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Fuzzy Prime Ideals in Γ-rings

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Abstract. In this paper, we study fuzzy prime ideal of a Γ-ring via its operator rings. We obtain some characterisations of fuzzy prime ideal of a Γ-ring.

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1. Introduction

The notion of fuzzy ideal in Γ-ring was introduced by Jun and Lee [10]. They studied some preliminary properties of fuzzy ideals of Γ-rings. Later Hong and Jun [11] defined normalised fuzzy ideal and fuzzy maximal ideal in Γ-ring and studied them. Dutta and Chanda [2], studied the structures of the set of fuzzy ideals of a Γ-ring and characterise Γ-field, Noetherian Γ-ring, etc. with the help of fuzzy ideals via operator rings of Γ-ring. Jun [12] defined fuzzy prime ideal of a Γ-ring and obtained a number of characterisations for a fuzzy ideal to be a fuzzy prime ideal. In this paper, we prove a characterisation of a fuzzy prime ideal, already obtained by Jun in [12], in different way and also get few more new characterisations of fuzzy prime ideal. Lastly, we obtain a one-one correspondence between the set of all fuzzy prime ideals of a Γ-ring and the set of all fuzzy prime ideals of the operator rings of the Γ-ring.

2. Some basic definitions and examples

Definition 2.1. [1] *Let M and Γ be two additive abelian groups. M is called a Γ-ring if there exists a mapping $f : M \times \Gamma \times M \longrightarrow M$, $f(a, \alpha, b)$ is denoted by $a\alpha b$, $a, b \in M$, $\alpha \in \Gamma$, satisfying the following conditions for all $a, b, c \in M$ and for all $\alpha, \beta, \gamma \in \Gamma$; $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$, and $a\alpha(b\beta c) = (a\alpha b)\beta c$.*

Definition 2.2. [1] *A subset A of a Γ-ring M is called a left (resp. right) ideal of M if A is an additive subgroup of M and $m\alpha a \in A$* (resp. $a\alpha m \in A$) *for all $m \in M$, $\alpha \in \Gamma$, $a \in A$. If A is a left and a right ideal of M, then A is called a two sided ideal of M or simply an ideal of M.*

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Definition 2.3. [9] *Let M be a Γ ring and F be the free abelian group generated by $\Gamma \times M$. Then $A = \sum_i n_i(\gamma_i, x_i) \in F : a \in M \Rightarrow \sum_i n_i a\gamma_i x_i = 0\}$ is a subgroup of F. Let $R = F/A$ be the factor group of F by A. Let us denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. It can be verified that $[\alpha, x] + [\beta, x] = [\alpha + \beta, x]$ and $[\alpha, x] + [\alpha, y] = [\alpha, x + y]$, for all $\alpha, \beta \in \Gamma$ and $x, y \in M$. We define a multiplication in R by $\sum_i[\alpha_i, x_i]\sum_j[\beta_j, y_j] = \sum_{i,j}[\alpha_i, x_i\beta_j y_j]$. Then R forms a ring. This ring R is called the right operator ring of the Γ ring M. Similarly we can construct left operator ring L of M. For the subsets $N \subseteq M$, $\Phi \subseteq \Gamma$, we denote by $[\Phi, N]$ the set of all finite sums $\sum_i[\gamma_i, x_i]$ in R where $\gamma_i \in \Phi$ and $x_i \in N$ and we denote by $[(\Phi, N)]$ the set of all elements $[\phi, x]$ in R, where $\phi \in \Phi$ and $x \in N$. Thus in particular, $R = [\Gamma, M]$ and $L = [M, \Gamma]$. If there exists an element $\sum_i[\delta_i, e_i] \in R$ such that $\sum_i x\delta_i e_i = x$ for every element x of M then it is called the right unity of M. It can be verified that $\sum_i[\delta_i, e_i]$ is the unity of R. Similarly we can define the left unity $\sum_j[f_j, \gamma_j]$ which is the unity of the left operator ring L.*

Definition 2.4. [10] *A nonempty fuzzy subset μ (i.e., $\mu(x) \neq 0$ for some $x \in M$) of a Γ ring M is called a fuzzy left (resp. right) ideal of M if (i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$, (ii) $\mu(x\alpha y) \geq \mu(y)$ (resp. $\mu(x\alpha y) \geq \mu(x)$) for all $x, y \in M$ and for all $\alpha \in \Gamma$.*

A non-empty fuzzy subset μ of a Γ-ring M is called a fuzzy ideal if it is a fuzzy left ideal and a fuzzy right ideal of M.

Let M be a Γ-ring and R and L be the right operator ring and the left operator ring of M respectively.

