BULLETIN *of the*
Malaysian Mathematical
Sciences Society
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **30***(1) (2007), 49–56*

Coalescence of Difans and Diwheels

Diego Bravo and Juan Rada
Departamento de Matemáticas, Facultad de Ciencias
Universidad de Los Andes, 5101 Mérida, Venezuela
dbravo@ula.ve, juanrada@ula.ve

Abstract. A directed graph G is nonderogatory if its adjacency matrix A is nonderogatory, i.e., the characteristic polynomial of A is equal to the minimal polynomial of A. We analyze the problem whether the coalescence of difans and diwheels is nonderogatory. Also, a formula for the characteristic polynomial of the coalescence of two directed graphs is presented.

2000 Mathematics Subject Classification: 05C50

Key words and phrases: Nonderogatory digraphs, coalescence of diagraph.

1. Introduction

A digraph (directed graph) $G = (V, E)$ is defined to be a finite set V and a set E of ordered pairs of elements of V. The sets V and E are called the set of vertices and arcs, respectively. If $(u, v) \in E$ then u and v are adjacent and (u, v) is an arc starting at vertex u and terminating at vertex v.

Suppose that $\{u_1, \ldots, u_n\}$ is the set of vertices of G. The adjacency matrix of G is denoted by A_G, and it is defined as the square matrix of order n whose entry ij is the number of arcs starting at u_i and terminating at u_j. The characteristic polynomial of the digraph G is denoted by $\Phi_G(x)$ (or simply Φ_G) and it is defined as the characteristic polynomial of the adjacency matrix A_G, i.e., $\Phi_G(x) = |xI - A_G|$, where I is the identity matrix and $|M|$ denotes the determinant of M.

By the Cayley-Hamilton theorem, Φ_G is an annihilating polynomial of A_G, which means that $\Phi_G(A_G) = 0$. The monic polynomial of least degree which annihilates A_G is called the minimal polynomial of G and will be denoted by $\mu_G(x) = \mu_G$. Recall that if

$$\Phi_G(x) = (x - \lambda_1)^{q_1} (x - \lambda_2)^{q_2} \cdots (x - \lambda_r)^{q_r}$$

where $q_1, \ldots, q_r$ are positive integers then

$$\mu_G(x) = (x - \lambda_1)^{p_1} (x - \lambda_2)^{p_2} \cdots (x - \lambda_r)^{p_r}$$

where $1 \le p_i \le q_i$ for all $i = 1, \ldots, r$. If $\Phi_G(x) = \mu_G(x)$ then G is a nonderogatory digraph. Otherwise, G is derogatory.

Received: December 19, 2005; *Accepted:* June 6, 2006.

For example, dipaths P_n, dicycles C_n and windmills $M_h(r)$ (with $r = 2$) are classes of nonderogatory digraphs [1,2]. Other examples are the difans F_n and the diwheels W_n, which were considered by Lam and Lim [4,5]. In particular, they treated the problem whether the complete product of nonderogatory digraphs is nonderogatory. More recently, Gan [5] showed that the complete product of difans and diwheels is nonderogatory. Motivated by these results we consider a well known operation between digraphs, the so-called coalescence or amalgamation between digraphs, and analyze the problem whether the coalescence of difans and diwheels is nonderogatory.

Although the characteristic polynomial of the coalescence of difans and diwheels can be calculated directly using the Coefficient Theorem for digraphs ([6],Theorem 1.2), we present in Section 2 a general formula for the characteristic polynomial of the coalescence of two digraphs.

2. Coalescence of digraphs

If G is a digraph, we denote the set of vertices of G by V_G and the set of arcs by E_G. Given $u \in V_G$, the digraph $G - u$ is the graph obtained from G by deleting the vertex u together with every arc which connects to u.

The definition of coalescense of non-directed graphs [6] can be extended to digraphs as follows.

Definition 2.1. *Let G and H be two digraphs such that $u \in V_G$ and $v \in V_H$. The coalescence of the digraphs G and H with respect to the vertices u, v, denoted by $G \cdot H$, is the graph obtained from G and H by merging the vertices u and v as w. In other words,*

$$V_{G \cdot H} = V_{G-u} \cup V_{H-v} \cup \{w\}$$

and two vertices in $G \cdot H$ are adjacent if they are adjacent in G or H, or if one is w and the other is adjacent to u or v in G or H.

