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On Strongly Prime Semiring

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Abstract. In this paper we introduce the notions of strongly prime ideal in a semiring and strongly prime semiring and study some of its properties. Also we characterize strongly prime radical by using the notion of super sp-system.

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1. Introduction

In 1975, Handelman and Lawrence [4] introduced the notion of (right) strongly prime ring motivated by the notion of primitive group ring and characterized strongly prime rings. According to Handelman and Lawrence, a ring is (right) strongly prime if for each nonzero element r of R, there is a finite subset $S(r)$(right insulator for r) of R such that for $t \in R$, $\{rst : s \in S(r)\} = \{0\}$ implies $t = 0$.

In this paper we generalize this notion for an arbitrary semiring and study some of its properties. In Section 2, we give some basic definitions and results. In Section 3, we introduce the notion of right strongly prime semiring and study some of its properties. In Section 4, we define super sp-system and characterize strongly prime radical by using the notion of super sp-system.

2. Preliminaries

Definition 2.1. *A nonempty set S is said to form a semiring with respect to two binary compositions, addition $(+)$ and multiplication $(.)$ defined on it, if the following conditions are satisfied.*

(i) *$(S, +)$ is a commutative semigroup with zero,*
(ii) *$(S, .)$ is a semigroup,*
(iii) *for any three elements $a, b, c \in S$ the left distributive law $a.(b+c) = a.b+a.c$ and the right distributive law $(b + c).a = b.a + c.a$ both hold and*
(iv) *$s.0 = 0.s = 0$ for all $s \in S$.*

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If S contains the multiplicative identity 1, then S is called a semiring with identity.

Definition 2.2. *A nonempty subset I of a semiring S is called a left ideal of S if* (i) $a, b \in I$ *implies* $a + b \in I$ *and* (ii) $a \in I$, $s \in S$ *implies* $s.a \in I$.

Similarly we can define right ideal of a semiring. A nonempty subset I of a semiring S is an ideal if it is a left ideal as well as a right ideal of S.

Definition 2.3. [5] *An ideal I of a semiring S is called a k-ideal if $b \in S$, $a+b \in I$ and $a \in I$ implies $b \in I$.*

Definition 2.4. [4] *Let A be a nonempty subset of a semiring S. Right annihilator of A in S, denoted by $ann_R(A)$, is defined by $ann_R(A) = \{s \in S : As = (0)\}$.*

Lemma 2.1. *$ann_R(A)$ is a right k-ideal of S.*

Definition 2.5. [1] *A semiring S is called a prime semiring if for any two ideals H and K of S and $HK = (0)$ implies either $H = (0)$ or $K = (0)$.*

Definition 2.6. *A non-trivial commutative semiring S is called a semiintegral domain if $ab = 0$ implies either $a = 0$ or $b = 0$.*

Throughout this paper S^* denotes the set of all nonzero elements of S, i.e. $S^* = S - \{0\}$.

3. Right strongly prime semiring

Definition 3.1. *Let $r \in S^*$. Right insulator for r is a finite subset $S(r)$ of S such that $ann_R(\{rs : s \in S(r)\}) = (0)$.*

Definition 3.2. *A semiring S is called a right strongly prime if every nonzero element of S has a right insulator. That is, for each $r \in S^*$ there is a finite subset $S(r)$ of S such that for $t \in S$, $\{rst : s \in S(r)\} = \{0\} \Rightarrow t = 0$.*

Proposition 3.1. *A right strongly prime semiring is prime.*

Proof. Suppose S is a right strongly prime semiring and $HK = (0)$, where H and K are two ideals of S. Suppose $H \neq (0)$. Then there exists an element $a \in H$ such that $a \neq 0$. Since a is a nonzero element of S, a has a right insulator $S(a)$. Let $b \in K$. Then $aS(a)b \subseteq Hb \subseteq HK = (0)$. Now $aS(a)b = (0)$ implies $b = 0$. Therefore $K = (0)$. Hence S is a prime semiring. ∎