Definition 2.5. [2] *For a fuzzy subset μ of R, we define a fuzzy subset μ^* of M by $\mu^*(a) = \inf_{\gamma\in\Gamma} \mu([\gamma, a])$, where $a \in M$. For a fuzzy subset σ of M we define a fuzzy subset $\sigma^{*'}$ of R by $\sigma^{*'}(\sum_i[\alpha_i, a_i]) = \inf_{m\in M} \sigma(\sum_i m\alpha_i a_i)$, where $\sum_i[\alpha_i, a_i] \in R$. For a fuzzy subset δ of L, we define a fuzzy subset δ^+ of M by $\delta^+(a) = \inf_{\gamma\in\Gamma} \delta([a, \gamma])$, where $a \in M$. For a fuzzy subset η of M we define a fuzzy subset $\eta^{+'}$ of L by $\eta^{+'}(\sum_i[a_i, \alpha_i]) = \inf_{m\in M} \eta(\sum_i a_i\alpha_i m)$, where $\sum_i[a_i, \alpha_i] \in L$.*

Definition 2.6. [2] *Let μ, σ be two fuzzy subsets of M. Then the sum $\mu \oplus \sigma$ and composition $\mu \circ \sigma$ of μ and σ are defined as follows:*

$$(\mu \oplus \sigma)(x) = \begin{cases} \sup\limits_{x=u+v} [\min[\mu(u),\ \sigma(v)]], & u, v \in M \\ 0, & \textit{otherwise.} \end{cases}$$

$$(\mu \circ \sigma)(x) = \begin{cases} \sup[\min[\min[\mu(u_i),\ \sigma(v_i)]]], & 1 \leq i \leq n, x = \sum\limits_{i=1}^{n} u_i\gamma_i v_i, \\ & u_i, v_i \in M \textit{ and } \gamma_i \in \Gamma, \\ 0, & \textit{otherwise.} \end{cases}$$

Definition 2.7. [12] *Let μ, σ be two fuzzy subsets of M. Then the product $\mu\Gamma\sigma$ of μ and σ is defined by*

$$(\mu\Gamma\sigma)(x) = \begin{cases} \sup\limits_{x=u\gamma v} [\min[\mu(u)\ \ \sigma(v)]], & \textit{for } u, v \in M \textit{ and } \gamma \in \Gamma \\ 0, & \textit{otherwise.} \end{cases}$$

We denote the set of all fuzzy ideals of M, the set of all fuzzy ideals of R, the set of all fuzzy prime ideals of M, the set of all fuzzy prime ideals of R by $FI(M)$, $FI(R)$, $FPI(M)$, $FPI(R)$ respectively.

Definition 2.8. [1] *Let M be a Γ-ring. A proper ideal P of M is called prime if for all pairs of ideals S and T of M, $S\Gamma T \subseteq P$ implies that $S \subseteq P$ or $T \subseteq P$.*

Remark 2.1. [7] If P is an ideal of a Γ-ring M, then the following conditions are equivalent:

(i) P is a prime ideal of M;
(ii) If $a, b \in M$ and $a\Gamma M\Gamma b \subseteq P$ then $a \in P$ or $b \in P$.

Definition 2.9. [7] *A fuzzy ideal μ of a ring R is said to be prime if μ is a non-constant function and for any two fuzzy ideals σ and δ of R, $\sigma\Gamma\delta \subseteq \mu$ implies that either $\sigma \subseteq \mu$ or $\delta \subseteq \mu$.*

Definition 2.10. [8] *Let f be a mapping from a Γ-ring M onto a Γ-ring N. Let $\mu \in FI(M)$. Now μ is said to be f-invariant if $f(x) = f(y)$ implies that $\mu(x) = \mu(y)$, for all $x, y \in M$.*

Definition 2.11. [1] *A function $f : M \to N$, where M, N are Γ-rings is said to be a Γ-homomorphism if $f(a + b) = f(a) + f(b)$, $f(a\alpha b) = f(a)\alpha f(b)$, for all $a, b \in M$, $\alpha \in \Gamma$.*

Definition 2.12. [8] *A fuzzy subset μ of a Γ-ring M is called a fuzzy point if $\mu(x) \in [0, 1]$ for some $x \in M$ and $\mu(y) = 0$ for all $y \in M \setminus \{x\}$. If $\mu(x) = \beta$, then the fuzzy point μ is denoted by x_β.*

Definition 2.13. [12] *A non-constant fuzzy ideal μ of a Γ-ring M is called a fuzzy prime ideal of M if for any two fuzzy ideals σ and θ of M, $\sigma\Gamma\theta \subseteq \mu$ implies that either $\sigma \subseteq \mu$ or $\theta \subseteq \mu$.*

