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Linear Differential Polynomials Sharing Three Values with Finite Weight

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Abstract. In the paper we study the uniqueness problem of two linear differential polynomials with weighted sharing of three values which improve and supplement a recent result of Lahiri-Banerjee [10].

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1. Introduction and definitions

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have same set of a-points with the same multiplicities, we say that f and g share the value a CM (counting multiplicities) and if we do not consider the multiplicities then f, g are said to share the value a IM (ignoring multiplicities). We do not explain the standard notations and definitions of the value distribution theory as these are available in [3].

Definition 1.1. [10] *We denote by $N(r, a; f| = 1)$ the counting function of simple a-points of f for $a \in \mathbb{C} \cup \{\infty\}$.*

Definition 1.2. [10] *Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $\overline{N}(r, a; f| \geq p)$ the counting function of those distinct a-points of f whose multiplicities are not less than p.*

Let $a_1, a_2, \ldots, a_n (a_n \neq 0)$ be finite complex numbers. In this paper we shall denote by F and G the following two linear differential polynomials unless otherwise stated.

$$F = \sum_{i=1}^{n} a_i f^{(i)} \text{ and } G = \sum_{i=1}^{n} a_i g^{(i)}$$

In [4] the following result is proved.

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Theorem 1.1. [4] *Let f and g be two nonconstant meromorphic functions. If*

(i) *f and g share ∞ CM;*
(ii) *F and G share $0, 1$ CM;*
(iii) $\dfrac{\sum_{a\neq\infty}\delta(a;f)}{1+n(1-\Theta(\infty;f))} - \dfrac{3(1-\Theta(\infty;f))}{2\sum_{a\neq\infty}\delta(a;f)} > \dfrac{1}{2}$,

where $\sum_{a\neq\infty}\delta(a;f) > 0$,

then either (a) $F \equiv G$ or (b) $F.G \equiv 1$. If further, f has at least one pole or F has at least one zero, the case (b) does not arise.

In [1] Fang and Lahiri improved Theorem 1.1. To state their result we require the following definition.

Definition 1.3. [5, 6] *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $1+k$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

With the notion of weighted sharing of values improving Theorem 1.1 the following result was proved in [1].

Theorem 1.2. [1] *Let f and g be two nonconstant meromorphic functions. If*

(i) *f and g share (∞, ∞),*
(ii) *F and G share $(0, 1), (1, \infty)$,*
(iii) $\sum_{a\neq\infty}\delta(a;f) > \frac{1}{2}$,

then either (a) $F \equiv G$ or (b) $F.G \equiv 1$. If, further, f has at least one pole or F has at least one zero, the case (b) does not arise.

Recently Lahiri and Banerjee [10] reduced the weight of sharing values in Theorem 1.2 and proved the following two theorems.

Theorem 1.3. *Let f and g be two nonconstant meromorphic functions. If*

(i) *f and g share $(\infty, 1)$,*
(ii) *F and G share $(0, 1), (1, 6)$,*
(iii) $\sum_{a\neq\infty}\delta(a;f) > \frac{1}{2}$

then either (a) $F \equiv G$ or (b) $F.G \equiv 1$. If, further, f has at least one pole or F has at least one zero, the case (b) does not arise.

Theorem 1.4. *Let f and g be two nonconstant meromorphic functions. If*

(i) *f and g share $(\infty, 0)$,*

(ii) *F and G share $(0,1),(1,6)$, where $F=\sum_{i=1}^{n} a_i f^{(i)}$, $G=\sum_{i=1}^{n} a_i g^{(i)}$ and $n\geq 2$,*

(iii) $\sum_{a\neq\infty} \delta(a;f) > \frac{1}{2}$

then either (a) $F\equiv G$ or (b) $F.G\equiv 1$. If further, f has at least one pole or F has at least one zero, the case (b) does not arise.

In the present paper we investigate the situation of further reducing the weight of the value 1 in the above two theorems.

