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Time-dependent Backward Stochastic Evolution Equations

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Abstract. We consider the following infinite dimensional backward stochastic evolution equation:

$$\begin{cases} -dY(t) = (A(t)\,Y(t) + f(t, Y(t), Z(t)))\,dt - Z(t)\,dW(t), \\ Y(T) = \xi, \end{cases}$$

where $A(t), t \geq 0$, are unbounded operators that generate a strong evolution operator $U(t, r),\ 0 \leq r \leq t \leq T$. We prove under non-Lipschitz conditions that such an equation admits a unique evolution solution. Some examples and regularity properties of this solution are given as well.

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1. Introduction

The aim of this paper is to study the following class of infinite dimensional equations:

$$\begin{cases} -\,dY(t) = (A(t)\,Y(t) + f(t, Y(t), Z(t)))\,dt - Z(t)\,dW(t), \\ Y(T) = \xi, \end{cases} \tag{1.1}$$

which we shall call briefly *backward stochastic evolution equations* and denote them by BSEEs. Here W is a cylindrical Wiener process on a separable Hilbert space H, and the operators $A(t), t \geq 0$, are unbounded linear operators on a separable Hilbert space K, depending measurably on t and generate a strong evolution operator $U(t, r),\ 0 \leq r \leq t \leq T$. The main hypothesis is the following: $\exists\ k > 0$ such that $\forall\, y, y' \in K$ and $\forall\ z, z' \in L_2(H; K)$

$$|f(t, y, z) - f(t, y', z')|^2 \leq c(|y - y'|^2) + k\ |z - z'|^2,$$

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where c is a continuous and nondecreasing concave function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $c(0) = 0,\ c(x) > 0$ for $x > 0$ and

$$\int_{0_+}^{a} \frac{dx}{c(x)} = \infty,$$

for any sufficiently small $a > 0$; cf. the hypotheses (H1) and (H2) in Section 3. The space $L_2(H;K)$ is the Hilbert space of all Hilbert-Schmidt operators from H to K endowed with the following norm: $|\Phi|_{L_2(H;K)} = (\sum_{j=1}^{\infty} |\Phi\ e_j|_K^2)^{1/2}$, for any arbitrary orthonormal base $\{e_j\}_{j=1}^{\infty}$ of H. This hypothesis is weaker than being f globally Lipschitz.

Our aim is to look for a pair (Y, Z) of progressively measurable processes taking values in $K \times L_2(H;K)$ such that

$$\begin{aligned} Y(t) &= U(T,t)\,\xi + \int_t^T U(s,t)\ f(s, Y(s), Z(s))\, ds \\ &\quad - \int_t^T U(s,t)\ Z(s)\, dW(s),\ 0 \le t \le T. \end{aligned} \tag{1.2}$$

In [10] equations of the type (1.1) with $A(t), t \geq 0$, taken as a second order differential operator A, i.e. $A(t) = A$ for each $t \geq 0$, were studied when the mapping f is globally Lipchitz. See for instance the BSEE (3.5) below. Thus our results here generalize those in [10]. This time independent case under non-Lipschitz conditions was studied in details in [1], which appeared also in the recent work of Mahmoudov and McKibben in [12] with essentially the same proof of the existence part.

BSEEs somehow can be looked at as a generalization of the usual backward stochastic differential equations introduced by Pardoux and Peng in [16] and also those in [3]. On the other hand, BSEEs should be linked with the study of infinite dimensional PDEs as suggested in [17] and as seen from the work in [3] and [5]. Moreover BSEEs are useful in studying stochastic Hamilton-Jacobi-Bellman equations as gleaned from the work in [19]. An application of the so-called backward stochastic partial differential equations is given in the Example 3.3 in Section 3.

We should remark at this point that the presence of the operator $A(t), t \geq 0$, prevents, in general, the solution Y of (1.1) from being a semimartingale. This is due to the unboundedness of $A(t)$ as operators. However in Section 4 we shall show that the solution Y of (1.1) is continuous. On the other hand, by adding more regularity conditions on the mappings ξ and f and A (when A is time independent) it was shown in [4] that the solution (Y, Z) of (1.1) is a strong solution and not just evolution. This makes Y a semimartingale. See also the discussion following Theorem 3.1. Independently, other restrictive conditions on $A(\cdot)$ are assumed in [5] to show also that such a solution Y is a semimartingale.