3. Fuzzy Prime ideal in Γ-ring

Theorem 3.1. *Let $\mu \in FI(M)$.Then μ is a fuzzy prime ideal of M if and only if μ is non-constant and $\sigma \circ \theta \subseteq \mu$ where $\sigma, \theta \in FI(M)$ implies that either $\sigma \subseteq \mu$ or $\theta \subseteq \mu$.*

Proof. The theorem follows since $\sigma \circ \theta \subseteq \mu$ if and only if $\sigma\Gamma\theta \subseteq \mu$ where σ, $\theta \in FI(M)$. ∎

Theorem 3.2. *Let M be a commutative Γ-ring and $\mu \in FI(M)$. Then the following are equivalent:*

(i) $x_r\Gamma y_t \subseteq \mu \Rightarrow x_r \subseteq \mu$ *or* $y_t \subseteq \mu$ *where* x_r *and* y_t *are two fuzzy points of* M.
(ii) μ *is a fuzzy prime ideal of* M.

Proof. (i) ⇒(ii) Let $\sigma, \theta \in FI(M)$ such that $\sigma\Gamma\theta \subseteq \mu$. Suppose $\sigma \not\subseteq \mu$. Then there exists $x \in M$ such that $\sigma(x) > \mu(x)$. Let $\sigma(x) = a$. Let $y \in M$ and $\theta(y) = b$. If $z = x\gamma y$ for some $\gamma \in \Gamma$, then $(x_a\Gamma y_b)(z) = \min\{a, b\}$. Hence $\mu(z) = \mu(x\gamma y) \geq (\sigma\Gamma\theta)(x\gamma y) \geq \min\{\sigma(x), \theta(y)\} = \min\{a, b\} = (x_a\Gamma y_b)(x\gamma y)$. If $(x_a\Gamma y_b)(z) = 0$ then $\mu(z) \geq (x_a\Gamma y_b)(z)$. Hence $x_a\Gamma y_b \subseteq \mu$. By (i) either $x_a \subseteq \mu$ or $y_b \subseteq \mu$. That is either $a \leq \mu(x)$ or $b \leq \mu(y)$. Since $a \nleq \mu(x)$, $\theta(y) = b \leq \mu(y)$. So $\theta \subseteq \mu$. Thus μ is a fuzzy

prime ideal of M.
(ii) $\Rightarrow$(i) Suppose that μ is a fuzzy prime ideal of a commutative Γ ring M. Suppose x_r and y_t be two fuzzy points of M such that $x_r\Gamma y_t \subseteq \mu$. Then

$$\text{(a)} \qquad (x_r\Gamma y_t)(x\gamma y) = \min\{r,t\} \leq \mu(x\gamma y) \text{ for all } \gamma \in \Gamma.$$

Let fuzzy subsets σ,θ be defined by

$$\sigma(z) = \begin{cases} r, & \text{if } z \in \langle x \rangle, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \theta(z) = \begin{cases} t, & \text{if } z \in \langle y \rangle, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly σ, θ are fuzzy ideals of M. Now $(\sigma\Gamma\theta)(z) = \sup_{z=u\gamma v}[\min(\sigma(u), \theta(v))] = \min\{r,t\}$, where $u \in \langle x \rangle$ and $v \in \langle y \rangle$. Hence $(\sigma\Gamma\theta)(z) = \min\{r,t\} \leq \mu(u\gamma v)$ from (a), when $z = u\gamma v$, where $u \in \langle x \rangle$, $v \in \langle y \rangle$. Otherwise $(\sigma\Gamma\theta)(z) = 0$. Hence $\sigma\Gamma\theta \subseteq \mu$. As μ is prime, $\sigma \subseteq \mu$ or $\theta \subseteq \mu$. Then $x_r \subseteq \sigma \subseteq \mu$ or $y_t \subseteq \theta \subseteq \mu$. Thus $x_r\Gamma y_t \subseteq \mu$ implies that either $x_r \subseteq \mu$ or $y_t \subseteq \mu$. ∎

Theorem 3.3. *Let I be an ideal of a Γ-ring M, $\alpha \in [0,1)$ and μ be a fuzzy subset of M defined by*

$$\mu(x) = \begin{cases} 1, & \text{if } x \in I, \\ \alpha, & \text{if } x \notin I. \end{cases}$$

Then μ is a fuzzy prime ideal of M if and only if I is a prime ideal of M.