We now give some more definitions.

Definition 1.4. [2] *For a meromorphic function f we put*

$$T_0(r,f)=\int_1^r \frac{T(t,f)}{t}dt, \quad N_0(r,a;f)=\int_1^r \frac{N(t,a;f)}{t}dt,$$

$$m_0(r,a;f)=\int_1^r \frac{m(t,a;f)}{t}dt, \quad S_0(r,f)=\int_1^r \frac{S(t,f)}{t}dt,$$

etc. where $a\in\mathbb{C}\cup\{\infty\}$

Definition 1.5. [2] *For a meromorphic function f we put*

$$\delta_0(a;f)=1-\limsup_{r\longrightarrow\infty}\frac{N_0(r,a;f)}{T_0(r,f)}=\liminf_{r\longrightarrow\infty}\frac{m_0(r,a;f)}{T_0(r,f)}.$$

Definition 1.6. [6, 9] *Let f,g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly $\overline{N}_*(r,a;f,g)\equiv\overline{N}_*(r,a;g,f)$.*

Definition 1.7. [11] *Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f|\,g=b)$ the counting function of those a-points of f, counted according to multiplicity, which are b-points of g.*

Definition 1.8. [11] *Let $a,b\in\mathbb{C}\cup\{\infty\}$. We denote by $N(r,a;f|\,g\neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.*

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Henceforth we shall denote by H, Φ_1, Φ_2, Φ_3 the following four functions.

$$H=\frac{F''}{F'}-\frac{2F'}{F-1}-\frac{G''}{G'}+\frac{2G'}{G-1},$$

$$\Phi_1=\frac{F'}{F(F-1)}-\frac{G'}{G(G-1)}=(\frac{F'}{F-1}-\frac{G'}{G-1})-(\frac{F'}{F}-\frac{G'}{G}),$$

$$\Phi_2 = \frac{F^{'}}{F-1} - \frac{G^{'}}{G-1} \quad \text{and} \quad \Phi_3 = \frac{F^{'}}{F} - \frac{G^{'}}{G}.$$

Lemma 2.1. [11] *For a meromorphic function f*

$$\lim_{r\longrightarrow\infty} \frac{S_0(r,f)}{T_0(r,f)} = 0$$

through all values of r.

Lemma 2.2. [8, 9] *If F, G share $(0,0),(1,0),(\infty,0)$ then*
(i) $T(r,F) \leq 3T(r,G) + S(r,F)$,
(ii) $T(r,G) \leq 3T(r,F) + S(r,G)$.

Lemma 2.2 shows that $S(r,F) = S(r,G)$ and we denote them by $S(r)$.

Lemma 2.3. [13] *Let F, G share $(0,0),(1,0),(\infty,0)$ and $H \equiv 0$ then F, G share $(0,\infty),(1,\infty),(\infty,\infty)$.*

Lemma 2.4. [7] *Let F, G share $(1,1)$ and $H \not\equiv 0$ then*

$$N(r,1;F| = 1) = N(r,1;G| = 1) \leq N(r,H) + S(r,F) + S(r,G).$$

Lemma 2.5. [14] *Let f, g share $(0,k_1)$, (∞,k_2) and $(1,k_3)$ where $k_j (j = 1,2,3)$ are positive integers satisfying $k_1k_2k_3 > k_1 + k_2 + k_3 + 2$. Then*

$$\overline{N}(r,0;f| \geq 2) + \overline{N}(r,\infty;f| \geq 2) + \overline{N}(r,1;f| \geq 2) = S(r).$$

Lemma 2.6. *Let F, G share $(0,1),(1,2),(\infty,1)$. If $F \not\equiv G$ and $\overline{N}(r,\infty;F) = \overline{N}(r,\infty;G) = S(r)$ then for $a = 0,1$ we get $\overline{N}(r,a;F| \geq 2) = \overline{N}(r,a;G| \geq 2) = S(r)$*