Theorem 3.1. *A semiring S with identity is right strongly prime if and only if every nonzero ideal of S contains a finitely generated left ideal whose right annihilator is zero.*

Proof. Suppose S is a right strongly prime semiring and I is a nonzero ideal of S. Let $r(\neq 0) \in I$. So $r \in S^*$. Since S is a right strongly prime semiring, so r has a right insulator $S(r)$(say). Then $rS(r) \subseteq I$. Here $rS(r)$ is finite. Let L be the left ideal of S generated by $rS(r)$, i.e. $L = SrS(r)$. So $L \subseteq I$. Let $Lt = 0$. Now $rS(r)t \subseteq SrS(r)t = Lt = 0$ (since $1 \in S$). Now $rS(r)t = 0 \Longrightarrow t = 0$, since $S(r)$ is a right insulator for r. Thus I contains the finitely generated left ideal L whose right annihilator is zero.

Conversely, suppose the condition holds. Let $r \in S^*$. Now $\langle r \rangle$ is a nonzero ideal of S. By the given condition there exists a finite subset F of $\langle r \rangle$, such that right annihilator of the left ideal L generated by F is zero. Since $F \subseteq \langle r \rangle$, so the elements of the set F are of the form $\sum r'_i r r_i$, where r'_i or r_i may be equal to 1. We construct a set $S(r)$ in such a way that if $r'_1 r r_1 + r'_2 r r_2 + ... + r'_m r r_m \in F$ then $r_1, r_2, ... r_m \in S(r)$. So let $S(r) = \{r_1, r_2, ... r_k\}$. We now prove that $S(r)$ is a right insulator for r. Let $rS(r)t = 0$, i.e. $r r_i t = 0$ for $i = 1, 2, ..., k$. Then $SFt = 0$. Since right annihilator of $L(= SF)$ is zero, so $SFt = 0 \Longrightarrow t = 0$. Therefore $rS(r)t = 0 \Longrightarrow t = 0$. Thus $S(r)$ is a right insulator for r. Hence S is a right strongly prime semiring. ∎

Theorem 3.2. *A semiring S is right strongly prime if and only if every nonzero ideal of S contains a finite subset G such that $ann_R(G) = (0)$.*

Proof. Suppose S is a right strongly prime semiring and I is any nonzero ideal of S. Let $a(\neq 0) \in I$. Since S is a right strongly prime semiring, a has a right insulator F . Let $G = aF$. Then G is a finite subset of I and $ann_R(G) = (0)$.

Conversely, suppose that every nonzero ideal of S contains a finite subset whose right annihilator is zero. Let a be any nonzero element of S. Then the ideal $\langle a \rangle$ contains a finite subset G such that $ann_R(G) = (0)$. Now the elements of G are of the forms $\sum x_i a x'_i, a x''_i, x'''_i a$ or na. Also $ay \neq 0$ for some $y \in S$, otherwise $Gy = 0$ for all nonzero elements of S, a contradiction, since $ann_R(G) = (0)$. Let us consider the ideal $\langle ay \rangle$ of S. Then by condition there exists a finite subset H of $\langle ay \rangle$ such that $ann_R(H) = (0)$. Here elements of H are of the forms $\sum x'_i a y x_i, a y x''_i, x'''_i a y$ or nay. We construct a set H' from the elements of H in such a way that if $x'_1 a y x_1 + x'_2 a y x_2 + ... + x'_m a y x_m \in H$ then $ayx_1, ayx_2, ... ayx_m \in H'$, also $ay \in H'$. So let $H' = \{ay, ayx_1, ayx_2, ..., ayx_k\}$. Then $ann_R(H') = (0)$. [Suppose $ann_R(H') \neq (0)$. Then there exists $x(\neq 0)$ such that $x \in ann_R(H')$. Then $ayx = 0 = ayx_i x$ for $i = 1, 2, ..., k$. So $Hx = 0$, a contradiction, since $ann_R(H) = (0)$]. Let $F = \{y, yx_1, yx_2, ..., yx_k\}$. Then F is a right insulator for a. Hence S is a right strongly prime semiring. ∎