The formula to calculate the characteristic polynomial of the coalescence of two non-directed graphs [1] also holds for digraphs, as we can see in Theorem 2.1. First we need a technical result about the determinant of a block matrix.

Lemma 2.1. *Let $A \in \mathcal{M}_p(\mathbb{C})$ and $B \in \mathcal{M}_q(\mathbb{C})$. Consider the block matrix*

$$D = \begin{pmatrix} A & y & 0 \\ \widetilde{y} & r & \widetilde{w} \\ 0 & w & B \end{pmatrix}$$

where y is a column vector of $\mathbb{C}^p$, $\widetilde{y}$ is a row vector of $\mathbb{C}^p$, w is a column vector of $\mathbb{C}^q$, $\widetilde{w}$ is a row vector of $\mathbb{C}^q$ and $r \in \mathbb{C}$. If

$$\widetilde{A} = \begin{pmatrix} A & y \\ \widetilde{y} & r \end{pmatrix} \text{ and } \widetilde{B} = \begin{pmatrix} r & \widetilde{w} \\ w & B \end{pmatrix},$$

then

$$|D| = \left|\widetilde{A}\right| |B| + \left|\widetilde{B}\right| |A| - r\,|A|\,|B|.$$

Proof. We assume that $D = (d_{ij})$. By the Laplace expansion by minors along the first row

$$|D| = \sum_{j=1}^{p+1} (-1)^{1+j} d_{1j} M_{1j}$$

where, for each $1 \leq j \leq p$,

$$M_{1j} = \begin{vmatrix} A_j^* & y^* & 0^* \\ \widetilde{y}_j & r & \widetilde{w} \\ 0_j & w & B \end{vmatrix} \text{ and } M_{1,p+1} = \begin{vmatrix} A^* & 0^* \\ \widetilde{y} & \widetilde{w} \\ 0 & B \end{vmatrix} = \begin{vmatrix} A^* \\ \widetilde{y} \end{vmatrix} |B|$$

where $*$ means that we deleted the first row and the subscript j means that we deleted the j-th column. An induction argument implies that

$$M_{1j} = \begin{vmatrix} A_j^* & y^* \\ \widetilde{y}_j & r \end{vmatrix} \left| \ B \ \right| + \left|\widetilde{B}\right| \left| \ A_j^* \ \right| - r \left| \ A_j^* \ \right| \left| \ B. \ \right|$$

The result follows from the facts that

$$\left|\widetilde{A}\right| = \sum_{j=1}^{p} (-1)^{1+j} d_{1j} \begin{vmatrix} A_j^* & y^* \\ \widetilde{y}_j & r \end{vmatrix} + (-1)^{2+p} d_{1,p+1} \begin{vmatrix} A^* \\ \widetilde{y} \end{vmatrix}$$

and

$$|A| = \sum_{j=1}^{p} (-1)^{1+j} d_{1j} \left|A_j^*\right|. \qquad \blacksquare$$

Now we can calculate the characteristic polynomial of the coalescence $G \cdot H$ of digraphs G and H.

Theorem 2.1. *Let G and H be two digraphs such that $u \in V_G$ and $v \in V_H$. If $G \cdot H$ is the coalescence of the digraphs G and H with respect to the vertices u, v then*

$$\Phi_{G \cdot H} = \Phi_G \Phi_{H-v} + \Phi_H \Phi_{G-u} - x \Phi_{G-u} \Phi_{H-v}.$$

Proof. We order the vertices of $G \cdot H$ as

$$\{u_1, u_2, \ldots, u_p = u = v = v_p, v_{p+1}, \ldots, v_{p+q-1}\}$$

where $V_G = \{u_1, \ldots, u_p = u\}$ and $V_H = \{v = v_p, v_{p+1}, \ldots, v_{p+q-1}\}$. Let $\mathbf{0}_{j,k}$ be the $j \times k$ matrix with 0 in all entries. Then the adjacency matrix of $G \cdot H$ with respect to u, v is of the form

$$A_{G \cdot H} = \begin{pmatrix} A_{G-u} & y & \mathbf{0}_{p-1,q-1} \\ \widetilde{y} & 0 & \widetilde{w} \\ \mathbf{0}_{q-1,p-1} & w & A_{H-v} \end{pmatrix}$$