The outline of the paper is as follows. In Section 2 we collect some necessary preliminary information on Wiener processes and stochastic integration. Section 3 is devoted to establishing the proof of existence and uniqueness of the solution of the BSEE (1.1). Some examples are given in Section 3. Then in Section 4 we provide some regularity properties of these solutions.

2. Notations and preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Denote by $\mathcal{N}$ the collection of $\mathbb{P}$-null sets of $\mathcal{F}$. Let $\{e_j\}_{j=1}^{\infty}$ be a complete orthonormal system in H. Suppose that $\{W(t),\, 0 \le t \le T\}$ is a cylindrical Wiener process on H, written formally as an infinite sum:

$$W(t) = \sum_{j=1}^{\infty} w_j(t)\, e_j \,,$$

where $w_j(\cdot)$, $j = 1, 2, \ldots,$ are i.i.d. Brownian motions in $\mathbb{R}$. By using this formal expansion we define what we call the completed natural filtration of W by $\sigma\{w_j(s),\, 0 \le s \le t,\, j = 1, \ldots, \infty\} \vee \mathcal{N},\ t \ge 0$. It was shown in [2] that $\sigma\{w_j(s),\, 0 \le s \le t,\, j = 1, \ldots, \infty\} \vee \mathcal{N} = \mathcal{F}_t$, for each $t \ge 0$, where $\mathcal{F}_t = \sigma\{l \circ W(s)\,,\ 0 \le s \le t\,, l \in H^*\} \vee \mathcal{N}$.

For a separable Hilbert space $\tilde{H}$ let $L^2_{\mathcal{F}}(0, T; \tilde{H})$ denote the space of all $\{\mathcal{F}_t, 0 \le t \le T\}$ - progressively measurable processes $\tilde{f}$ with values in $\tilde{H}$ such that

$$\mathbb{E}\left[\int_0^T \left|\tilde{f}(s)\right|^2_{\tilde{H}}\, ds\right] < \infty.$$

Thus $L^2_{\mathcal{F}}(0, T; \tilde{H})$ is a Hilbert space with the norm:

$$||\tilde{f}|| = \left(\mathbb{E}\left[\int_0^T \left|\tilde{f}(s)\right|^2_{\tilde{H}}\, ds\right]\right)^{1/2}.$$

For elements Ψ of $L^2_{\mathcal{F}}(0, T; L_2(H; K))$ we define the stochastic integral

$$\int_0^T \Psi(s)\, dW(s) := \lim_{N\to\infty} \sum_{j=1}^{N} \int_0^T (\Psi(s)\, e_j)\, dw_j(s).$$

This limit is known to exists $\mathbb{P}$- $a.s.$ since

$$\begin{aligned} \mathbb{E}\left[\left|\sum_{j=1}^{N} \int_0^T (\psi(s)\, e_j)\, dw_j(s)\right|^2_K\right] &= \sum_{j=1}^{N} \mathbb{E}\left[\int_0^T |\Psi(s)\, e_j|^2_K\, ds\right] \\ &\to \sum_{j=1}^{\infty} \mathbb{E}\left[\int_0^T |\Psi(s)\, e_j|^2_K ds\right] < \infty, \end{aligned} \tag{2.1}$$

as $N \to \infty$. Thus $\int_0^T \Psi(s)\, dW(s)$ is well-defined and belongs to $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$. Moreover $\int_0^{\cdot} \Psi(s)\ dW(s)$ can also be constructed as a limit in the above respect and is a square integrable martingale with values in K. More details on stochastic integration can be found for example in [8,14,15,22]. The following martingale representation theorem was proved in [2, Theorem 3.1].