Proposition 3.2. *Every simple semiring with unity is right strongly prime.*

Proof. Let S be a simple semiring with unity 1. Since S is simple, so S is the only nonzero ideal of S. Now $G = \{1\}$ is a finite subset of S and $ann_R(G) = (0)$ which implies that S is a right strongly prime semiring. ∎

Definition 3.3. [2] *A class ρ of semirings is called hereditary if I is an ideal of a semiring S and $S \in \rho$ implies $I \in \rho$.*

Proposition 3.3. *The class of all right strongly prime semirings is hereditary.*

Proof. Let S be a right strongly prime semiring and I be an ideal of S. We now prove that I is a right strongly prime semiring. Let a be a nonzero element of I. If $aI = (0)$ then for any finite subset G of S we have $(0) \neq I \subseteq ann_R(aG)[x \in I \Rightarrow aGx \subseteq aI = (0)$, so $x \in ann_R(aG)]$, a contradiction by Theorem 3.2, since S is a right strongly prime semiring. Hence there exists an element $y \in I$ such that $ay \neq 0$. Then by Theorem 3.2 there exists a finite subset H of $\langle ay \rangle$ such that $ann_R(H) = (0)$. As in the proof of the Theorem 3.2 we may assume $H = \{ay, ayx_1, ayx_2, ..., ayx_k\}$ for some $x_1, x_2, ..., x_k \in S$. Hence the subset $F = \{y, yx_1, yx_2, ..., yx_k\}$ of I is a right insulator for $a \in I$. Hence I is a right strongly prime semiring. ∎

Definition 3.4. *When a particular finite subset F of a semiring S is a insulator for every nonzero element of S, then F is called a uniform insulator for S. If S contains a uniform insulator then S is called a uniformly strongly prime semiring.*

Definition 3.5. *A semiring S is said to be a bounded right strongly prime semiring of bound n (denoted by $SP_r(n)$) if each nonzero element of S has an insulator containing not more than n elements and at least one element has no insulator with fewer than n elements. Here n is called the uniform bound of S.*

Proposition 3.4. *A semi-integral domain is a bounded right strongly prime of bound 1.*

Proof. For each nonzero element r of a semi-integral domain, $\{r\}$ is a right insulator for r. ∎

Definition 3.6. *A semiring S is said to satisfy the descending chain condition (DCC) for left (right) ideals if for each sequence of left (right) ideals $A_1, A_2, \ldots$ of S with $A_1 \supseteq A_2 \supseteq \ldots$, there exists a positive integer n (depending on the sequence) such that $A_n = A_{n+1} = \ldots$.*

Proposition 3.5. *If S is a prime semiring with DCC on right annihilators, then S is a right strongly prime semiring.*

Proof. Let $s \in S^*$ and consider the collection of right annihilator ideals of the form $ann_R(\{sr : r \in I\})$ where I runs over all finite subsets of S. By the condition, above collection has a minimal element M and let I be the corresponding finite subset. We now prove that $M = (0)$. If possible let $M \neq (0)$ and $m(\neq 0) \in M$. Since S is prime there exists an element $q \in S$ such that $sqm \neq 0$. Let $I' = I \cup \{q\}$ and $M' = ann_R(\{sr : r \in I'\})$. Then $M' \nsubseteq M$, a contradiction. So $M = (0)$. Therefore I is a right insulator for s. Thus every nonzero element of S has a right insulator. Hence S is a right strongly prime semiring. ∎

Proposition 3.6. *If S is an $n \times n$ matrix semiring over a semi-integral domain, then S is bounded strongly prime of bound n. Also S has uniform bound n^2.*