Proof. Let I be a prime ideal of M. Obviously μ is non-constant. If $\min\{\mu(a), \mu(b)\} = \alpha$, then $\mu(a-b) \geq \min\{\mu(a), \mu(b)\}$. If $\min\{\mu(a), \mu(b)\} = 1$, then $\mu(a) = \mu(b) = 1$. So $a,\ b \in I$ which implies that $a - b \in I$. So $\mu(a-b) = 1$. Hence for all $a, b \in M$, $\mu(a-b) \geq \min\{\mu(a),\ \mu(b)\}$. Similarly $\mu(a\gamma b) \geq \mu(a), \mu(b)$. Thus μ is a fuzzy ideal of M. Let $\sigma,\ \theta \in FI(M)$ be such that $\sigma\Gamma\theta \subseteq \mu$ and $\sigma \not\subseteq \mu, \theta \not\subseteq \mu$. Then there exist $x,\ y \in M$ such that $\sigma(x) > \mu(x),\ \theta(y) > \mu(y)$. This implies that $\mu(x) = \mu(y) = \alpha$. Therefore $x,\ y \notin I$. Since I is a prime ideal of M, $x\Gamma M\Gamma y \not\subseteq I$ [6]. Then there exist $m \in M, \gamma_1, \gamma_2 \in \Gamma$, such that $x\gamma_1 m\gamma_2 y \notin I$. Hence $\mu(x\gamma_1 m\gamma_2 y) = \alpha$. Now $(\sigma\Gamma\theta)(x\gamma_1 m\gamma_2 y) \geq \min\{\sigma(x), \theta(m\gamma_2 y)\} \geq \min\{\sigma(x), \theta(y)\} > \min\{\mu(x), \mu(y)\} = \alpha = \mu(x\gamma_1 m\gamma_2 y)$, a contradiction. Thus μ is prime.

Conversely let μ be a fuzzy prime ideal and P, Q be two ideals of M such that $P\Gamma Q \subseteq I$. Let $P \not\subseteq I$ and $Q \not\subseteq I$ and let $p \in P \setminus I$ and $q \in Q \setminus I$. We define fuzzy subsets σ, θ of M as follows

$$\sigma(x) = \begin{cases} 1, & \text{if } x \in P, \\ \alpha, & \text{if } x \notin P \end{cases} \quad \text{and} \quad \theta(x) = \begin{cases} 1, & \text{if } x \in Q, \\ \alpha, & \text{if } x \notin Q. \end{cases}$$

Then σ, θ are fuzzy ideals of M. Since $\sigma(p) = 1 > \alpha = \mu(p)$, $\sigma \not\subseteq \mu$. Similarly $\theta \not\subseteq \mu$. But $\sigma\Gamma\theta \subseteq \mu$, a contradiction. So I is a prime ideal of M. ∎

Corollary 3.1. [12, Theorem 1, Corollary 1] *Let I be an ideal of a Γ-ring M. Then the characteristic function χ_I of I is a fuzzy prime ideal of M if I is a prime ideal of M.*

Theorem 3.4. [12, Theorem 2, Theorem 3] *If μ is a fuzzy prime ideal of M then the following conditions hold:*

(i) $\mu(O_M) = 1$,
(ii) $Im\mu = \{1, \alpha\},\ \alpha \in [0,1)$,

(iii) $\mu_o = \{x \in M : \mu(x) = \mu(O_M)\}$ *is a prime ideal of* M.

Proof. (i) Let μ be a fuzzy prime ideal of M. Suppose $\mu(O_M) < 1$. Since μ is non-constant, there exist $a \in M$ such that $\mu(a) < \mu(O_M)$. Let $\sigma, \theta \in FI(M)$ be defined by

$$\sigma(x) = \begin{cases} 1, & \text{if } x \in \mu_0, \\ 0, & \text{if } x \notin \mu_0 \end{cases} \quad \text{and} \quad \theta(x) = \mu(O_M)$$

for all $x \in M$. Then $\sigma\Gamma\theta \subseteq \mu$. Since $\sigma(O_M) = 1 > \mu(O_M)$ and $\theta(a) = \mu(O_M) > \mu(a), \sigma \nsubseteq \mu$ and $\theta \nsubseteq \mu$. This contradicts the fact that μ is a fuzzy prime ideal of M. Hence $\mu(O_M) = 1$.

