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On Some Maps Concerning gα-Open Sets

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Abstract. In this paper, we consider a new generalization of α-open maps via the concept of gα-closed sets which we call approximately α-open maps. We study some of its fundamental properties. It turns out that we can use this notion to obtain a new characterization of α-$T_{1/2}$ spaces.

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1. Introduction and preliminaries

The notion of α-open set in topological spaces was introduced in 1965 by Njåstad [16]. Since then, many mathematicians turned their attention to the generalizations of various concepts in topology by considering α-open sets (see for example [1], [4], [5], [6], [12], [13], [14], [15], [17]). In 1994 Maki, Devi and Balachandran [15] generalized the notion of closed sets to generalized α-closed sets (briefly gα-closed sets) with the help of α-open sets. In this direction, we introduce the notion of ap-α-open maps by using gα-closed sets and study some of their basic properties. Finally we characterize the class of α-$T_{1/2}$ spaces in terms of ap-α-open maps.

Throughout this paper $(X,\tau),(Y,\sigma)$ and (Z,γ) represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. For a subset A of a space (X,τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure and the interior of A, respectively.

In order to make the contents of the paper as self contained as possible, we briefly describe certain definitions, notations and some properties. For those not described, we refer the reader to [16]. A subset A of a space (X,τ) is said to be α-open [16] if, $A\subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$.

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The α-interior of A, denoted by $\alpha\,\mathrm{Int}(A)$, is the union of all α-open sets of (X,τ) contained in A. A is α-open [1] if and only if $\alpha\,\mathrm{Int}(A) = A$. Also, we have $\alpha\,\mathrm{Int}(A) = A \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$ [1]. A subset B of (X,τ) is said to be α-closed if its complement B^c is α-open in (X,τ). By $\alpha O(X,\tau)$ (resp. $\alpha C(X,\tau)$) we mean the collection of all α-open sets (resp. α-closed sets) in (X,τ). The α-closure of a set B of (X,τ) denoted by $\alpha\,\mathrm{Cl}_X(B)$ (briefly $\alpha\,\mathrm{Cl}(B)$), is defined to be the intersection of all α-closed sets of (X,τ) containing B. B is α-closed [1] if and only if $\alpha\,\mathrm{Cl}(B) = B$. Also, we have $\alpha\,\mathrm{Cl}(B) = B \cup \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(B)))$ [1].

A subset F of (X,τ) is said to be generalized α-closed (briefly gα-closed) in (X,τ) [15] if, $\alpha\,\mathrm{Cl}(F) \subseteq O$ whenever $F \subseteq O$ and O is α-open in (X,τ). A subset B is said to be generalized α-open (briefly gα-open) in (X,τ) if its complement B^c is gα-closed in (X,τ).

A map $f : (X,\tau) \to (Y,\sigma)$ is called α-irresolute [13] (resp. gα-irresolute) if $f^{-1}(O)$ is α-open (resp. gα-closed) in (X,τ) for every $O \in \alpha O(Y,\sigma)$ (resp. gα-closed in (Y,σ)).

A map $f : (X,\tau) \to (Y,\sigma)$ is called pre-α-closed (resp. pre-α-open) [9] if for every α-closed (resp. α-open) set B of (X,τ), $f(B)$ is α-closed (resp. α-open) in (Y,σ).

2. Ap-α-open maps

We introduce the following notion:

Definition 2.1. *A map $f : (X,\tau) \to (Y,\sigma)$ is said to be approximately α-open (briefly ap-α-open) if $\alpha\,\mathrm{Cl}(B) \subseteq f(A)$ whenever B is a gα-closed subset of (Y,σ), A is an α-open subset of (X,τ) and $B \subseteq f(A)$.*

Theorem 2.1. *$f : (X,\tau) \to (Y,\sigma)$ is ap-α-open if $f(O) \in \alpha C(Y,\sigma)$ for every α-open subset O of (X,τ).*

Proof. Let $B \subseteq f(A)$, where A is an α-open subset of (X,τ) and B is a gα-closed subset of (Y,σ). Therefore $\alpha\,\mathrm{Cl}(B) \subseteq \alpha\,\mathrm{Cl}(f(A)) = f(A)$. Thus f is ap-α-open. ∎

Clearly pre-α-open maps are ap-α-open, but not conversely.

Example 2.1. Let $X = \{a,b\}$ be the Sierpinski space with the topology, $\tau = \{\emptyset, \{a\}, X\}$. Let $f : X \to X$ be defined by $f(a) = b$ and $f(b) = a$. Since the image of every α-open set is α-closed, then f is ap-α-open. However $\{a\}$ is α-open in (X,τ) but $f(\{a\})$ is not α-open in (Y,σ). Therefore f is not pre-α-open.