Theorem 2.1. *Suppose that W is a cylindrical Wiener process on H with natural filtration $\{\mathcal{F}_t\,,\, t \ge 0\}$. Assume that $\{M(t), 0 \le t \le T\}$ is a martingale in K with respect to $\{\mathcal{F}_t\,,\, t \ge 0\}$, which is square integrable, i.e.* $\sup_{0\le t\le T} \mathbb{E}\,[\,|M(t)|^2_K] < \infty.$

Then there exists a unique stochastic process $R \in L^2_{\mathcal{F}}(0,T; L_2(H;K))$ *such that, for all* $0 \le t \le T$, *we have* $\mathbb{P}$ *- a.s.*

$$M(t) = M(0) + \int_0^t R(s)\, dW(s). \tag{2.2}$$

In particular, M *has a continuous modification.*

3. Backward stochastic evolution equations

In this section we are concerned with finding a unique solution to the BSEE (1.1). Before going into that business we need to introduce the following definitions.

Definition 3.1. *A two parameter family of bounded linear operators* $\{U(s,t),\ 0 \le t \le s \le T\}$ *on* K *is called an* evolution system *if the following holds.*

(i) $U(s,s) = I,\ 0 \le s \le T,$
(ii) $U(s,r)\,U(r,t) = U(s,t),\ \ 0 \le t \le r \le s,$
(iii) $(s,t) \mapsto U(s,t)$ *is strongly continuous for* $0 \le t \le s$.

The mapping U *(or* $U(s,t),\ 0 \le t \le s \le T$*) is usually called an evolution operator or a two-parameters semigroup.*

Definition 3.2. *A* strong evolution operator *is an evolution operator* $U(s,t),\ 0 \le t \le s \le T$, *for which there exists a closed and densely defined linear operator* $A(s)$, *with domain* $\mathcal{D}(A(s)),\ s \ge 0$, *such that*

$$U(s,t)(\mathcal{D}(A(t))) \subset \mathcal{D}(A(s)),\ \ s > t,$$

and

$$\frac{\partial}{\partial s}\,(U(s,t)\,y) = A(s)\,U(s,t)\,y,\ \ s > t,\ \ y \in \mathcal{D}(A(t)).$$

The family $\{A(s),\ 0 \le s \le T\}$ *is called the infinitesimal generator of* $U(s,t),\ 0 \le t \le s \le T$.

For example, the C_0 - semigroup $\{e^{A\,s},\ 0 \le s \le T\}$ of infinitesimal generator A, defines a strong evolution operator by $U(s,t) := e^{A\,(s-t)}$ with infinitesimal generator $A(s) = A,\ 0 \le s \le T$.

The operator A appearing in the BSEE (1.1) will be assumed to depend in a measurable way on time, i.e. $[0,T] \ni s \mapsto A(s)\, y \in K$ is Borel measurable for all $y \in K$. Moreover all $\{A(s), s \ge 0\}$ will be assumed to be closed and densely defined linear operators, which generate a strong evolution operator $U(s,t),\ 0 \le t \le s \le T$. We assume also that $[0,T]^2 \to L(K),\ (s,t) \mapsto U(s,t)$ is measurable. We refer the reader to [24,20,18] for the existence of such evolution operators and to [11,18,9] for the properties and the examples. See also the Example 3.3 in Section 3. In fact a general method of obtaining evolution operators is given in the following perturbation result, which is taken from [20].

Proposition 3.1. *Let* $U(s,t)$ *be an evolution operator and let* $C : [0,T] \to L(K)$ *be point measurable and essentially bounded. Then the operator integral equation*

$$S(s,t)y = U(s,t)y + \int_t^s U(s,r)C(r)S(r,t)y\, dr,\ \ y \in K,$$

has a unique solution $S(s,t)$ in the class of strongly continuous bounded linear operators on K, and moreover, $S(s,t)$ is the unique solution of

$$S(s,t)y = U(s,t)y + \int_t^s S(s,r)C(r)U(r,t)y\,dr, \quad y \in K.$$

The operator S is called the perturbation of U by C.

An evolution solution (or simply a solution) of (1.1) is a pair $(Y,Z) \in L^2_{\mathcal{F}}(0,T;K) \times L^2_{\mathcal{F}}(0,T;L_2(H;K))$ such that the following equality holds *a.s.*

$$\begin{aligned} Y(t) \;&=\; U(T,t)\,\xi + \int_t^T U(s,t)\,f(s,Y(s),Z(s))\,ds \\ &\quad - \int_t^T U(s,t)\,Z(s)\,dW(s),\ 0 \le t \le T. \end{aligned} \tag{3.1}$$

Our assumptions are the following.

