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The Symmetric Group of Degree Six can be Covered by 13 and no Fewer Proper Subgroups

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Abstract. In this note, we prove that the symmetric group of degree six can be covered by 13 and no fewer proper subgroups. This partially answers a question of M. J. Tomkinson [Groups as the union of proper subgroups, *Math. Scand.* **81**(1997), 189–198] and gives a negative answer to a part of a conjecture of L. Serena [On finite covers of groups by subgroups, *Advances in group theory* **2002**, 173–190, Aracne, Rome, 2003].

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1. Introduction

Cohn [2] studied the function σ from the set of all groups G to the set $\mathbb{N} \cup \{\infty\}$, where $\sigma(G)$ is defined to be the least number n (if it exists) of proper subgroups of G whose union is equal to G, and $\sigma(G) = \infty$ whenever there is no such an integer n. It is an easy exercise that there is no group G with $\sigma(G) = 2$. In Cohn [2], all groups G with $\sigma(G) \in \{3, 4, 5\}$ are characterized and it was conjectured that there is no group G with $\sigma(G) = 7$. The latter conjecture has been confirmed by Tomkinson [5]. In the same paper, Tomkinson suggested that there does not exist a group G with $\sigma(G) = 11, 13$, or 15. Bryce, Fedri, and Serena [1] showed that $\sigma\big((\mathrm{P})\mathrm{SL}(2,7)\big) = 15$ (we must note that in their paper they attribute this result to Joseph Shieh). Later Serena [4, Conjecture 1] conjectured that there is no group G with $\sigma(G) = 11$ or 13. In this note, we prove that for the symmetric group S_6 of degree 6, we have $\sigma(S_6) = 13$.

2. The symmetric group S_6

We prove the following theorem.

Theorem. *For the symmetric group S_6 of degree 6, we have $\sigma(S_6) = 13$; and S_6 has only one cover with 13 proper subgroups, which consists of all twelve maximal*

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subgroups of S_6 isomorphic to the symmetric group S_5 of degree 5 and the alternating group A_6 of degree 6.

Proof. The symmetric group S_6 of degree 6 has 12 maximal subgroups $B_1, \ldots, B_{12}$ isomorphic to S_5. It is easy to see (e.g. by GAP [3]) that $S_6 = \big(\cup_{i=1}^{12} B_i\big) \cup A_6$. It follows that $\sigma(S_6) \leq 13$. Put $k = \sigma(S_6) \leq 13$, and suppose that $S_6 = \cup_{i=1}^{k} M_i$, where $M_1, \ldots, M_k$ are proper subgroups of S_6. Thus $B_j = \bigcup_{i=1}^{k} \big(M_i \cap B_j\big)$ for all $j \in \{1, \ldots, 12\}$. By [2, Lemma 7], we have $\sigma(B_j) = 16$ (for all $j \in \{1, \ldots, 12\}$); and it follows that $B_j = M_i$ for some $i \in \{1, \ldots, k\}$. On the other hand we have that $\sigma(A_6) = 16$ (see Bryce, Fedri and Serena [1, p. 238]), so that, by a similar argument, $M_\ell = A_6$ for some $\ell \in \{1, \ldots, k\}$. Therefore $\{M_1, \ldots, M_k\} = \{B_1, \ldots, B_{12}, A_6\}$ and $\sigma(S_6) = k = 13$. This completes the proof. ∎

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