Remark 2.1. Let (X,τ) be the topological space defined in Example 2.1. Then the identity map on (X,τ) is ap-α-open. It is clear that the converse of Theorem 2.1 does not hold (see Example 2.2).

In the following theorem, we show that under certain conditions the converse of Theorem 2.1 is true.

Theorem 2.2. *Let $f : (X,\tau) \to (Y,\sigma)$ be a map. If the α-open and α-closed sets of (Y,σ) coincide, then f is ap-α-open if and only if $f(A) \in \alpha C(Y,\sigma)$ for every α-open subset A of (X,τ).*

Proof. Suppose that f is ap-α-open. Let A be an arbitrary subset of (Y,σ) such that $A \subseteq Q$ where $Q \in \alpha O(Y,\sigma)$. Then by hypothesis $\alpha\,\mathrm{Cl}(A) \subseteq \alpha\,\mathrm{Cl}(Q) = Q$. Therefore all subset of (Y,σ) are gα-closed and hence all are gα-open. So for any $O \in \alpha O(X,\tau)$,$f(O)$ is gα-closed in (Y,σ). Since f is ap-α-open $\alpha\,\mathrm{Cl}(f(O)) \subseteq f(O)$. Therefore $\alpha\,\mathrm{Cl}(f(O)) = f(O)$, i.e., $f(O)$ is α-closed in (Y,σ). The converse is obvious by Theorem 2.1. ∎

As an immediate consequence of Theorem 2.2, we have the following:

Corollary 2.1. *Let $f : (X,\tau) \to (Y,\sigma)$ be a map. If the α-closed and α-open sets of (Y,σ) coincide then f is ap-α-open if and only if f is pre-α-open.*

Definition 2.2. *A map $f : (X,\tau) \to (Y,\sigma)$ is called contra pre-α-open if $f(O)$ is α-closed in (Y,σ) for each set $O \in \alpha O(X,\tau)$.*

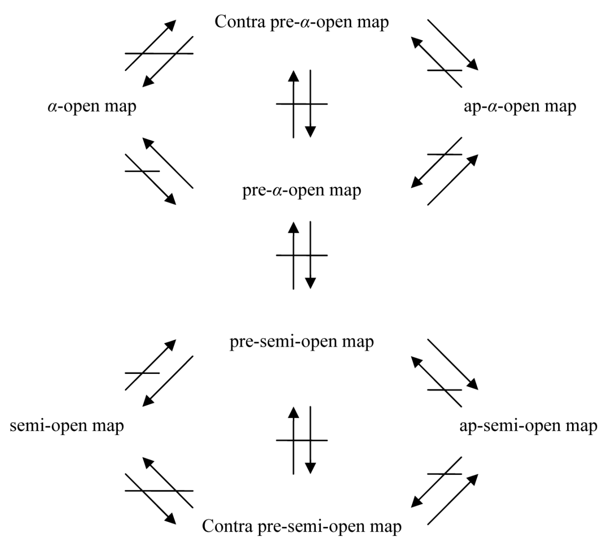
Remark 2.2. In fact contra pre-α-open maps and pre-α-open are independent notions. Example 2.1 shows that contra-pre-α-openness does not imply pre-α-openness while the converse is shown in the following example.

Example 2.2. The identity map on the same topological space (X,τ) where $\tau = \{\emptyset, \{a\}, X\}$ is an example of a pre-α-open map which is not contra pre-α-open.

Remark 2.3. By Theorem 2.1 and Remark 2.1 we have that every contra pre-α-open map is ap-α-open but the converse does not hold.

For the definitions of ap-semi-open, contra pre-semi-open, pre-semi-open and semi-open maps see Caldas and Baker [8], Sundaram, Maki and Balachandran [18] and Biswas [3].

The following diagram holds:



The next theorem establishes conditions under which the inverse map of every gα-open set from codomain is a gα-open set.

Theorem 2.3. *If a map $f : (X,\tau) \to (Y,\sigma)$ is surjective α-irresolute and ap-α-open, then $f^{-1}(A)$ is gα-open whenever A is gα-open subset of (Y,σ).*