Proof. Let $A = (a_{ij})_{n\times n} \in S^*$. Then at least one $a_{ij} (1 \leq i, j \leq n)$ is nonzero. Suppose $a_{pq} \neq 0$. We now prove that $\{E_{qi}\}_{i=1}^{n}$ is a right insulator for a_{pq}, where E_{ij} denotes the $n \times n$ elementary matrix with (i, j)-th component 1 and all other elements is zero. Let $B = (b_{ij})_{n\times n}$. Now $AE_{qi}B = 0 \Longrightarrow a_{pq}b_{ij} = 0$ for $1 \leq j \leq n \Longrightarrow b_{ij} = 0 (1 \leq j \leq n)$, since S is a matrix semiring over a semi-integral domain, which shows that $\{E_{qi}\}_{i=1}^{n}$ is a right insulator for A. Also the element $E_{11} \in S^*$ has an insulator $\{E_{1j}\}_{j=1}^{n}$ and no insulator of E_{11} contains less than n elements. Hence S is a bounded strongly prime semiring of bound n. Here $\{E_{ij}\}_{i,j=1}^{n}$ is a uniform insulator of every element of S. So S has uniform bound n^2. ∎

Remark 3.1. If D is a division semiring then $M_n(D)$ is a bounded strongly prime with bound exactly n and $M_n(D)$ is also uniformly strongly prime of bound n^2.

Proposition 3.7. *Right strongly primeness is a Morita invariant.*

Proof. First we prove that if e is a nonzero element of a semiring S, then eSe is a subsemiring of S. Clearly eSe is a nonempty subset of S. Let $es_1e, es_2e \in eSe$.

Then $es_1e + es_2e = e(s_1 + s_2)e \in S$ and $(eSe)(eSe) \subseteq eSe$. So eSe is a subsemiring of S. Again let S be a right strongly prime semiring. We now prove that eSe is a right strongly prime semiring. Let $ese \in (eSe)^*$. Then $ese \in S^*$. Since S is a right strongly prime semiring, so ese has a right insulator $\{f_i\}$ (say) in S,
i.e $esef_it = 0$ for all $i, t \in S \Longrightarrow t = 0$....(1).
Then $\{ef_ie\}$ is a right insulator for ese in eSe, since $(ese)(ef_ie)(ete) = 0$, where $ete \in eSe \Longrightarrow esef_iete = 0$ for all $i \Longrightarrow ete = 0$ by (1). So eSe is a right strongly prime semiring.

Now we prove that if S is a right strongly prime semiring then $M_n(S)$ is also a right strongly prime semiring. Let B be a nonzero matrix in $M_n(S)$ and let its (p, q)-th component b_{pq} is nonzero. Let $\{t_k\}$ be a right insulator for b_{pq} in S. Let A be a nonzero matrix with nonzero (i, j)-th component a_{ij}. Then $b_{pq}t_ka_{ij} \neq 0$ for some $t_k \in \{t_k\}$. Now (p, j)-th component of $B(t_kE_{qj})A$ is $b_{pq}t_ka_{ij}$. So $A \neq 0 \Longrightarrow B(t_kE_{qj})A \neq 0$ for some $t_k \in \{t_k\}$. Contrapositively, $B(t_kE_{qj})A = 0$ for all $t_k \in \{t_k\} \Longrightarrow A = 0$. Therefore $\{t_ke_{ij}\}_{i,j,k}$ is a right insulator for B. Hence $M_n(S)$ is a right strongly prime semiring. ∎

Definition 3.7. [2] *A nonzero ideal I of a semiring S is called an essential ideal of S if for any nonzero ideal J of S, $I \cap J \neq (0)$.*

Definition 3.8. *Singular ideal of a semiring S is (denoted by $Z(S)$) the ideal composed of elements whose right annihilator is an essential right ideal.*