Proof. We prove $\overline{N}(r,a;F| \geq 2) = S(r)$ for $a = 0,1$ because the other can similarly be proved. We suppose that $\overline{N}(r,a;F) \neq S(r)$ for $a = 0,1$ because otherwise the case is trivial. Since $F \not\equiv G$, it follows that $\Phi_i \not\equiv 0$ for $i = 2,3$. Now

$$\begin{aligned} \overline{N}(r,0;F| \geq 2) &\leq N(r,0;\Phi_2) \\ &\leq T(r,\Phi_2) + O(1) \\ &= N(r,\infty;\Phi_2) + S(r) \\ &\leq \overline{N}(r,1;F| \geq 3) + \overline{N}(r,\infty;F| \geq 2) + S(r) \\ &= \overline{N}(r,1;F| \geq 3) + S(r). \end{aligned} \tag{2.1}$$

Again

$$\begin{aligned} 2\,\overline{N}(r,1;F| \geq 3) &\leq \overline{N}(r,1;F| \geq 3) + \overline{N}(r,1;F| \geq 2) \\ &\leq N(r,0;\Phi_3) \\ &\leq N(r,\infty;\Phi_3) + S(r) \\ &\leq \overline{N}(r,0;F| \geq 2) + \overline{N}(r,\infty;F| \geq 2) + S(r) \\ &= \overline{N}(r,0;F| \geq 2) + S(r). \end{aligned} \tag{2.2}$$

From (2.1) and (2.2) we get $\overline{N}(r,0;F| \geq 2) = S(r)$ and hence from (2.2) we get $\overline{N}(r,1;F| \geq 2) = S(r)$. This proves the lemma. ∎

Lemma 2.7. *Let F, G share $(0,1),(1,1),(\infty,2)$. If $F \not\equiv G$ and $\overline{N}(r,1;F|\geq 2) = \overline{N}(r,1;G|\geq 2) = S(r)$ then for $a = 0,\infty$ we get $\overline{N}(r,a;F|\geq 2) = \overline{N}(r,a;G|\geq 2) = S(r)$.*

Proof. We prove $\overline{N}(r,a;F|\geq 2) = S(r)$ for $a = 0,\infty$ because the other can similarly be proved.We suppose that $\overline{N}(r,a;F) \neq S(r)$ for $a = 0,\infty$ because otherwise the case is trivial. Since $F \not\equiv G$, it follows that $\Phi_i \not\equiv 0$ for $i = 1,2$. Now

$$
\begin{aligned}
2\,\overline{N}(r,\infty;F|\geq 3) &\leq \overline{N}(r,\infty;F|\geq 3) + \overline{N}(r,\infty;F|\geq 2) \\
&\leq N(r,0;\Phi_1) \\
&\leq N(r,\infty;\Phi_1) + S(r) \\
&\leq \overline{N}(r,0;F|\geq 2) + \overline{N}(r,1;F|\geq 2) + S(r) \\
&= \overline{N}(r,0;F|\geq 2) + S(r).
\end{aligned}
\tag{2.3}
$$

Again

$$
\begin{aligned}
\overline{N}(r,0;F|\geq 2) &\leq N(r,0;\Phi_2) \\
&= N(r,\infty;\Phi_2) + S(r) \\
&\leq \overline{N}(r,\infty;F|\geq 3) + \overline{N}(r,1;F|\geq 2) + S(r) \\
&= \overline{N}(r,\infty;F|\geq 3) + S(r).
\end{aligned}
\tag{2.4}
$$

From (2.3) and (2.4) we get $\overline{N}(r,\infty;F|\geq 3) = S(r)$.
So from (2.4) we get $\overline{N}(r,0;F|\geq 2) = S(r)$ and hence using (2.3) we have $\overline{N}(r,\infty;F|\geq 2) = S(r)$. This proves the lemma. ∎