(ii) Now we shall show that $|Im\mu| = 2$. Let $x,\ y \in M \setminus \mu_0$ and $\mu(x) = c,\ c \neq 0$. We define fuzzy ideal $C_{\langle x\rangle}$ by

$$C_{\langle x\rangle}(a) = \begin{cases} c, & \text{if } a \in \langle x\rangle \\ 0, & \text{if } a \notin \langle x\rangle. \end{cases}$$

For $a \in \langle x\rangle\, C_{\langle x\rangle}(a) = c \leq \mu(a)$. For $a \notin \langle x\rangle$, $C_{\langle x\rangle}(a) = 0 \leq \mu(a)$. Hence $C_{\langle x\rangle} \subseteq \mu$. Clearly $1_{\langle x\rangle},\ C_M \in FI(M)$. Now $1_{\langle x\rangle} \nsubseteq \mu$ as $1_{\langle x\rangle}(x) = 1 > c = \mu(x)$. Now $(1_{\langle x\rangle}\Gamma C_M)(a) = 0$ or c for any $a \in M$. If $(1_{\langle x\rangle x>}\Gamma C_M)(a) = 0$, then clearly $\mu(a) \geq (1_{\langle x\rangle}\Gamma C_M)...$ and if $(1_{\langle x\rangle}\Gamma C_M)(a) = c$, then $a \in \langle x\rangle$; hence $\mu(a) \geq c = (1_{\langle x\rangle}\Gamma C_M)(a)$. Thus $(1_{\langle x\rangle}\Gamma C_M) \subseteq \mu$. Now since μ is a fuzzy prime ideal of M and $1_{\langle x\rangle} \nsubseteq \mu$, $C_M \subseteq \mu$. Now $\mu(x) = c = C_M(y) \leq \mu(y)$. Hence $\mu(x) \leq \mu(y)$. Similarly we can show that $\mu(y) \leq \mu(x)$. Hence $\mu(x) = \mu(y)$ for all $x,\ y \in M \setminus \mu_0$. This proves that $|Im\mu| = 2$. (iii) Clearly from (i) and (ii) it follows

$$\mu(x) = \begin{cases} 1, & \text{for } x \in \mu_0, \\ \alpha, & \text{for } x \notin \mu_0. \end{cases}$$

Then from Theorem 3.3, it follows that μ_0 is a prime ideal of M as μ is a fuzzy prime ideal of M. ∎

The converse of the above theorem is also true. We shall prove it later using operator rings of a Γ-ring.

Lemma 3.1. [2] *If $\mu \in FI(R)$ (resp. FLI(R), FRI(R)) then $\mu^* \in FI(M)$(resp. FLI(M), FRI(M)), where μ^* is defined by $\mu^*(m) = \inf_{\gamma\in\Gamma} \mu([\gamma, m]),\ m \in M$.*

Lemma 3.2. [8, Theorem 1.2.48] *μ is a fuzzy prime ideal of a ring R, if and only if $\mu(O_R) = 1$, μ_0 is prime ideal of R and $\mu(R) = \{1, \alpha\},\ \alpha \in [0, 1)$.*

Theorem 3.5. *If μ be a fuzzy prime ideal of the right operator ring R of a Γ-ring M, then μ^* is a fuzzy prime ideal of M.*

Proof. Since μ is a fuzzy prime ideal of R, $\mu(O_R) = 1$, μ_0 is prime ideal of R and $\mu(R) = \{1, \alpha\},\ \alpha \in [0, 1)$ [7]. By definition of μ^*, it follows that $|Im\mu^*| = 2,\ \mu^*(M) = \{1, \alpha\},\ \mu^*(O_M) = \inf_{\gamma\in\Gamma} \mu([\gamma, O_M]) = \mu(O_R) = 1$. Now we shall prove $(\mu^*)_0 = (\mu_0)^*$. Let $x \in (\mu^*)_0$. Now

$$\begin{aligned} x \in (\mu^*)_0 &\Leftrightarrow \mu^*(x) = \mu^*(O_M) = 1 \\ &\Leftrightarrow \inf_{\gamma\in\Gamma} \mu[\gamma, x] = 1 \\ &\Leftrightarrow \mu[\gamma, x] = 1 = \mu(O_R), \text{ for all } \gamma \in \Gamma \\ &\Leftrightarrow [\gamma, x] \in \mu_0 \\ &\Leftrightarrow \gamma \in \Gamma \\ &\Leftrightarrow x \in (\mu_0)^*. \end{aligned}$$

Therefore $(\mu^*)_0 = (\mu_0)^*$. Since μ is a fuzzy prime ideal of R, μ_0 is a prime ideal of R and hence $(\mu^*)_0 = (\mu_0)^*$ is a prime ideal of M [5]. Then from Theorem 3.3, it follows that μ^* is a fuzzy prime ideal of M. ∎

Lemma 3.3. [2] *If $\sigma \in FI(M)(resp. FLI(M), FRI(M))$, then $\sigma^{*'} \in FI(R)$ (resp. $FLI(R)$, $FRI(R)$), where $\sigma^{*'}$ is defined by $\sigma^{*'}(\sum_i[\gamma_i, a_i]) = \inf_{m\in M} \sigma(\sum_i m\gamma_i a_i)$.*