Proposition 3.8. *The singular ideal of a right strongly prime semiring is zero.*

Proof. Suppose S is a right strongly prime semiring and $s \in Z(S) \cap S^*$. Since S is a right strongly prime semiring, so s has a right insulator $\{s_i\}_{i=1}^{k}$(say). Since $Z(S)$ is an ideal, $ss_i \in Z(S)$. So right annihilator E_i of ss_i is an essential right ideal (i.e. $ss_iE_i = (0)$). Thus $ss_i(\cap_{j=1}^{k} E_j) \subseteq ss_iE_i = (0)$ for all i, since E_i is the right insulator for ss_i. We know the intersection of finitely many essential ideals is nonzero i.e. $\cap_{j=1}^{k} E_j \neq (0)$. This contradicts the fact that $\{s_i\}_{i=1}^{k}$ is a right insulator for s. Hence $Z(S) \cap S^* = \phi$ i.e. $Z(S) = (0)$. ∎

Definition 3.9. *An ideal I of a semiring S is said to be right strongly prime if $a \notin I$, then there is a finite set $F \subseteq \langle a \rangle$ such that $Fb \subseteq I \Longrightarrow b \in I$.*

Definition 3.10. [3] *Let I be a proper ideal of a semiring S. Then the congruence on S, denoted by ρ_I and defined by $s\rho_I s'$ if and only if $s + a_1 = s' + a_2$ for some $a_1, a_2 \in I$, is called the Bourne congruence on S defined by the ideal I.*

We denote the Bourne congruence (ρ_I) class of an element r of S by r/ρ_I or simply by r/I and denote the set of all such congruence classes of S by S/ρ_I or simply by S/I.

It should be noted that for any $s \in S$ and for any proper ideal I of S, s/I is not necessarily equal to $s + I = \{s + a : a \in I\}$ but surely contains it.

Definition 3.11. [3] *For any proper ideal I of S if the Bourne congruence ρ_I, defined by I, is proper i.e. $0/I \neq S$ then we define the addition and multiplication on S/I by $a/I + b/I = (a + b)/I$ and $(a/I)(b/I) = (ab)/I$ for all $a, b \in S$. With these two operations S/I forms a semiring and is called the Bourne factor semiring or simply the factor semiring.*

Proposition 3.9. *A k-ideal I of a semiring S is a right strongly prime ideal if and only if S/I is a right strongly prime semiring.*

Proof. Suppose I is a right strongly prime ideal of S. Let P/I be any nonzero ideal of S/I. Then there exists $a/I(\neq 0/I) \in S/I$ such that $a/I \in P/I$. Then $a \notin I$. Since I is a right strongly prime ideal, so there exists a finite subset F of $\langle a \rangle$ such that $Fb \subseteq I \Longrightarrow b \in I$. Now F/I is a finite subset of P/I. Let $F/I.b/I = 0/I$. Then $Fb/I = 0/I \Longrightarrow Fb \subseteq I$, since I is a k-ideal $\Longrightarrow b \in I$ i.e. $b/I = 0/I$. So by Theorem 3.2 S/I is a right strongly prime semiring.

Conversely, suppose S/I is a right strongly prime semiring and let $a \notin I$. Then $a/I \neq 0/I$ and $\langle a \rangle/I$ is a nonzero ideal of S/I. So there exists a finite subset F/I of $\langle a \rangle/I$ such that right annihilator of F/I is zero. Let $F/I = \{f_1/I, f_2/I, ..., f_k/I\}$ and $F^* = \{f_1, f_2, ..., f_k\}$. Then F^* is a finite subset of $\langle a \rangle$. Suppose $F^*b \subseteq I$. Then $F^*b/I = 0/I$ i.e. $F^*/I.b/I = 0/I \Longrightarrow F/I.b/I = 0/I$ which implies that $b/I = 0/I$ i.e. $b \in I$. Hence I is a right strongly prime ideal of S. ∎

