be an α-set if $A \subset \operatorname{int}\operatorname{cl}\operatorname{int} A$ [6]. Denote the collection of all α-sets as τ^{α}.

In Example 3.3 it has been shown that intersection of two Ψ-C sets may not be a Ψ-C set. However we show that the intersection of a Ψ-C set and an α-set is also a Ψ-C set.

Theorem 3.5. *Let $(X, \tau, \mathbf{I})$ be a topological space and $A \in \Psi\,(X, \tau)$. If $U \in \tau^{\alpha}$, then $U \cap A \in \Psi\,(X, \tau)$.*

Proof. First we note that if G is open, for any $A \subset X, G \cap \operatorname{cl} A \subset \operatorname{cl}(G \cap A)$, as well as that $\Psi(A \cap B) = \Psi(A) \cap \Psi(B)$. Hence if $U \in \tau^{\alpha}$ and $A \in \Psi(X, \tau)$ we have therefore $U \cap A \subset \operatorname{int}(\operatorname{cl}(\operatorname{int} U)) \cap \operatorname{cl}\Psi(A) \subset \operatorname{int}(\operatorname{cl}(\Psi(U)) \cap \operatorname{cl}\Psi(A) \subset \operatorname{cl}(\operatorname{int}(\operatorname{cl}\Psi(U)) \cap \Psi(A)) = \operatorname{cl}(\operatorname{int}(\operatorname{cl}(\Psi(U) \cap \Psi(A)))) = \operatorname{cl}(\Psi(U) \cap \Psi(A)) = \operatorname{cl}(\Psi(U \cap A))$ and hence $U \cap A \in \Psi(X, \tau)$. ∎

From Theorem 3.5 we get the following corollary.

Corollary 3.2. *Let $(X, \tau, \mathbf{I})$ be a topological space and $A \in \Psi\,(X, \tau)$. If $U \in \tau$, then $U \cap A \in \Psi\,(X, \tau)$.*

Proof. It follows from the fact that $\tau \subset \tau^{\alpha}$. ∎

It is obvious that if $A \in \mathbf{I}$ is nonempty, where $\mathbf{I}$ is codense, then $A \notin \Psi\,(X, \tau)$. [It follows from (vi) of Theorem 2.3].

However the following example shows that the converse need not hold.

Example 3.5. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$, $\mathbf{I} = \{\emptyset, \{c\}\}$ and $C\,(\tau) = \{\emptyset, X, \{b, c, d\}, \{a, d\}, \{d\}\}$. Now $\Psi\,(\{a, c\}) = X - \{b, d\}^{*} = X - \{b, c, d\} = \{a\}$. Therefore $\operatorname{cl}\Psi\,(\{a, c\}) = \{a, d\}$. Thus $\{a, c\} \notin \Psi\,(X, \tau)$,where as $\{a, c\} \notin \mathbf{I}$. Also recalling that $\Psi\,(A) = X - (X - A)^{*}$, from the definition of $\mathbf{I}$ - dense set it follows that $\Psi\,(A) = \emptyset$ if and only if $(X - A)$ is $\mathbf{I}$ - dense. Therefore for a topological space $(X, \tau, \mathbf{I})$ if $\mathbf{I}$ is codense $A \neq \emptyset$, $A \notin \Psi\,(X, \tau)$ if $A \in \mathbf{I}$ or $(X - A)$ is $\mathbf{I}$-dense.

Calling a set D to be relatively $\mathbf{I}$-dense in a set A if for every relatively nonempty open set $U \cap A$, $U \in \tau$, it is true that $(U \cap A) \cap D \notin \mathbf{I}$. We now prove Theorm 3.6 giving a necessary and sufficient condition for $A \notin \Psi\,(X, \tau)$.

Theorem 3.6. *A set A does not belong to $\Psi\,(X, \tau)$ if and only if there exists $x \in A$ such that there is a neighborhood V_x of x for which $X - A$ is relatively $\mathbf{I}$ -dense in V_x.*

Proof. Let $A \notin \Psi\,(X, \tau)$. We are to show that there exists an element $x \in A$ and a neighborhood V_x of x satisfying that $(X - A)$ is relatively $\mathbf{I}$-dense in V_x. Since $A \not\subset \operatorname{cl}\Psi\,(A)$, there exists $x \in X$ such that $x \in A$ but $x \notin \operatorname{cl}\Psi\,(A)$. Hence there exists a neighborhood V_x of x such that $V_x \cap \Psi\,(A) = \emptyset$. This implies that $V_x \cap \left(X - (X - A)^{*}\right) = \emptyset$, therefore $V_x \subset (X - A)^{*}$. Let U be any nonempty open set in V_x. Since $V_x \subset (X - A)^{*}$, therefore $U \cap (X - A) \notin \mathbf{I}$. This implies that $(X - A)$ is relatively $\mathbf{I}$-dense in V_x.

Converse part follows by reversing the argument. ∎

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