Lemma 3.4. [5] *If P is a prime ideal of a Γ-ring M, then $P^{*'}$ is a prime ideal of the right operator ring R of the Γ-ring M.*

Theorem 3.6. *If σ be a fuzzy prime ideal of M, then $\sigma^{*'}$ is a fuzzy prime ideal of R.*

Proof. Since σ is a fuzzy prime ideal of M, $\sigma(O_M) = 1$, σ_0 is prime ideal of M and $\sigma(M) = \{1, \alpha\}$, $\alpha \in [0, 1)$. Now $\sigma^{*'}(\sum[\gamma_i, a_i]) = \inf_{m\in M} \sigma(\sum_i m\gamma_i a_i)$. So $\sigma^{*'}(O_R) = 1$, $\sigma^{*'}(R) = \{1, \alpha\}$. We shall now show that $(\sigma^{*'})_0 = (\sigma_0)^{*'}$. Now $\sum_i[\gamma_i, a_i] \in (\sigma^{*'})_0$ if and only if $\sigma^{*'}(\sum_i[\gamma_i, a_i]) = \sigma^{*'}(O_R)$ if and only if $\inf_{m\in M} \sigma(\sum_i m\gamma_i a_i) = 1$ if and only if $\sigma(\sum_i m\gamma_i a_i) = 1 = \sigma(O_M)$ for all $m \in M$ if and only if $\sum_i m\gamma_i a_i \in \sigma_0$ for all $m \in M$ if and only if $\sum_i[\gamma_i, a_i] \in (\sigma_0)^{*'}$. Thus $(\sigma^{*'})_0 = (\sigma_0)^{*'}$. Since σ_0 is a prime ideal of M, $(\sigma^{*'})_0 = (\sigma_0)^{*'}$ is a prime ideal of R by Lemma 3.4. Hence $\sigma^{*'}$ is a fuzzy prime ideal of R by Lemma 3.2. ∎

Theorem 3.7. *The mapping $\mu \to \mu^*$ defines a one-one correspondence between the set of all fuzzy prime ideals of R and the set of all fuzzy prime ideals of M, where μ is a fuzzy prime ideal of R.*

Proof. Let μ be a fuzzy prime ideal of R. Then μ^* is a fuzzy prime ideal of M by Theorem 3.5, and $(\mu^*)^{*'}$ is a fuzzy prime ideal of R by Theorem 3.6. We shall show that $\mu = (\mu^*)^{*'}$. For this we first show that $\mu_0 = ((\mu^*)^{*'})_0$. Clearly by definition of $(\mu^*)^{*'}$, $Im\mu = Im(\mu^*)^{*'}$. Let $\sum_i[\gamma_i, a_i] \in \mu_0$. Now

$$(\mu^*)^{*'}(\sum[\gamma_i, a_i]) = \inf_{\gamma\in\Gamma} \inf_{m\in M} \mu([\gamma, \sum_i m\gamma_i a_i) = \inf_{\gamma\in\Gamma} \inf_{m\in M} \mu([\gamma, m] \sum_i [\gamma_i, a_i]) = 1.$$

Thus $\mu_0 \subseteq ((\mu^*)^{*'})_0$. Let $\sum[\gamma_i, a_i] \in ((\mu^*)^{*'})_0$. Then

$$(\mu^*)^{*'}(\sum_i[\gamma_i, a_i]) = 1 = \inf_{\gamma\in\Gamma} \inf_{m\in M} \mu([\gamma, m] \sum_i [\gamma_i, a_i]).$$

Hence for all $\gamma \in \Gamma, m \in M$, $[\gamma, m] \sum_i[\gamma_i, a_i] \in \mu_0$. Now as μ_o is prime ideal of R, either $[\gamma, m] \in \mu_0$ or $\sum_i[\gamma_i, a_i] \in \mu_0$. Now as μ is non-constant, $[\gamma, m] \notin \mu_0$ for all $\gamma \in \Gamma$, $m \in M$. Hence $\sum_i[\gamma_i, a_i] \in \mu_0$. Thus $((\mu^*)^{*'})_0 \subseteq \mu_0$. Hence $\mu_0 = ((\mu^*)^{*'})_0$. As μ is a fuzzy prime ideal of R, $Im\mu = \{1, \alpha\} = Im(\mu^*)^{*'}$ where $\alpha \in [0, 1)$. Hence $\mu = (\mu^*)^{*'}$. Now let σ be a fuzzy prime ideal of M. We shall show that