where

$$A_G = \begin{pmatrix} A_{G-u} & y \\ \widetilde{y} & 0 \end{pmatrix} \text{ and } A_H = \begin{pmatrix} 0 & \widetilde{w} \\ w & A_{H-v} \end{pmatrix}$$

are the adjacency matrices of G and H, respectively. Note that $y \in \mathbb{C}^{p-1}$ is a column vector whose *j-th* coordinate is 1 if there exists an arc from u_j to u and 0 otherwise. On the other hand, $\widetilde{y} \in \mathbb{C}^{p-1}$ is a row vector whose *j-th* coordinate is 1 if there exists an arc from u to u_j and 0 otherwise. Similarly for $w, \widetilde{w} \in \mathbb{C}^{q-1}$. It follows from Lemma 2.1 that

$$\begin{aligned} |xI - A_{G \cdot H}| &= \begin{vmatrix} xI - A_{G-u} & -y & 0 \\ -\widetilde{y} & x & -\widetilde{w} \\ 0 & -w & xI - A_{H-v} \end{vmatrix} \\ &= |xI - A_G|\,|xI - A_{H-v}| + |xI - A_H|\,|xI - A_{G-u}| \\ &\quad - x\,|xI - A_{G-u}|\,|xI - A_{H-v}| \\ &= \Phi_G \Phi_{H-v} + \Phi_H \Phi_{G-u} - x \Phi_{G-u} \Phi_{H-v}. \end{aligned}$$ ∎

3. Coalescence of difans and diwheels

In this section we consider the problem whether the coalescence of difans and diwheels is nonderogatory. Recall the following definitions:

- A dipath P_n is a digraph with vertex set $\{v_1, \ldots, v_n\}$ and arcs (v_i, v_{i+1}) for $i = 1, \ldots, n-1$.
- A dicycle C_n is a digraph with vertex set $\{v_1, \ldots, v_n\}$ having arcs (v_i, v_{i+1}) for $i = 1, \ldots, n-1$, and (v_n, v_1).
- A difan F_n is a digraph consisting of a dipath P_{n-1} of $n-1$ vertices labelled $1, 2, \ldots, n-1$, and an additional vertex n, where there is an arc from n to each of the other vertices of P_{n-1}. The vertex n is called the hub of the difan whereas the arc (n, k) from the hub to the vertex k of P_{n-1} is called a spoke. The vertices with labelling $1, \ldots, n-1$ are called the rim vertices of the difan.
- The diwheel W_n consists of a directed cycle C_{n-1} with additional arcs from an additional vertex n to each of the vertices in C_{n-1}. Vertex n is called the hub of the diwheel whereas the arc (n, k) from the hub to the vertex k of C_{n-1} is called a spoke. The vertices with labelling $1, \ldots, n-1$ are called the rim vertices of the diwheel.

The characteristic polynomials of these digraphs are well known:

$$\begin{aligned} \Phi_{P_n}(x) &= x^n, \\ \Phi_{C_n}(x) &= x^n - 1, \\ \Phi_{F_n}(x) &= x^n, \\ \Phi_{W_n}(x) &= x^n - x. \end{aligned}$$

We will strongly rely on the following well known result from algebraic graph theory.

Theorem 3.1. [6, Theorem 1.9] *Let A_G be the adjacency matrix of the digraph G with set of vertices $\{u_1, \ldots, u_n\}$. If $a_{ij}^{(k)}$ denotes the entry ij of the power matrix $(A_G)^k$, then $a_{ij}^{(k)}$ is the number of diwalks of length k starting at vertex u_i and terminating at u_j.*

Recall that a diwalk of length k in G is a sequence of vertices $v_0 v_1 \cdots v_k$ in which each (v_{i-1}, v_i) is an arc of G.

