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Cordial Deficiency

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Abstract. We introduce a new measure of the noncordiality of a graph and discuss a previously defined measure of the noncordiality of a graph. We then calculate the values of these measures for various families of noncordial graphs. We also determine exactly which of the Möbius ladders are cordial.

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1. Introduction and definitions

Cahit [2] introduced cordial graph labelings twenty years ago, and they have remained the focus of a steady if scant stream of papers, for a summary and listing of which see Gallian [3]. We allow graphs to have multiple edges but not loops. A *binary labeling* of a graph G is a function $f : V(G) \to \{0,1\}$. Two real numbers x and y are *roughly equal* if $0 \leq |x-y| \leq 1$. A binary labeling is said to be *friendly* if $|f^{-1}(0)|$ is roughly equal to $|f^{-1}(1)|$. A binary labeling f of G induces a binary labeling $f_e : E(G) \to \{0,1\}$ by $f_e(uv) = f(u) + f(v)$, where the sum is calculated modulo 2. A friendly labeling of G is *cordial* when $|f_e^{-1}(0)|$ is roughly equal to $|f_e^{-1}(1)|$. After the tradition of those who author new methods of graph labeling, Cahit's seminal paper contains a number of theorems concerning the cordiality and noncordiality of various families of graphs, the most salient of which to our purpose is Theorem 1.1.

Theorem 1.1. *The complete graph K_n is cordial if and only if $n \leq 3$.*

We introduce a measure of the noncordiality of a graph, which was inspired by Kotzig's and Rosa's notion of *edge-magic deficiency* [4]. Note that every friendly labeling of a graph G can be made into a cordial labeling of a graph G' by adding no more than $|f_e^{-1}(0) - f_e^{-1}(1)| - 1$ edges between appropriate pairs of vertices so that the number of edges labeled 0 becomes roughly equal to the number of edges labeled 1 in the augmented graph G'. The minimum number of edges, taken over all

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friendly labelings of G, which it is necessary to add in order that G' become cordial is the *cordial edge deficiency* of G, denoted by $ced(G)$. This is essentially the same as Boxwala and Limaye's *index of cordiality* [1]. In fact $i(G) = ced(G) + 1$.

If it is possible to find a binary labeling of G so that $|f_e^{-1}(0)|$ and $|f_e^{-1}(1)|$ are roughly equal, then G can be made cordial by adding no more than $|f^{-1}(0) - f^{-1}(1)| - 1$ vertices, labeled appropriately. The minimum number of vertices, taken over all such binary labelings of G, which it is necessary to add in order that G' become cordial is the *cordial vertex deficiency* of G, denoted by $cvd(G)$. If there is no such binary labeling of G we say that G is *strictly noncordial*, and write $cvd(G) = \infty$.

2. Results

Our first two theorems have to do with the cordial deficiencies of the complete graph.

Theorem 2.1. *The cordial edge deficiency of K_n for $n > 1$ is $\lfloor \frac{n}{2} \rfloor - 1$.*

Proof. Let f be a friendly labeling of K_n and suppose that $n = 2j$. Then $|f^{-1}(0)| = |f^{-1}(1)| = j$. Thus $|f_e^{-1}(0)| = 2\binom{j}{2} = j^2 - j$ and $|f_e^{-1}(1)| = j^2$. The difference between them is j, and so $cev(K_n) = j - 1$. A similar calculation yields the result when $n = 2j + 1$. ∎

Theorem 2.2. *The cordial vertex deficiency of K_n is $j - 1$ if $n = j^2 + \delta$, where $\delta \in \{-2, 0, 2\}$. Otherwise K_n is strictly noncordial.*

Proof. Let f be a binary labeling of K_n. Suppose that $|f^{-1}(0)| = \ell$, so that $|f^{-1}(1)| = n - \ell$. Then $|f_e^{-1}(0)| = \binom{\ell}{2} + \binom{n-\ell}{2}$ and $|f_e^{-1}(1)| = \ell(n - \ell)$. The difference between these is $\left|2\ell^2 - 2n\ell + \binom{n}{2}\right|$. This equals 0 if and only if $\ell = \frac{n \pm \sqrt{n}}{2}$ and it equals 1 if and only if $\ell = \frac{n \pm \sqrt{n \pm 2}}{2}$. Since ℓ is a whole number, we must have $n = j^2 + \delta$ for $\delta \in \{-2, 0, 2\}$ in order to have $|f_e^{-1}(0)|$ roughly equal to $|f_e^{-1}(1)|$. If $\delta = 0$ we have $\ell = \frac{j^2 \pm j}{2}$, in which case the difference between $|f^{-1}(0)|$ and $|f^{-1}(1)|$ is j, so that $cvd(K_n) = j - 1$. Similar calculations yield the result in the other two cases. ∎

We next consider the cordiality of Möbius ladders. The Möbius ladder M_k consists of the cycle C_{2k} (the *canonical* $2k$*-cycle*) with k additional edges (the *cross-edges*) joining opposite pairs of vertices. These graphs have a natural grid-like embedding into the Möbius strip, from which they take their name, and of which it is fruitful to think while reading the proofs which follow.