$\sigma = (\sigma^{*'})^*$. Clearly $Im\sigma = Im(\sigma^{*'})^*$ and both are fuzzy prime ideals of M. We shall first show that $\sigma_0 = ((\sigma^{*'})^*)_0$. Let $a \in \sigma_0$. Then $\sigma(a) = 1$. Now $\sigma(m\gamma a) \geq \sigma(a)$, for all $m \in M$, for all $\gamma \in \Gamma$. So $\sigma(m\gamma a) = 1$ for all $m \in M$, for all $\gamma \in \Gamma$. Now $(\sigma^{*'})^*(a) = \inf_{m \in M} \inf_{\gamma \in \Gamma} \sigma(m\gamma a) = 1$. So $a \in ((\sigma^{*'})^*)_0$. Thus $\sigma_0 \subseteq ((\sigma^{*'})^*)_0$. Now let $a \in ((\sigma^{*'})^*)_0$. Thus $(\sigma^{*'})^*(a) = 1 = \inf_{m \in M} \inf_{\gamma \in \Gamma} \sigma(m\gamma a)$. This implies that $\sigma(m\gamma a) = 1$, for all $m \in M$, $\gamma \in \Gamma$. So $m\gamma a \in \sigma_0$ i.e., $m_1 \Gamma M \Gamma a \subseteq \sigma_0$ for all $m_1 \in M$. Now as σ_0 is a prime ideal of M, either $m_1 \in \sigma_0$ or $a \in \sigma_0$. Now $m_1 \notin \sigma_0$ for all $m_1 \in M$ as σ is non-constant. Thus $a \in \sigma_0$. Hence $((\sigma^{*'})^*)_0 \subseteq \sigma_0$. Hence $\sigma_0 = ((\sigma^{*'})^*)_0$. As σ is a fuzzy prime ideal of M, $\sigma^{*'}$ is also a fuzzy prime ideal of R. By Lemma 3.2, $Im\sigma^{*'} = \{1, \alpha\}$, $\alpha \in [0, 1)$. Since $(\sigma^{*'})^*(m) = \inf_{\gamma \in \Gamma} \sigma^{*'}([\gamma, m])$ where $m \in M$, $Im(\sigma^{*'})^* = \{1, \alpha\} = Im\sigma$ where $\alpha \in [0, 1)$. This proves that $\sigma = (\sigma^{*'})^*$. Thus $\mu \to \mu^*$ is a one-to-one correspondence between the set of all fuzzy prime ideals of R and the set of all fuzzy prime ideals of M. ∎

Similar result holds for the Γ-ring M and the left operator ring L of M. As a converse of Theorem 3.5, we have the following theorem.

Theorem 3.8. [12, *Theorem* 4] *Let μ be a fuzzy ideal of M. Then μ is fuzzy prime ideal of M if the following conditions hold*

(i) $\mu(O_M) = 1$,
(ii) $Im\mu = \{1, \alpha\}$, $\alpha \in [0, 1)$,
(iii) $\mu_0 = \{x \in M : \mu(x) = \mu(O_M)\}$ *is a prime ideal of M.*

Proof. As μ is a fuzzy ideal of M, so $\mu^{*'}$ is a fuzzy ideal of R, where $\mu^{*'}$ is defined by $\mu^{*'}(\sum_i [\gamma_i, \alpha_i]) = \mu(\sum_i m\gamma_i\alpha_i)$. Clearly if (i) $\mu(O_M) = 1$ then $\mu^{*'}(O_R) = 1$, (ii) $Im\mu = \{1, \alpha\}$, $\alpha \in [0, 1)$ implies $Im\mu^{*'} = \{1, \alpha\}$, $\alpha \in [0, 1)$. From Theorem 3.6, $(\mu^{*'})_0 = (\mu_0)^{*'}$. Now as μ_0 is prime ideal of M, $(\mu_0)^{*'}$ is a prime ideal of R by Lemma 3.4. Hence $(\mu^{*'})_0$ is prime ideal of R. Hence $\mu^{*'}$ is a fuzzy prime ideal of R by Lemma 3.2. So $(\mu^{*'})^* = \mu$ is a fuzzy prime ideal of M. ∎

Lemma 3.5. *If f is a homomorphism of a Γ-ring M onto a Γ-ring N and μ be an f-invariant fuzzy ideal of M, then $f(\mu_0) = [f(\mu)]_0$.*

Proof. Clearly

$$[f(\mu)](O_N) = \sup_{f(x)=O_N} \mu(x) = \sup_{f(x)=f(O_M)} \mu(x) = \sup_{f(x)=f(O_M)} \mu(O_M) = \mu(O_M),$$

since μ is f-invariant. Let $y \in f(\mu_0)$. Then $y = f(x)$ for some $x \in \mu_0$. Hence $\mu(x) = \mu(O_M) = [f(\mu)](O_N)$. Now

$$[f(\mu)](y) = \sup_{f(z)=y} \mu(z) = \sup_{f(z)=f(x)} \mu(z) = \mu(x) = \mu(O_M) = f(\mu)(O_N).$$