4. Right strongly prime radical

Definition 4.1. *A subset G of a semiring S is called an sp-system if for any $g \in G$ there is a finite subset $F \subseteq \langle g \rangle$ such that $Fz \cap G \neq \phi$ for all $z \in G$.*

Proposition 4.1. *An ideal I of a semiring S is a right strongly prime if and only if $S - I$ is an sp-system.*

Proof. Let I be a right strongly prime ideal of S. Let $g \in S - I$. Then $g \notin I$. So there exists a finite subset F of $\langle g \rangle$ such that $Fb \subseteq I$ implies $b \in I$ i.e. $Fz \cap (S - I) \neq \phi$ for all $z \in S - I$.

Conversely, suppose $S - I$ is an sp-system. Let $a \notin I$. Then $a \in S - I$. So there exists a finite subset F of $\langle a \rangle$ such that $Fz \cap (S - I) \neq \phi$ for all $z \in S - I$. Let $Fb \subseteq I$. Then $Fb \cap (S - I) = \phi$. If possible let $b \notin I$. Then $b \in S - I$ which implies that $Fb \cap (S - I) \neq \phi$, a contradiction. Hence $b \in I$. Therefore I is a right strongly prime ideal of S. ∎

Definition 4.2. *Right strongly prime radical of a semiring S is defined by $SP(S) = \cap\{I : I$ is a right strongly prime ideal of $S\}$.*

Definition 4.3. *A pair of subsets (G, P) where P is an ideal of a semiring S and G is a nonempty subset of S is called a super sp-system of S if $G \cap P$ contains no nonzero element of S and for any $g \in G$ there is a finite subset F of $\langle g \rangle$ such that $Fz \cap G \neq \phi$ for all $z \notin P$.*

Remark 4.1. An ideal I of a semiring S is a right strongly prime ideal if and only if $(S - I, I)$ is a super sp-system of S.

Theorem 4.1. *For any semiring S, $SP(S) = \{x \in S :$ whenever $x \in G$ and (G, P) is a super sp-system for some ideal P of S then $0 \in G\}...(*)$.*

Proof. Let $x \in SP(S)$. If possible let $x \in G$ where (G, P) is a super sp-system and $0 \notin G$. Then $G \cap P = \phi$. By Zorn's lemma choose an ideal Q with $P \subseteq Q$ and Q is maximal with respect to $G \cap Q = \phi$. We now prove that Q is a right strongly prime ideal of S. Let $a \notin Q$. Then there is a $g \in G$ such that $\langle g \rangle \subseteq Q + \langle a \rangle$. Since (G, P)

is a super sp-system there exists a finite subset $F = \{f_1, f_2, ..., f_k\} \subseteq \langle g \rangle$ such that $Fz \cap G \neq \phi$ for all $z \notin P$. Since $F \subseteq Q + \langle a \rangle$ each f_i is of the form $f_i = q_i + a_i$ for some $q_i \in Q$ and $a_i \in \langle a \rangle$. Let $F^* = \{a_1, a_2, ..., a_k\}$. Then $F^* \subseteq \langle a \rangle$. Let $z \in S$ such that $F^*z \subseteq Q$. Then $Fz \subseteq Q$. If $z \notin Q$ then $Fz \cap G \neq \phi$, because $P \subseteq Q$. But this contradicts $G \cap Q = \phi$. Hence $z \in Q$ must hold. So Q is a right strongly prime ideal. But $x \notin Q$, since $x \in G$, which is a contradiction. Hence $0 \in G$.

Conversely, suppose x belongs to the R.H.S. of $(*)$. If possible, let $x \notin SP(S)$. Then there exists a right strongly prime ideal I of S such that $x \notin I$. Then $(S - I, I)$ is a super sp-system where $x \in S - I$ but $0 \notin S - I$, a contradiction. Hence the converse inclusion follows. ∎

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