- (H1) f is a mapping from $[0,T]\times\Omega\times K\times L_2(H;K)$ to K that is $\mathcal{P}\otimes\mathcal{B}(K)\otimes\mathcal{B}(L_2(H;K))/\mathcal{B}(K)$ - measurable and satisfies
$$f(\cdot,0,0) \in L^2_{\mathcal{F}}(0,T;K),$$
where $\mathcal{P}$ is the σ - algebra of all $\mathcal{F}_*$ - progressively measurable subsets of $[0,T]\times\Omega$.
- (H2) $\exists\, k>0$ such that $\forall\, y,y' \in K$ and $\forall\, z,z' \in L_2(H;K)$
$$|f(t,y,z) - f(t,y',z')|_K^2 \le c(|y-y'|_K^2) + k\,|z-z'|^2_{L_2(H;K)}\,,$$
for a.e. $(\omega,t) \in \Omega\times[0,T]$, where c is a concave nondecreasing continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $c(0)=0, c(x)>0$ for all $x>0$ and
$$\int_{0^+}^a \frac{dx}{c(x)} = \infty,$$
for any sufficiently small (and so for all) $a>0$.

Examples of such a function c is $c(x) = \alpha\,x$, for some $\alpha>0$, and the following two functions which are introduced in [13]:

$$c_1(x) = \begin{cases} x\,\log(x^{-1}) & (0 \le x \le \delta) \\ \delta\,\log(\delta^{-1}) + \dot{c}_1(\delta^-)\,(x-\delta) & (x > \delta) \end{cases}$$
$$c_2(x) = \begin{cases} x\,\log(x^{-1})\,\log\log(x^{-1}) & (0 \le x \le \delta) \\ \delta\,\log(\delta^{-1})\,\log\log(\delta^{-1}) + \dot{c}_2(\delta^-)\,(x-\delta) & (x > \delta) \end{cases}$$

with $\delta \in (0,1)$ being sufficiently small. The following example will be used in the Example 3.3. Assume that $h:\mathbb{R}\to[0,\infty)$ is defined by:

$$h(x) = \begin{cases} 0 & (x=0) \\ |x|\sqrt{\log(1+\frac{1}{|x|})} & (0<|x|<\delta) \\ \delta\,\sqrt{\log(1+\frac{1}{\delta})} & (|x| \ge \delta). \end{cases} \tag{3.2}$$

Define

$$c_3(x) = \begin{cases} 0 & (x = 0) \\ x \log(1 + \frac{1}{x}) & (0 < x < 1) \\ \log 2 & (x \geq 1). \end{cases} \tag{3.3}$$

Then c_3 satisfies the properties in (H2), is not Lipschitz and, moreover,

$$|h(x) - h(x')|^2 \leq c_3(|x - x'|^2) \quad \forall\, x, x' \in \mathbb{R}. \tag{3.4}$$

We refer the reader to [7] for the proof.

To the best of our knowledge introducing a condition as in (H2) in the study of the uniqueness of solutions of stochastic differential equations is due to Yamada and Watanabe in [25,26].

Our main theorem is the following.

Theorem 3.1. *Let $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$ be given. Assume that f satisfies (H1) and (H2). Then there exists a unique solution of (1.1) in $L^2_{\mathcal{F}}(0, T; K) \times L^2_{\mathcal{F}}(0, T; L_2(H; K))$.*

Note that if the operators $A(t), t \geq 0$, in (1.1) are taken to be a second order differential operator A, i.e. $A(t) = A$ for all $t \geq 0$, the equation (1.1) becomes

$$\begin{cases} -\, dY(t) = (A\, Y(t) + f(t, Y(t), Z(t)))\, dt - Z(t)\, dW(t), \\ \quad Y(T) = \xi, \end{cases} \tag{3.5}$$

with a solution (Y, Z) given by

$$\begin{aligned} Y(t) \;&=\; e^{A\,(T-t)}\, \xi + \int_t^T e^{A\,(s-t)}\, f(s, Y(s), Z(s))\, ds \\ &-\; \int_t^T e^{A\,(s-t)}\, Z(s)\, dW(s)\,, \quad 0 \leq t \leq T. \end{aligned} \tag{3.6}$$