Theorem 3.2. *The coalescence $F_r \cdot W_s$ with respect to the hub u of the difan and the hub v of the diwheel is nonderogatory (see Figure 1).*

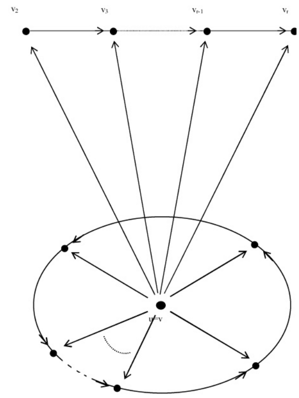

Figure 1. Coalescence of F_r and W_s

Proof. First we note that $F_r - u = P_{r-1}$ and $W_s - v = C_{s-1}$. It follows from Theorem 2.1 that

$$\begin{aligned}\Phi_{F_r \cdot W_s} &= \Phi_{F_r}\Phi_{C_{s-1}} + \Phi_{W_s}\Phi_{P_{r-1}} - x\Phi_{C_{s-1}}\Phi_{P_{r-1}} \\ &= x^r\left(x^{s-1}-1\right) + \left(x^s - x\right)x^{r-1} - x\left(x^{s-1}-1\right)x^{r-1} \\ &= x^r\left(x^{s-1}-1\right).\end{aligned}$$

Hence the minimal polynomial is of the form

$$\mu_{F_r \cdot W_s} = x^k\left(x^{s-1}-1\right)$$

for some $1 \le k \le r$. We will show that $A^{r-1}\left(A^{s-1}-I\right) \neq 0$ or equivalently, $A^{r+s-2} \neq A^{r-1}$. To see this note that there exists a unique diwalk

$$u \to v_2 \to v_3 \to \cdots \to v_{r-1} \to v_r$$

of length $r-1$ in $F_r \cdot W_s$ from the vertex $u = v$ to v_r (see Fig. 1). Therefore, $a_{uv_r}^{(r-1)} = 1$. On the other hand, it is clear that there are no diwalks from u to v_r of length $r+s-2$. Hence $a_{uv_r}^{(r+s-2)} = 0$. Consequently, $\mu_{F_r \cdot W_s} = \Phi_{F_r \cdot W_s}$ and $F_r \cdot W_s$ is nonderogatory. ∎

Theorem 3.3. *The coalescence of F_r and F_s with respect to the hub u of F_r and the hub v of F_s is derogatory (see Fig. 2).*

Proof. The characteristic polynomial of $F_r \cdot F_s$ is given by

$$\begin{aligned}\Phi_{F_r \cdot F_s} &= \Phi_{F_r}\Phi_{P_{s-1}} + \Phi_{F_s}\Phi_{P_{r-1}} - x\Phi_{P_{s-1}}\Phi_{P_{r-1}} \\ &= x^r x^{s-1} + x^s x^{r-1} - x x^{s-1} x^{r-1} = x^{r+s-1}.\end{aligned}$$

Since the longest diwalk in $F_r \cdot F_s$ has length $\gamma = \max\{r-1, s-1\}$ then $A^\gamma \neq 0$ and $A^{\gamma+1} = 0$. Consequently, $\mu_{F_r \cdot F_s} = x^{\gamma+1} \neq x^{r+s-1} = \Phi_{F_r \cdot F_s}$ and $F_r \cdot F_s$ is derogatory. ∎

Theorem 3.4. *The coalescence of W_r and W_s with respect to the hub vertices u and v of W_r and W_s, respectively, is derogatory (see Fig. 3).*

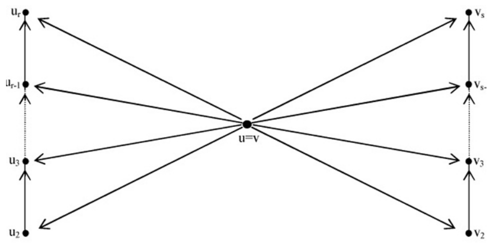

Figure 2. Coalescence of F_r and F_s

Proof. The characteristic polynomial of $W_r \cdot W_s$ is given by

$$\begin{aligned}
\Phi_{W_r \cdot W_s} &= \Phi_{W_r}\Phi_{C_{s-1}} + \Phi_{W_s}\Phi_{C_{r-1}} - x\Phi_{C_{s-1}}\Phi_{C_{r-1}} \\
&= \left(x^r - x\right)\left(x^{s-1} - 1\right) + \left(x^s - x\right)\left(x^{r-1} - 1\right) - x\left(x^{s-1} - 1\right)\left(x^{r-1} - 1\right) \\
&= x\left(x^{r-1} - 1\right)\left(x^{s-1} - 1\right).
\end{aligned}$$