Proof. Let A be a gα-open subset of (Y,σ). Suppose that $F \subseteq f^{-1}(A)$, where $F \in \alpha C(X,\tau)$. Taking complements we obtain $f^{-1}(A^c) \subseteq F^c$ or $A^c \subseteq f(F^c)$. Since f is an ap-α-open and $\alpha\,\mathrm{Int}(A) = A \cap \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$ and $\alpha\,\mathrm{Cl}(A) = A \cup \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$, then $(\alpha\,\mathrm{Int}(A))^c = \alpha\,\mathrm{Cl}(A^c) \subseteq f(F^c)$. It follows that $(f^{-1}(\alpha\,\mathrm{Int}(A)))^c \subseteq F^c$ and hence $F \subseteq f^{-1}(\alpha\,\mathrm{Int}(A))$. Since f is α-irresolute $f^{-1}(\alpha\,\mathrm{Int}(A))$ is α-open. Thus, we have $F \subseteq f^{-1}(\alpha\,\mathrm{Int}(A)) = \alpha\,\mathrm{Int}(f^{-1}(\alpha\,\mathrm{Int}(A))) \subseteq \alpha\,\mathrm{Int}(f^{-1}(A))$. This implies that $f^{-1}(A)$ is gα-open in (X,τ). ∎

Theorem 2.4. *Let $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\gamma)$ be two maps and $g \circ f : (X,\tau) \to (Z,\gamma)$. Then*

(i) *$g \circ f$ is ap-α-open, if f is pre-α-open and g is ap-α-open,*

(ii) *$g \circ f$ is ap-α-open, if f is ap-α-open and g is bijective pre-α-closed and gα-irresolute.*

Proof. (i) Suppose that A is an arbitrary α-open subset in (X,τ) and B a gα-closed subset of (Z,γ) for which $B \subseteq g \circ f(A)$. Then $f(A)$ is α-open in (Y,σ) since f is pre-α-open. Since g is ap-α-open, $\alpha\,\mathrm{Cl}(B) \subseteq g(f(A))$. This implies that $g \circ f$ is ap-α-open. (ii) Let A be an arbitrary α-open subset of (X,τ) and B a gα-closed subset of (Z,γ) for which $B \subseteq g \circ f(A)$. Hence $g^{-1}(B) \subseteq f(A)$. Then $\alpha\,\mathrm{Cl}(g^{-1}(B)) \subseteq f(A)$ because $g^{-1}(B)$ is gα-closed and f is ap-α-open. Hence we have $\alpha\,\mathrm{Cl}(B) \subseteq \alpha\,\mathrm{Cl}(gg^{-1}(B)) \subseteq g(\alpha\,\mathrm{Cl}(g^{-1}(B))) \subseteq g(f(A)) = (g \circ f)(A)$. This implies that $g \circ f$ is ap-α-open. ∎

Now we state the following theorem whose proof is straightforward and hence omitted.

Theorem 2.5. *Let $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\gamma)$ be two maps and $g \circ f : (X,\tau) \to (Z,\gamma)$. Then*

(i) *$g \circ f$ is contra pre α-open, if f is pre α-open and g is contra pre α-open,*

(ii) *$g \circ f$ is contra pre α-open , if f is contra pre α-open and g is pre α-closed.*

Theorem 2.6. *Let $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\gamma)$ be two maps such that $g \circ f : (X,\tau) \to (Z,\gamma)$ is a contra pre α-open map. Then*

(i) *If f is an α-irresolute surjection, then g is contra pre α-open,*

(ii) *If g is an α-irresolute injection, then f is contra pre α-open.*

Proof. (i) Suppose A is any arbitrary α-open set in Y. Since f is α-irresolute, $f^{-1}(A)$ is α-open in X. Moreover $g \circ f$ is contra pre α-open and f is surjective, then $(g \circ f)(f^{-1}(A)) = g(A)$ is α-closed in Z. This implies that g is a contra pre α-open map. (ii) Suppose A is any arbitrary α-open set in X. Since $g \circ f$ is contra pre α-open, $(g \circ f)(A)$ is α-closed in Z. Since g is an α-irresolute injection, $g^{-1}(g \circ f)(A) = f(A)$ is α-closed in Y. This implies that f is a contra pre α-open map. ∎

Lemma 2.1. (i) [14] *Let A and Y be subsets of a topological space X. If $A \in \alpha O(Y,\tau_Y)$ and $Y \in \alpha O(X,\tau)$, then $A \in \alpha O(X,\tau)$.*
(ii) [13] *If $A \in \alpha O(X,\tau)$ and $B \in \alpha O(Y,\sigma)$, then $A \times B \in \alpha O(X \times Y, \tau \times \sigma)$.*

Definition 2.3. *Let* (X, τ) *and* (Y, σ) *be topological spaces. A map* $f : (X, \tau) \to (Y, \sigma)$ *is said to have an* α*-closed graph if its* $\mathrm{G(f)} = \{(x, y) : y = f(x), x \in X\}$ *is* α*-closed in the product space* $(X \times Y, \tau_p)$*, where* τ_p *denotes the product topology.*

It is well-known that the graph $\mathrm{G}(f)$ of f is a closed set of $X \times Y$, whenever f is continuous and Y is Hausdorff. The following theorem is a modification of this fact, i.e., we give a condition under which a contra pre α-open map has α-closed graph.