Thus solutions of the equations (3.5) are actually mild solutions. In [4] we studied the regularity of such mild solutions. In particular it was shown that a weak solution and a strong solution exist for the equation (3.5). Unfortunately, it is not clear how one can get the same for the evolution case (the equation (1.1)). In fact the existence of strong solutions requires usually the commutativity between the operator $A(s)$ and the evolution operator $U(s, t)$, which does not hold in general. We mention here that for an operator which generates a weak forward adjoint and a weak backward adjoint the existence of weak and strong solutions of some forward stochastic evolution equations is derived in [9]. Now since we are dealing here with time dependent backward equations such an approach seems not to work, and so a different approach is really needed. To the best of our knowledge such a problem has not yet been studied for BSEEs. However we can still deal with the case of the following example.

Example 3.1. Let A be a second order operator on K, possibly unbounded, which generates a C_0-semigroup $\{e^{A\,t},\ t \geq 0\}$ on K. Assume that for each $t \in [0, T],\ A(t) = A + C(t)$, where $C : [0, T] \to L(K)$ is point measurable[1] and essentially bounded. Then $A(t)$ generates a strong evolution operator $U(t, r),\ 0 \leq r \leq t \leq T$, given by the perturbation of e^{tA} by $C(t)$ in the sense of the Proposition 3.1.

[1] e.g. Borel measurable.

Consider the following BSEE:

$$(3.7)\begin{cases} -dY(t) = (\,(\,A + C(t)\,)\,Y(t) + f(t, Y(t), Z(t))\,)\,dt - Z(t)\,dW(t), \\ Y(T) = \xi. \end{cases}$$

If the conditions in Theorem 3.1 hold, we conclude by applying Proposition 3.1 that the unique evolution solution

$$\begin{aligned} Y(t) &= U(T,t)\,\xi + \int_t^T U(s,t)\,f(s, Y(s), Z(s))\,ds \\ &\quad - \int_t^T U(s,t)\,Z(s)\,dW(s), \quad 0 \leq t \leq T, \end{aligned}$$

is the unique (up to modification) solution of the equation

$$\begin{aligned} Y(t) &= e^{A\,(T-t)}\,\xi + \int_t^T e^{A\,(s-t)}\,C(s)\,Y(s)\,ds \\ &+ \int_t^T e^{A\,(s-t)}\,f(s, Y(s), Z(s))\,ds - \int_t^T e^{A\,(s-t)}\,Z(s)\,dW(s), \end{aligned}$$

where $0 \leq t \leq T$. That is (Y, Z) is a mild solution of the following BSEE:

$$\begin{cases} -dY(t) = (\,A\,Y(t) + \tilde{f}(t, Y(t), Z(t))\,)\,dt - Z(t)\,dW(t), \ 0 \leq t \leq T, \\ Y(T) = \xi, \end{cases}$$

where $\tilde{f}(t, y, z) := f(t, y, z) + C(t)\,y$. Those mild solutions are discussed in details in [1].

Now before we go directly to the proof of Theorem 3.1 let us introduce some lemmas which will help to establish it. The following lemma is a special case of the theorem when the mappings f and g in (3.1) do not depend on Y and Z.

Lemma 3.1. *If $f \in L^2_{\mathcal{F}}(0, T; K)$ and $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$, there exists a unique pair $(Y, Z) \in L^2_{\mathcal{F}}(0, T; K) \times L^2_{\mathcal{F}}(0, T; L_2(H; K))$ such that*

$$(3.8) \qquad Y(t) = U(T,t)\xi + \int_t^T U(s,t)\,f(s)\,ds - \int_t^T U(s,t)\,Z(s)\,dW(s),$$

for each $t \in [0, T]$ a.s. Furthermore $\forall\ t \in [0, T]$,

$$(3.9) \qquad \mathbb{E}\,[\,|Y(t)|^2_K] \leq 2M^2(T-t)\,\mathbb{E}\left[\int_t^T |f(s)|^2_K ds\right] + 2M^2\,\mathbb{E}\,[\,|\xi|^2_K]$$