Lemma 2.8. *Let F, G share $(0,1),(1,m),(\infty,k)$ and $\overline{N}(r,0;F|\geq 2) = \overline{N}(r,0;G|\geq 2) = S(r)$. If $F \not\equiv G$ and $mk-1 > 0$ then for $a = 1,\infty$ we get $\overline{N}(r,a;F|\geq 2) = \overline{N}(r,a;G|\geq 2) = S(r)$.*

Proof. We prove $\overline{N}(r,a;F|\geq 2) = S(r)$ for $a = 1,\infty$ because the other can similarly be proved. We suppose that $\overline{N}(r,a;F) \neq S(r)$ for $a = 1,\infty$ because otherwise the case is trivial. Since $F \not\equiv G$, it follows that $\Phi_i \not\equiv 0$ for $i = 1,3$. Now

$$
\begin{aligned}
m\,\overline{N}(r,1;F|\geq m+1) &\leq (m-1)\,\overline{N}(r,1;F|\geq 1+m) + \overline{N}(r,1;F|\geq 2) \\
&\leq N(r,0;\Phi_3) \\
&\leq N(r,\infty;\Phi_3) + S(r) \\
&\leq \overline{N}(r,\infty;F|\geq k+1) + \overline{N}(r,0;F|\geq 2) + S(r) \\
&= \overline{N}(r,\infty;F|\geq k+1) + S(r).
\end{aligned}
\tag{2.5}
$$

Also

$$
\begin{aligned}
k\,\overline{N}(r,\infty;F|\geq k+1) &\leq (k-1)\,\overline{N}(r,\infty;F|\geq k+1) + \overline{N}(r,\infty;F|\geq 2) \\
&\leq N(r,0;\Phi_1) \\
&\leq N(r,\infty;\Phi_1) + S(r) \\
&\leq \overline{N}(r,1;F|\geq m+1) + \overline{N}(r,0;F|\geq 2) + S(r) \\
&= \overline{N}(r,1;F|\geq m+1) + S(r).
\end{aligned}
\tag{2.6}
$$

From (2.5) and (2.6) we get $(m-\frac{1}{k})\ \overline{N}(r,1;F|\geq m+1)\leq S(r)$
i.e. $N(r,1;F|\geq m+1)=S(r)$. So from (2.6) we obtain $\overline{N}(r,\infty;F|\geq 2)=S(r)$. Again from (2.5) we get $\overline{N}(r,1;F|\geq 2)=S(r)$.This completes the proof of the lemma. ∎

Lemma 2.9. [12] *If $N(r,0;f^{(k)}|f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)}|f\neq 0)\leq k\overline{N}(r,\infty;f)+N(r,0;f|<k)+k\overline{N}(r,0;f|\geq k)+S(r,f).$$

Lemma 2.10. *For a meromorphic function F*

$$\overline{N}(r,1;F\Big|\geq k+1)\leq \frac{1}{k}\ \overline{N}(r,0;F)+\frac{1}{k}\ \overline{N}(r,\infty;F)-\frac{1}{k}\ N_{\otimes}(r,0;F^{'})+S(r,F),$$

where $N_{\otimes}(r,0;F^{'})$ is the counting functions of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$.

Proof. Using Lemma 2.9 we get

$$\begin{aligned}&\overline{N}(r,1;F|\geq k+1)\\ &\leq \frac{1}{k}\ N(r,0;F^{'}\Big|F=1)\\ &\leq \frac{1}{k}\ N(r,0;F^{'}\Big|F\neq 0)-\frac{1}{k}\ N_{\otimes}(r,0;F^{'})\\ &\leq \frac{1}{k}\ \overline{N}(r,0;F)+\frac{1}{k}\ \overline{N}(r,\infty;F)-\frac{1}{k}\ N_{\otimes}(r,0;F^{'})+S(r,F).\end{aligned}$$