Let us introduce some notations: denote by X and Y the adjacency matrices of the cycles C_{r-1} and C_{s-1}, respectively. Let J_k denote the $1 \times k$ matrix with 1 in all entries. Then the adjacency matrix A of $W_r \cdot W_s$ is given by the block matrix

$$A = \begin{pmatrix} 0 & J_{r-1} & J_{s-1} \\ \mathbf{0}_{r-1,1} & X & \mathbf{0}_{r-1,s-1} \\ \mathbf{0}_{s-1,1} & \mathbf{0}_{s-1,r-1} & Y \end{pmatrix}.$$

Since $J_{r-1}X = J_{r-1}$ and $J_{s-1}Y = J_{s-1}$, it can be easily shown using induction that for every integer $k \geq 1$

$$A^k = \begin{pmatrix} 0 & J_{r-1} & J_{s-1} \\ \mathbf{0}_{r-1,1} & X^k & \mathbf{0}_{r-1,s-1} \\ \mathbf{0}_{s-1,1} & \mathbf{0}_{s-1,r-1} & Y^k \end{pmatrix}. \tag{3.1}$$

Consider two cases:

(a) $r = s$. In this case $\Phi_{W_r \cdot W_r} = x\left(x^{r-1} - 1\right)^2$. Let $f(x) = x\left(x^{r-1} - 1\right)$. Since $X^{r-1} = I$ it follows that

$$A^r = A^{r-1}A = \begin{pmatrix} 0 & J_{r-1} & J_{s-1} \\ \mathbf{0}_{r-1,1} & I & \mathbf{0}_{r-1,s-1} \\ \mathbf{0}_{s-1,1} & \mathbf{0}_{s-1,r-1} & I \end{pmatrix} A = A$$

and then

$$f(A) = A\left(A^{r-1} - I\right) = A^r - A = 0$$

which implies that $\mu_{W_r \cdot W_r} = x\left(x^{r-1} - 1\right)$.

(b) $r \neq s$. Assume that $r < s$. Clearly,

$$\Phi_{W_r \cdot W_s} = x\left(x^{r-1} - 1\right)(x - 1)\left(1 + x + \cdots + x^{s-2}\right).$$

Let $g(x) = x\left(x^{r-1} - 1\right)\left(1 + x + \cdots + x^{s-2}\right)$. Since $Y^{s-1} = I$ it follows that

$$Y^{r-1}\sum_{k=1}^{s-1} Y^k = Y^r + Y^{r+1} + \cdots + Y^{s-1} + Y + Y^2 + \cdots + Y^{r-1} = \sum_{k=1}^{s-1} Y^k.$$

Consequently

$$
\begin{aligned}
&A^{r-1}\sum_{k=1}^{s-1}A^k\\
&=\begin{pmatrix} 0 & J_{r-1} & J_{s-1}\\ \mathbf{0}_{r-1,1} & I & \mathbf{0}_{r-1,s-1}\\ \mathbf{0}_{s-1,1} & \mathbf{0}_{s-1,r-1} & Y^{r-1}\end{pmatrix}\begin{pmatrix} 0 & (s-1)\,J_{r-1} & (s-1)\,J_{s-1}\\ \mathbf{0}_{r-1,1} & \sum\limits_{k=1}^{s-1}X^k & \mathbf{0}_{r-1,s-1}\\ \mathbf{0}_{s-1,1} & \mathbf{0}_{s-1,r-1} & \sum\limits_{k=1}^{s-1}Y^k\end{pmatrix}\\
&=\begin{pmatrix} 0 & J_{r-1}\sum\limits_{k=1}^{s-1}X^k & J_{s-1}\sum\limits_{k=1}^{s-1}Y^k\\ \mathbf{0}_{r-1,1} & \sum\limits_{k=1}^{s-1}X^k & \mathbf{0}_{r-1,s-1}\\ \mathbf{0}_{s-1,1} & \mathbf{0}_{s-1,r-1} & Y^{r-1}\sum\limits_{k=1}^{s-1}Y^k\end{pmatrix}\\
&=\begin{pmatrix} 0 & (s-1)\,J_{r-1} & (s-1)\,J_{s-1}\\ \mathbf{0}_{r-1,1} & \sum\limits_{k=1}^{s-1}X^k & \mathbf{0}_{r-1,s-1}\\ \mathbf{0}_{s-1,1} & \mathbf{0}_{s-1,r-1} & \sum\limits_{k=1}^{s-1}Y^k\end{pmatrix}\\
&=\sum_{k=1}^{s-1}A^k.
\end{aligned}
$$