and

$$(3.10) \qquad \mathbb{E}\left[\int_t^T |Z(s)|^2_{L_2(H;K)}\,ds\right] \leq 8\,M^2(T-t)\,\mathbb{E}\left[\int_t^T |f(s)|^2_K\,ds\right] + 8\,M^2\,\mathbb{E}\,\left[\,|\xi|^2_K\right],$$

where

$$M := \sup_{0 \leq t \leq s \leq T} ||\,U(s,t)\,||\,.$$

The proof of this lemma can be achieved in a similar way to that in [10, Lemma 2.1]. We shall sketch it here for completeness.

Proof. Uniqueness: Let both (Y_1, Z_1) and (Y_2, Z_2) be two solutions of (3.8). Then for arbitrary $t \in [0, T]$

$$Y_1(t) - Y_2(t) = \int_t^T U(s,t)\ (Z_1(s) - Z_2(s))\ dW(s). \tag{3.11}$$

By applying conditional expectation $\mathbb{E}\,[\,\cdot \mid \mathcal{F}_t]$ to both sides of (3.11) and using the continuity of Y_1 and Y_2, cf. Proposition 4.1 below, we obtain

$$Y_1(t) = Y_2(t), \quad \forall\, t \in [0,T]\ \ a.s.$$

Hence by a simple use of (2.1), we find that $Z_1(t) = Z_2(t)$, for all $t \in [0, T]$ $a.s.$

Existence: Define

$$Y(t) = \mathbb{E}\,[\,U(T,t)\xi + \int_t^T U(s,t)\ f(s)\ ds \mid \mathcal{F}_t\,],\ \ 0 \le t \le T. \tag{3.12}$$

Hence (3.9) follows immediately from Jensen's inequality (see [21]) and the assumption (H1).

To construct $Z(\cdot)$ we mimic the method of [10] as follows. Since for each $s \in [0,T]$, $f(s)$ and ξ belong to $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K)$, it follows by using the martingale representation theorem (Theorem 2.1) that there exist two processes $z_1(s)$ and z_2 in $L^2_{\mathcal{F}}(0,T; L_2(H;K))$, such that

$$\mathbb{E}\,[f(s)|\mathcal{F}_t] = \mathbb{E}\,[f(s)] + \int_0^t z_1(s)(r)\ dW(r),\ \ 0 \le t \le s, \tag{3.13}$$

and

$$\mathbb{E}\,[\xi|\mathcal{F}_t] = \mathbb{E}\,[\xi] + \int_0^t z_2(r)\ dW(r),\ \ 0 \le t \le T. \tag{3.14}$$

It is not difficult to see that $z_1(\cdot)(\cdot)$ is $\mathcal{B}([0,T]) \otimes \mathcal{P}$ - measurable.

Now let

$$Z(t) := U(T,t)\ z_2(t) + \int_t^T U(s,t)\ z_1(s)(t)\ ds, \tag{3.15}$$

for any $0 \le t \le T$. It is then easy to check that (3.8) holds. The estimate (3.10) follows from (3.15), (3.13) and (3.14). ∎

From here on, unless if it is necessarily needed, we will suppress writing norms subscripts. The following example illustrates the above lemma.

Example 3.2. Coping with the above setting, let $H = \mathbb{R}^d, K = L^2(\mathbb{R}^d; \mathbb{R})$ and $A = \frac{1}{2}\,\Delta$. Consider the following BSEE:

$$\begin{aligned} -dY(t) &= \frac{1}{2}\,\Delta Y(t)\ dt + Z(t)\ dW(t), \\ Y(T) &= \Phi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; K). \end{aligned}$$

In this case,

$$Y(t,x) = (\,e^{\frac{1}{2}\,\Delta(T-t)}\ \mathbb{E}\,[\Phi]\,)(x) + (\int_0^t (e^{\frac{1}{2}\,\Delta(T-t)}\ z_2(s))\ dW(s)\,)(x)$$

and

$$Z(t)(\cdot) = (e^{\frac{1}{2}\,\Delta(T-t)}\ z_2(t))(\cdot),$$

$t \in [0,T],\ x \in \mathbb{R}^d$. The process z_2 is given as in the lemma through the martingale representation theorem. Note that if Φ is regular enough, z_2 can be calculated explicitly according to the Clark-Ocone theorem as in [2].