Lemma 2.1. *If $k \equiv 2 \pmod 4$ then M_k is not cordial.*

Proof. Suppose $k = 4n + 2$. Then M_k has $8n + 4$ vertices and $12n + 6$ edges. If f is a cordial labeling of M_k we have

$$\sum_{uv \in E(M_k)} f_e(uv) = 6n + 3 \equiv 1 \pmod 2$$

Furthermore,

$$\sum_{uv \in E(M_k)} f_e(uv) = 3 \sum_{v \in V(M_k)} f(v) = 12n + 6 \equiv 2 \pmod 2$$

and we have obtained a contradiction. ∎

Theorem 2.3. *If $k \not\equiv 2 \pmod 4$ and $k \geq 3$ then M_k is cordial.*

Proof. The cordiality of M_3 follows from the labeling of the vertices of the canonical 6-cycle with 1, 1, 0, 1, 0, 0 in this order. That of M_4 from 1, 1, 0, 1, 1, 0, 0, 0, and that of M_5 from 1, 1, 1, 1, 0, 1, 0, 0, 0, 0. Now, if we have a cordially labeled M_k which has a cross-edge uv with $f(u) = f(v) = 1$ then it is possible to separate the labeled graph along this edge, obtaining a (non-cordially) labeled copy of $P_2 \times P_k$ in the process, and do the same with a cordially labeled copy of M_4, which can then be grafted into the modified M_k with an appropriate twist, yielding a cordially labeled copy of M_{k+4}. Since all three of the labelings given above have such a cross-edge, the result follows by induction. ∎

Theorem 2.4. *If $k \equiv 2 \pmod 4$ and $k \geq 6$ we have $ced(M_k) = cvd(M_k) = 1$.*

Proof. Consider the friendly labeling of M_6 obtained by assigning the labels 1, 1, 1, 1, 1, 0, 1, 0, 0, 0, 0, 0 in this order to the vertices of the canonical 12-cycle. This labeling has $|f_e^{-1}(0)| = 10$ and $|f_e^{-1}(1)| = 8$. Hence $ced(M_6) = 1$. The fact that $ced(M_k) = 1$ follows by induction exactly as in the proof of Theorem 2.3. The result for $cvd(M_k)$ follows by the same method using the binary labeling 1, 1, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0. ∎

The *wheel graph* W_n is obtained from the cycle C_n (the *canonical n-cycle*) by adding another vertex (the *central vertex*) and joining it to the n vertices of the canonical n-cycle. The n edges incident with the central vertex are called *central edges* and the other n edges are called *cycle edges*. Cahit [2] showed that W_k is cordial if and only if $n \not\equiv 3 \pmod 4$ and that C_n is cordial if and only if $n \not\equiv 2 \pmod 4$.

Theorem 2.5. *If $n \equiv 3 \pmod 4$ then $ced(W_k) = cvd(W_k) = 1$.*

Proof. If $n = 4k+3$ then $|V(W_n)| = 4k+4$ and $|E(W_n)| = 8k+6$. Let f be a friendly labeling of W_n. Note that we may assume without loss of generality that if w is the central vertex then $f(w) = 0$. This leaves $2k+1$ cycle vertices labeled 0 and $2k+2$ labeled 1, which in turn yields $2k+1$ central edges labeled 0 and $2k+2$ labeled 1. Since C_n is cordial, it is possible to arrange the vertex labels on the canonical n-cycle so that $|f_{e^*}^{-1}(0)|$ is roughly equal to $|f_{e^*}^{-1}(1)|$, where f_{e^*} represents the restriction of f_e to cycle edges. This implies that $|f_{e^*}^{-1}(0)| = 2k+1$ and $|f_{e^*}^{-1}(1)| = 2k+2$, since otherwise we'd have a cordial labeling of W_n. Therefore $ced(W_n) = 1$.

Now, if we begin with a cordial labeling of the canonical n-cycle then we may assume without loss of generality that we have $2k+1$ cycle vertices labeled 0 and $2k+2$ labeled 1. As above this implies that we have $2k+1$ cycle edges labeled 0 and $2k+2$ labeled 1. Hence if we label the central vertex with 1, we obtain a binary labeling f in which $|f_e^{-1}(0)| = |f_e^{-1}(1)| = 4k+3$, and in which $|f^{-1}(0)| = |f^{-1}(1)| - 2$. The result follows immediately. ∎

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