This proves the lemma. ∎

Lemma 2.11. *Let f, g share $(\infty;1)$, F, G share $(0,1),(1,3)$ where $F=\sum\limits_{i=1}^{n}a_i f^{(i)}$, $G=\sum\limits_{i=1}^{n}a_i g^{(i)}$ and $n\geq 2$. If $F\not\equiv G$, $N(r,0;F\Big|=1)=N(r,0;G\Big|=1)=S(r)$ and $\overline{N}(r,\infty;F|\geq 4)=\overline{N}(r,\infty;G|\geq 4)=S(r)$ then for $a=0,1,\infty$ we get $\overline{N}(r,a;F|\geq 2)=\overline{N}(r,a;G|\geq 2)=S(r)$.*

Proof. We prove $\overline{N}(r,a;F|\geq 2)=S(r)$ for $a=0,1,\infty$ because the other can similarly be proved. We suppose that $\overline{N}(r,a;F)\neq S(r)$ for $a=0,1,\infty$ because otherwise the case is trivial. Since $F\not\equiv G$, it follows that $\Phi_i\not\equiv 0$ for $i=1,2,3$. Since f, g share $(\infty;1)$ it follows that F, G share $(\infty,3)$ and F, G has no simple or double pole. i.e. $\overline{N}(r,\infty;F)=\overline{N}(r,\infty;G)=\overline{N}(r,\infty;F|\geq 3)=\overline{N}(r,\infty;G|\geq 3)$.
So from Lemma 2.10 we get

$$\begin{aligned}&3\ \overline{N}(r,\infty;F|\geq 4)+2\ \overline{N}(r,\infty;F|=3)\\ &\leq N(r,0;\Phi_1)\\ &\leq N(r,\infty;\Phi_1)+S(r)\\ &\leq \overline{N}(r,1;F|\geq 4)+\overline{N}(r,0;F|\geq 2)+S(r)\\ &\leq \frac{1}{3}\ N(r,0;F\Big|=1)+\frac{4}{3}\ \overline{N}(r,0;F\Big|\geq 2)\end{aligned}$$

$$+\frac{1}{3}\,\overline{N}(r,\infty;F\big|=3)+\frac{1}{3}\,\overline{N}(r,\infty;F\big|\geq 4)+S(r),$$

i.e.

$$\frac{5}{3}\,\overline{N}(r,\infty;F\big|=3)\leq \frac{4}{3}\,\overline{N}(r,0;F\big|\geq 2)+S(r). \tag{2.7}$$

Again using Lemma 2.10 we get

$$\begin{aligned}\overline{N}(r,0;F|\geq 2) &\leq N(r,0;\Phi_2)\\ &\leq N(r,\infty;\Phi_2)+S(r)\\ &\leq \overline{N}(r,1;F|\geq 4)+\overline{N}(r,\infty;F|\geq 4)+S(r)\\ &\leq \frac{1}{3}\,N(r,0;F\big|=1)+\frac{1}{3}\,\overline{N}(r,0;F\big|\geq 2)\\ &+\frac{1}{3}\,\overline{N}(r,\infty;F\big|=3)+S(r)\\ &\leq \frac{1}{3}\,\overline{N}(r,0;F\big|\geq 2)+\frac{1}{3}\,\overline{N}(r,\infty;F\big|=3)+S(r),\end{aligned}$$

i.e.

$$\overline{N}(r,0;F\big|\geq 2)\leq \frac{1}{2}\;\overline{N}(r,\infty;F\big|=3)+S(r). \tag{2.8}$$

Using (2.8) in (2.7) we get

$$\frac{5}{3}\,\overline{N}(r,\infty;F\big|=3)\leq \frac{2}{3}\,\overline{N}(r,\infty;F\big|=3)+S(r)$$

i.e. $\overline{N}(r,\infty;F|=3)=S(r)$, which implies $\overline{N}(r,\infty;F|\geq 2)=S(r)$. Since $\overline{N}(r,\infty;F)=\overline{N}(r,\infty;G)=\overline{N}(r,\infty;F|\geq 3)$ the remaining part of the lemma follows from Lemma 2.6. This completes the proof of the lemma. ∎