This example is actually a very special case of the Example 3.3 below.

Remark 3.1. We have also some high moments inequalities for the solution of the BSEE (3.8) as follows. Let $p > 2$. By using the fact that

$$Y(t) = \mathbb{E}\,[\,U(T,t)\xi + \int_t^T U(s,t)\ f(s)\ ds \mid \mathcal{F}_t\,],\ \ 0 \le t \le T,$$

we see that

$$\mathbb{E}\,[\,|Y(t)|^p] \le 2^{p-1}M^p\ \mathbb{E}\,[\,|\xi|^p] + 2^{p-1}M^p(T-t)^{p-1}\ \mathbb{E}\ \left[\int_t^T |f(s)|^p\ ds\right],$$

which yields

$$\mathbb{E}\ \left[\,|\int_t^T U(s,t)\ Z(s)\ dW(s)|^p\right]\ \le\ 3^{p-1}\ M^p(2^{p-1}+1)\ \mathbb{E}\,[\,|\xi|^p]$$
$$+\ 3^{p-1}\ M^p(T-t)^{p-1}(2^{p-1}+1)\ \mathbb{E}\ \left[\int_t^T |f(s)|^p\ ds\right],$$

for all $t \in [0,T]$.

These two inequalities become useful when the right hand side of each is finite.

Proposition 3.2. *Let $\xi \in L^2(\Omega,\mathcal{F}_T,\mathbb{P};K)$ and let $f : \Omega \times [0,T] \times L_2(H;K) \to K$ be a mapping satisfying (H1) and (H2). Then the following BSEE:*

$$(3.16)\qquad \begin{cases} -\,dY(t) = (A(t)\ Y(t) + f(t,Z(t)))\ dt - Z(t)\ dW(t), \\ \ \ Y(T) = \xi, \end{cases}$$

has a unique evolution solution (Y,Z).

By using Lemma 3.1 the proof of this proposition is direct and is similar to that of [10, Proposition 2.5] since the mapping f in (3.16) does not depend on the variable Z, so we prefer to omit it and tell briefly about it. It is simply achieved by showing that the following sequence $\{(Y_n(t), Z_n(t)) : 0 \le t \le T,\ n \ge 1\}$ of elements of $L^2_{\mathcal{F}}(0,T;K) \times L^2_{\mathcal{F}}(0,T;L_2(H;K))$, which is defined in the following equation (3.17), is Cauchy and then proving that its limit is the solution of (3.16). This sequence is defined recursively as follows: $Z_0 \equiv 0$,

$$Y_n(t) = U(T,t)\ \xi + \int_t^T U(s,t)\ f(s,Z_{n-1}(s))\ ds -$$
$$(3.17)\qquad \int_t^T U(s,t)\ Z_n(s)\ dW(s),\ 0 \le t \le T.$$

We now study the BSEE (1.1). With the help of Proposition 3.2 we first introduce the following iteration scheme, from which we will be able to construct the solution of (1.1). Let $Y_0(t) \equiv 0$ and let $\{(Y_n(t), Z_n(t)) : 0 \leq t \leq T,\ n \geq 1\}$ be a sequence in $L^2_{\mathcal{F}}(0,T;K) \times L^2_{\mathcal{F}}(0,T;L_2(H;K))$ defined by

$$Y_n(t) = U(T,t)\,\xi + \int_t^T U(s,t)\; f(s, Y_{n-1}(s), Z_n(s))\; ds - \int_t^T U(s,t)\; Z_n(s)\; dW(s),\ 0 \leq t \leq T. \tag{3.18}$$

Our aim is to show that these solutions $\{(Y_n,\ Z_n)\}$ converge as $n \to \infty$ to derive the solution of the original BSEE (1.1). Before doing so we need to give some vital estimates as in the following two lemmas.