Theorem 2.7. *If* $f : (X, \tau) \to (Y, \sigma)$ *is a contra pre* α*-open map with* α*-closed fibers, then the graph* $\mathrm{G}(f)$ *of* f *is* α*-closed in the product space* $X \times Y$.

Proof. Let $(x, y) \in X \times Y \backslash \mathrm{G}(f)$. Then $x \in (f^{-1}\{y\})^c$. Since fibers are α-closed, there is an α-open set O for which $x \in O \subseteq (f^{-1}\{y\})^c$. Set $A = (f(O))^c$. Then A is an α-open set in Y containing y, since f is contra pre α-open. Therefore, we obtain that $(x, y) \in O \times A \subseteq X \times Y \backslash \mathrm{G}(f)$, where $O \times A \in \alpha O(X \times Y, \tau_p)$. This implies that $X \times Y \backslash \mathrm{G}(f)$ is α-open in $X \times Y$. Hence $\mathrm{G}(f)$ is a α-closed set of $X \times Y$. ∎

Regarding the restriction f_A of a map $f : (X, \tau) \to (Y, \sigma)$ to a subset A of X, we have the following:

Theorem 2.8. *If* $f : (X, \tau) \to (Y, \sigma)$ *is ap-*α*-open and* A *is an* α*-open set of* (X, τ)*, then its restriction* $f_A : (A, \tau_A) \to (Y, \sigma)$ *is ap-*α*-open.*

Proof. Let O be an arbitrary α-open subset of (A, τ_A) and B a gα-closed subset of (Y, σ) for which $B \subseteq f_A(O)$. $O \in \alpha O(X, \tau)$ since $A \in \alpha O(X, \tau)$. Then $B \subseteq f(O) = f_A(O)$. Using Definition 2.1, we have that $\alpha \, \mathrm{Cl}(B) \subseteq f_A(O)$. Thus f_A is an ap-α-open map. ∎

Observe that restrictions of ap-α-open maps can fail to be ap-α-open. Let X be an indiscrete space. Then X and $\emptyset$ are the only α-closed subsets of X. Hence the α-open subsets of X are X and $\emptyset$. Let A be a nonempty proper subset of X. The identity map $f : X \to X$ is ap-α-open, but $f_A : A \to X$ fails to be ap-α-open. In fact, $f(A)$ is gα-closed (every subset of X is gα-closed) and A is open in A. Therefore α-open in (A, τ_A). But $\alpha \, \mathrm{Cl}(f(A)) \nsubseteq f(A)$.

In recent years, the class of α-$\mathrm{T}_{1/2}$ spaces has been of some interest (see for example [15,9,7]). In the following theorem, we give a new characterization of α-$\mathrm{T}_{1/2}$ spaces by using the notion of ap-α-open maps. We recall that a topological space (X, τ) is said to be α-$\mathrm{T}_{1/2}$ space [15, Definition 5 and Theorem 2.3], if every αg-closed set is α-closed.

Theorem 2.9. *Let* (Y, σ) *be a topological space. Then the following statements are equivalent.*

(i) (Y, σ) *is a* α-$T_{1/2}$ *space,*
(ii) *For every space* (X, τ) *and every map* $f : (X, \tau) \to (Y, \sigma), f$ *is ap-*α*-open.*

Proof. (i)→(ii): Let B be a gα-closed subset of (Y, σ) and suppose that $B \subseteq f(A)$ where $A \in \alpha O(X, \tau)$. Since (Y, σ) is a α-$\mathrm{T}_{1/2}$ space, B is α-closed (i.e., $B = \alpha \, \mathrm{Cl}(B)$). Therefore $\alpha \, \mathrm{Cl}(B) \subseteq f(A)$. Then f is ap-α-open.
(ii)→(i): Let B be a gα-closed subset of (Y, σ) and let X be the set Y with the topology $\tau = \{\emptyset, B, X\}$. Finally let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. By assumption f is ap-α-open. Since B is gα-closed in (X, τ) and α-open in (X, τ)

and $B \subseteq f(B)$, it follows that $\alpha\,\mathrm{Cl}(B) \subseteq f(B) = B$. Hence B is α-closed in (Y, σ). Therefore (Y, σ) is a α-$\mathrm{T}_{1/2}$ space. ∎

As a consequence of Theorem 2.9 [10, Theorem 2.1] and [11, Theorem 2.1] we have:

Corollary 2.2. *Let (Y, σ) be a topological space. Then the following statements are equivalent.*

(i) *(Y, σ) is a semi-$T_{1/2}$ space* [2,18]*,*

(ii) *For every space (X, τ) and every map $f : (X, \tau) \to (Y, \sigma)$, f is ap-α-open.*

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