Lemma 2.12. *Let F, G share $(0,1)$, $(1,1)$, $(\infty,1)$ and $\overline{N}(r,\infty;F)=\overline{N}(r,\infty;G)=S(r)$ and $\sum\limits_{a\neq\infty}\delta(a;f)>\frac{1}{2}$ then $\delta_0(0;F)>\frac{1}{2}$.*

Proof. The lemma can be proved in the similar manner as followed in p. 34 [10]. ∎

Lemma 2.13. [9] *Let F, G share $(0,0)$, $(1,0)$, $(\infty,0)$ and $H\not\equiv 0$. Then*

$$\begin{aligned}N(r,H)\leq\ &\overline{N}_*(r,0;F,G)+\overline{N}_*(r,\infty;F,G)+\overline{N}_*(r,1;F,G)\\ &+\overline{N}_{\otimes}(r,0;F^{'})+\overline{N}_{\otimes}(r,0;G^{'}),\end{aligned}$$

where $\overline{N}_{\otimes}(r,0;F^{'})$ is the reduced counting function of those zeros of $F^{'}$ which are not the zeros of $F(F-1)$ and $\overline{N}_{\otimes}(r,0;G^{'})$ is similarly defined.

Lemma 2.14. *Let F, G share $(0,1)$, $(1,1)$, $(\infty,1)$ and $H\not\equiv 0$. If $\overline{N}(r,a;F|\geq 2)=\overline{N}(r,a;G|\geq 2)=S(r)$ for $a=0,1,\infty$ then $\delta_0(0;F)\leq\frac{1}{2}$.*

Proof. Since F, G have only multiple poles, by the second fundamental theorem we get

$$\begin{aligned}T(r,F)+T(r,G)\leq\ &\overline{N}(r,0;F)+\overline{N}(r,1;F)+\overline{N}(r,\infty;F)\\ &+\overline{N}(r,0;G)+\overline{N}(r,1;G)+\overline{N}(r,\infty;G)\end{aligned} \tag{2.9}$$

$$
\begin{aligned}
&- N_{\bigotimes}(r,0;F^{'}) - N_{\bigotimes}(r,0;G^{'}) + S(r)\\
&\leq 2\ \overline{N}(r,0;F) + 2\ \overline{N}(r,\infty;F| \geq 2) + N(r,1;F| = 1)\\
&+ \overline{N}(r,1;F| \geq 2) + T(r,G) - m(r,1;G)\\
&- N_{\bigotimes}(r,0;F^{'}) - N_{\bigotimes}(r,0;G^{'}) + S(r).
\end{aligned}
$$

So using Lemmas 2.4 and 2.13 we note that

$$
\begin{aligned}
(2.10) \qquad N(r,1;F| = 1) &\leq N(r,H) + S(r)\\
&\leq \overline{N}_{*}(r,0;F,G) + \overline{N}_{*}(r,\infty;F,G) + \overline{N}_{*}(r,1;F,G)\\
&+ \overline{N}_{\bigotimes}(r,0;F^{'}) + \overline{N}_{\bigotimes}(r,0;G^{'}) + S(r)\\
&\leq \overline{N}(r,0;F| \geq 2) + \overline{N}(r,\infty;F| \geq 2) + \overline{N}(r,1;F| \geq 2)\\
&+ \overline{N}_{\bigotimes}(r,0;F^{'}) + \overline{N}_{\bigotimes}(r,0;G^{'}) + S(r)\\
&\leq \overline{N}_{\bigotimes}(r,0;F^{'}) + \overline{N}_{\bigotimes}(r,0;G^{'}) + S(r).
\end{aligned}
$$

Combining (2.9) and (2.10) we see that

$$
\begin{aligned}
T(r,F) &\leq 2\ \overline{N}(r,0;F) + S(r)\\
&\leq 2\ N(r,0;F) + S(r).
\end{aligned}
$$

On integration we get $T_0(r,F) \leq 2\ N_0(r,0;F) + S_0(r,F)$
So by Lemma 2.1 we get $\delta_0(0;F) \leq \frac{1}{2}$. This proves the lemma. ∎

3. Theorems

This section discusses the main result of the paper.