Hence $y \in [f(\mu)]_0$. Again let $f(x) \in [f(\mu)]_0$. Then

$$f(\mu)(O_N) = [f(\mu)](f(x)) = \sup_{f(t)=f(x)} \mu(t) = \mu(x). So \mu(x) = [f(\mu)](O_N) = \mu(O_M).$$

So $x \in \mu_0$. Hence $f(x) \in f(\mu_0)$. Thus $f(\mu_0) = [f(\mu)]_0$. ∎

Lemma 3.6. [13] *Let f be a homomorphism of a Γ-ring M onto a Γ-ring N. If μ is an f-invariant fuzzy ideal of M, then $f(\mu)$ is a fuzzy ideal of N.*

Theorem 3.9. *Let f be a homomorphism of a Γ-ring M onto a Γ-ring N. If μ is an f-invariant fuzzy prime ideal of M, then $f(\mu)$ is a fuzzy prime ideal of N.*

Proof. Let μ be an f-invariant fuzzy prime ideal of M. Then $f(\mu)$ is a fuzzy ideal of N by Lemma 3.15. Since μ is fuzzy prime:

(i) $\mu(O_M) = 1$,
(ii) $\mu(M) = \{1, \alpha\},\ \alpha \in [0, 1)$,
(iii) $\mu_0 = \{\, x \in M : \mu(x) = \mu(O_M)\}$ is a prime ideal of M.

From the proof of the Lemma 3.5, $[f(\mu)](O_N) = \mu(O_M) = 1$(a). Also by Lemma 3.5, $[f(\mu)]_0 = f(\mu_0)$ is a prime ideal of N. Now we prove $[f(\mu)](N) = \{1, \alpha\},\ \alpha \in [0, 1)$. Let $x \in M$ be such that $\mu(x) = \alpha$. Then $[f(\mu)](f(x)) = \sup\limits_{f(z)=f(x)} \mu(z) = \mu(x) = \alpha$, as μ is f-invariant. Also $(f(\mu))(O_N) = 1$. So $(f(\mu))(N) = \{1, \alpha\}$. By Theorem 3.8, it follows that $f(\mu)$ is a fuzzy prime ideal of N. ∎

Lemma 3.7. *Let f be a homomorphism of a Γ-ring M to a Γ-ring N. If $\eta \in FI(N)$, then $f^{-1}(\eta_0) = [f^{-1}(\eta)]_0$.*

Proof. Let $x \in M$. Now

$$\begin{aligned} x \in f^{-1}(\eta_\circ) &\Leftrightarrow f(x) \in \eta_0 \\ &\Leftrightarrow \eta(f(x)) = \eta(O_N) = \eta(f(O_M)) \\ &\Leftrightarrow f^{-1}(\eta)(x) = f^{-1}(\eta)(O_M) \\ &\Leftrightarrow x \in [f^{-1}(\eta)]_0. \end{aligned}$$

Hence $f^{-1}(\eta_0) = [f^{-1}(\eta)]_0$. ∎

Lemma 3.8. [10] *Let f be a homomorphism of a Γ-ring M onto a Γ-ring N and $\eta \in FI(N)$. If η is a fuzzy ideal of N, then $f^{-1}(\eta)$ is a fuzzy ideal of M.*

Theorem 3.10. *Let f be a homomorphism of a Γ-ring M onto a Γ-ring N and $\eta \in FI(N)$. If η is a fuzzy prime ideal of N, then $f^{-1}(\eta)$ is a fuzzy prime ideal of M.*

Proof. By Lemma 3.8, $f^{-1}(\eta)$ is a fuzzy ideal of M. $f^{-1}(\eta)(O_M) = \eta(f(O_M)) = \eta(O_N) = 1$ as η is a fuzzy prime ideal of N. Now $\eta(N) = \{1, \alpha\}$, where $\alpha \in [0, 1)$. Let $y \in N$ be such that $\eta(y) = \alpha$, then there exists $x \in M$ such that $f(x) = y$ as f is onto. Now $f^{-1}(\eta)(x) = \eta(f(x)) = \alpha$. Thus $f^{-1}(\eta)(M) = \{1, \alpha\}, \alpha \in [0, 1)$. Hence by Lemma 3.5

(i) $f^{-1}(\eta)(O_M) = 1$,
(ii) $|f^{-1}(\eta)(M)| = 2$,
(iii) $[f^{-1}(\eta)]_0$ is a prime ideal of M.

Hence from (i), (ii), (iii) it follows from Theorem 3.8 that $f^{-1}(\eta)$ is a fuzzy prime ideal of M. ∎

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