Hence

$$g\left(A\right)=\left(A^{r-1}-I\right)\left(\sum_{k=1}^{s-1}A^k\right)=A^{r-1}\left(\sum_{k=1}^{s-1}A^k\right)-\sum_{k=1}^{s-1}A^k=0$$

and so $\mu_{W_r\cdot W_s}\neq\Phi_{W_r\cdot W_s}$. In both cases, $W_r\cdot W_s$ is derogatory. ∎

Remark 3.1. In the proof of Theorem 3.4, it was shown that $\mu_{W_r\cdot W_s}\neq\Phi_{W_r\cdot W_s}$ in the case $r\neq s$. Actually, we can determine the minimal polynomial of $W_r\cdot W_s$ in this case. If $d=gcd\left(r-1,s-1\right)$ then $\Phi_{W_r\cdot W_s}$ has exactly d repeated roots. In particular, if $d=1$, then

$$\mu_{W_r\cdot W_s}=g\left(x\right)=x\left(x^{r-1}-1\right)\left(1+x+\cdots+x^{s-2}\right)$$

Suppose that $d\geq 2$. Clearly $\Phi_{W_r\cdot W_s}$ has exactly $r+s-1-d$ distinct roots. Consider $(r+s-1-d)$-degree polynomial

$$h\left(x\right)=-\sum_{i=0}^{\frac{r-1}{d}-1}x^{1+id}+\sum_{j=0}^{\frac{r-1}{d}-1}x^{r+s-1-d(1+j)}.$$

It is not difficult to show that for each $0\leq i\leq\frac{r-1}{d}-1$ there exists a unique $0\leq j\leq\frac{r-1}{d}-1$ such that

$$X^{1+id}=X^{r+s-1-d(1+j)}.$$

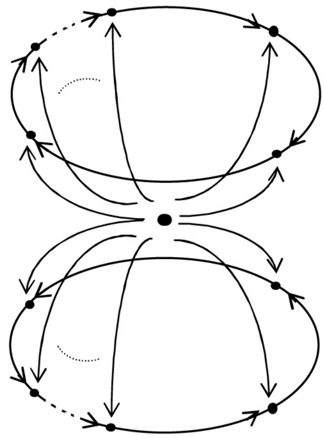

Figure 3. Coalescence of W_f and W_s

Similarly for Y. Consequently, $h(X) = h(Y) = 0$ which implies from relation (3.1), that $h(A) = 0$. Hence $h(x) = \mu_{W_r \cdot W_s}$.

For example,

$$\Phi_{W_9 \cdot W_{13}} = x\left(x^8 - 1\right)\left(x^{12} - 1\right)$$

and

$$\mu_{W_9 \cdot W_{13}} = -x - x^5 + x^{13} + x^{17}.$$

Since $r = 9$ and $s = 13$, $X^8 = I$ and $Y^{12} = I$, where X and Y are the adjacency matrices of C_8 and C_{12}, respectively. Note that

$$X^{13} = X^5, \quad X^{17} = X \quad \text{and} \quad Y^{13} = Y, \quad Y^{17} = Y^5.$$

Acknowledgement. Financial support was received from Consejo de Desarrollo Científico, Humanístico y Tecnológico de la Universidad de Los Andes (CDCHT, C-13490505B).

References

[1] C. S. Gan and V. C. Koo, On annihilating uniqueness of directed windmills, *Proceedings of the ATCM (ATCM 2002)*, Melaka, Malaysia.

[2] J. Rada, Nonderogatory directed windmills, submitted.

[3] K. S. Lam, *On digraphs with unique annihilating polynomial,* PhD Thesis, University of Malaya, Kuala Lumpur, 1990.

[4] K. S. Lam and C. K. Lim, The characteristic polynomial of ladder digraph and an annihilating uniqueness theorem, *Discrete Mathematics 151,* (1996,) 161–167.

[5] C. S. Gan, The complete product of annihilatingly unique digraphs, *Int. J. Math. Math. Sci.*, (2005), 1327–1331.

[6] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of graphs.*, Academic Press, New York, 1980.