Theorem 3.1. *Let f and g be two nonconstant meromorphic functions. If*

(i) *f and g share $(\infty,0)$,*
(ii) *F and G share $(0,2),(1,3)$,*
(iii) $\sum\limits_{a\neq\infty} \delta(a;f) > \frac{1}{2}$,

then either (a) $F \equiv G$ or (b) $F.G \equiv 1$. If, further, f has at least one pole or F has at least one zero, the case (b) does not arise.

Theorem 3.2. *Let f and g be two nonconstant meromorphic functions. If*

(i) *f and g share $(\infty,0)$,*
(ii) *F and G share $(0,2),(1,3)$, where $F = \sum\limits_{i=1}^{n} a_i f^{(i)}$, $G = \sum\limits_{i=1}^{n} a_i g^{(i)}$ and $n \geq 2$,*
(iii) $\sum\limits_{a\neq\infty} \delta(a;f) > \frac{1}{2}$,

then either (a) $F \equiv G$ or (b) $F.G \equiv 1$. If, further, f has at least one pole or F has at least one zero, the case (b) does not arise.

Theorem 3.3. *Let f and g be two nonconstant meromorphic functions. If*

(i) *f and g share $(\infty,0)$,*
(ii) *F and G share $(0,1),(1,2)$,*
(iii) $\sum\limits_{a\neq\infty} \delta(a;f) > \frac{1}{2}$,

(iv) $\overline{N}(r,\infty;F)=\overline{N}(r,\infty;G)=S(r)$,

then either (*a*) $F\equiv G$ *or* (*b*) $F.G\equiv 1$. *If, further,* f *has at least one pole or* F *has at least one zero, the case* (*b*) *does not arise.*

Theorem 3.4. *Let* f *and* g *be two nonconstant meromorphic functions. If*

(i) f *and* g *share* $(\infty,1)$,
(ii) F *and* G *share* $(0,1),(1,1)$,
(iii) $\sum\limits_{a\neq\infty}\delta(a;f)>\frac{1}{2}$,
(iv) $\overline{N}(r,1;F|\geq 2)=\overline{N}(r,1;G|\geq 2)=S(r)$,

then either (*a*) $F\equiv G$ *or* (*b*) $F.G\equiv 1$. *If, further,* f *has at least one pole or* F *has at least one zero, the case* (*b*) *does not arise.*

Theorem 3.5. *Let* f *and* g *be two nonconstant meromorphic functions. If*

(i) f *and* g *share* $(\infty,k-1)$,
(ii) F *and* G *share* $(0,1),(1,m)$, *where* $mk-1>0$,
(iii) $\sum\limits_{a\neq\infty}\delta(a;f)>\frac{1}{2}$,
(iv) $\overline{N}(r,0;F|\geq 2)=\overline{N}(r,0;G|\geq 2)=S(r)$,

then either (*a*) $F\equiv G$ *or* (*b*) $F.G\equiv 1$. *If, further,* f *has at least one pole or* F *has at least one zero, the case* (*b*) *does not arise.*

Theorem 3.6. *Let* f *and* g *be two nonconstant meromorphic functions. If*

(i) f *and* g *share* $(\infty,1)$,
(ii) F *and* G *share* $(0,1),(1,3)$, *where* $F=\sum\limits_{i=1}^{n}a_if^{(i)}$, $G=\sum\limits_{i=1}^{n}a_ig^{(i)}$ *and* $n\geq 2$,
(iii) $\sum\limits_{a\neq\infty}\delta(a;f)>\frac{1}{2}$,
(iv) $N(r,0;F|=1)=N(r,0;G|=1)=S(r)$ *and* $\overline{N}(r,\infty;F|\geq 4)=\overline{N}(r,\infty;G|\geq 4)=S(r)$,

then either (*a*) $F\equiv G$ *or* (*b*) $F.G\equiv 1$. *If, further,* f *has at least one pole or* F *has at least one zero, the case* (*b*) *does not arise.*

Example 3.1. Let $f=\frac{1}{2}e^{z}\left(e^{z}-1\right)$, $g=\frac{1}{2}e^{-z}\left(\frac{1}{2}-\frac{1}{5}e^{-z}\right)$ Then $F=f^{''}-3f^{'}=e^{z}\left(1-e^{z}\right)$ $G=g^{''}-3g^{'}=e^{-z}\left(1-e^{-z}\right)$. Clearly F, G share $(0,\infty)$, $(1,\infty)$ and f, g share (∞,∞). Also $\delta(0;f)=\sum\limits_{a\neq\infty}\delta(a;f)=\frac{1}{2}$ and $\overline{N}(r,\infty;F)=\overline{N}(r,\infty;F|\geq 2)=\overline{N}(r,1;F|\geq 2)=\overline{N}(r,0;F|\geq 2)=0$. Hence we see that the condition (iii) in Theorems 3.1–3.5 is sharp.

4. Proofs of the theorems

Proof of Theorem 3.1. Since f, g share $(\infty,1)$, clearly F, G share $(\infty,2)$. Suppose $H\not\equiv 0$ then $F\not\equiv G$. So by Lemma 2.5 and Lemma 2.14 $\delta_0(0;F)\leq\frac{1}{2}$. But by Lemma 2.12 this leads to a contradiction. So $H\equiv 0$. Hence by Lemma 2.3 F, G share $(0,\infty)$, $(1,\infty)$, $(\infty;\infty)$. Now the theorem follows from Theorem 1.2. This proves the theorem. ∎

Proof of Theorem 3.2. Noting that F and G have no simple and double pole the theorem can be proved in the line of the proof of Theorem 3.1. This proves the theorem. ▮

Proof of Theorem 3.3. Since f, g share $(\infty, 0)$, it follows that F, G share $(\infty, 1)$. Suppose $H \not\equiv 0$ then $F \not\equiv G$. So by Lemma 2.6 and Lemma 2.14 $\delta_0(0; F) \leq \frac{1}{2}$. But by Lemma 2.12 we have a contradiction. So $H \equiv 0$. Hence the theorem follows from Lemma 2.3 and Theorem 1.2. This proves the theorem. ▮

Proof of Theorem 3.4. Clearly F, G share $(\infty, 2)$. Suppose $H \not\equiv 0$ then $F \not\equiv G$. Using Lemma 2.7 and Lemma 2.14 we have $\delta_0(0; F) \leq \frac{1}{2}$. Now the theorem can be proved in the line of the proof of Theorem 3.3. This proves the theorem. ▮

Proof of Theorem 3.5. It is clear from the given condition of the theorem F, G share (∞, k). Suppose $H \not\equiv 0$ then $F \not\equiv G$.Using Lemma 2.8 and Lemma 2.14 we get $\delta_0(0; F) \leq \frac{1}{2}$. Now proceeding in the same way as done in Theorem 3.3 we can prove the theorem. This completes the proof of the theorem. ▮

Proof of Theorem 3.6. According to the hypothesis F, G share $(\infty, 3)$. Suppose $H \not\equiv 0$ then $F \not\equiv G$. Using Lemma 2.11 and Lemma 2.14 we get $\delta_0(0; F) \leq \frac{1}{2}$. Now proceeding in the same manner as done in Theorem 3.3 we can prove the theorem. This completes the proof of the theorem. ▮

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