Lemma 3.2. *Assume that the hypotheses (H1) and (H2) hold. Then there exist two positive constants C_1 and C_2 such that, for all $t \in [0,T]$ and for all $n \geq 1$, the solution of (3.18) satisfies*

$$\mathbb{E}\,[\,|Y_n(t)|^2\,] \leq C_1 \ \ \textit{and}\ \ \mathbb{E}\left[\int_0^T |Z_n(s)|^2\; ds\right] \leq C_2\,.$$

Proof. By applying Lemma 3.1 to the equation (3.18) we see that

$$\mathbb{E}\,[\,|Y_n(t)|^2\,] \ \leq\ 2M^2(T-t)\;\mathbb{E}\left[\int_t^T |f(s,Y_{n-1}(s),Z_n(s))|^2\; ds\right] + 2M^2\;\mathbb{E}\,[\,|\xi|^2\,] \tag{3.19}$$

and

$$\mathbb{E}\;\left[\int_t^T |Z_n(s)|^2\; ds\right] \ \leq\ 8M^2(T-t)\;\mathbb{E}\;\left[\int_t^T |f(s,Y_{n-1}(s),Z_n(s))|^2\; ds\right] + 8M^2\;\mathbb{E}\,[\,|\xi|^2\,], \tag{3.20}$$

for all $t \in [0,T]$. Since c is concave, there exist $a, b\ > 0$ such that $c(x) \leq a + b\,x$. Thus by using (H2) we find that

$$\mathbb{E}\;\left[\int_t^T |f(s,Y_{n-1}(s),Z_n(s))|^2\; ds\right] \leq C_3 + 2\,b\;\mathbb{E}\;\left[\int_t^T |Y_{n-1}(s)|^2\; ds\right] + 2k\;\mathbb{E}\;\left[\int_t^T |Z_n(s)|^2\; ds\right], \tag{3.21}$$

where $C_3 := 2\;a\;T + 2\,\mathbb{E}\;[\int_0^T |f(s,0,0)|^2\; ds]$. In particular (3.19) and (3.20) take the following shapes:

$$\mathbb{E}\,[\,|Y_n(t)|^2\,] \ \leq\ 2M^2(T-t)C_3 + 4M^2(T-t)\,b\;\mathbb{E}\;\left[\int_t^T |Y_{n-1}(s)|^2\; ds\right] + 4M^2(T-t)\,k\,\mathbb{E}\left[\int_t^T |Z_n(s)|^2\, ds\right] + 2M^2\,\mathbb{E}\;\left[\,|\xi|^2\,\right] \tag{3.22}$$

and

$$\mathbb{E}\left[\int_t^T |Z_n(s)|^2\, ds\right] \leq$$

$$8M^2\,\mathbb{E}\,[\,|\xi|^2\,] + 8M^2(T-t)C_3 + 16M^2(T-t)\,b\,\mathbb{E}\left[\int_t^T |Y_{n-1}(s)|^2\, ds\right]$$

$$+16M^2(T-t)\,k\,\mathbb{E}\left[\int_t^T |Z_n(s)|^2\, ds\right]. \tag{3.23}$$

Take $\eta \in (0,T)$ with $32M^2k\,\eta < 1$. Assume that $t \in [T-\eta\,, T]$. We observe from (3.23) that

$$\mathbb{E}\left[\int_t^T |Z_n(s)|^2\, ds\right] \leq$$

$$8M^2\,\mathbb{E}\,[\,|\xi|^2\,] + 8M^2\eta\, C_3 + 16M^2\,\eta\, b\,\mathbb{E}\left[\int_t^T |Y_{n-1}(s)|^2\, ds\right]$$

$$+\frac{1}{2}\,\mathbb{E}\left[\int_t^T |Z_n(s)|^2\, ds\right], \tag{3.24}$$

or in particular

$$\mathbb{E}\left[\int_t^T |Z_n(s)|^2\, ds\right] \leq \frac{C_3}{2k} + 16M^2\,\mathbb{E}\,[\,|\xi|^2\,] + \frac{b}{k}\,\mathbb{E}\left[\int_t^T |Y_{n-1}(s)|^2\, ds\right]. \tag{3.25}$$

Similarly from (3.22) we get

$$\begin{aligned}\mathbb{E}\,[\,|Y_n(t)|^2\,] \quad &\leq \quad \frac{C_3}{16k} + \frac{b}{8k}\,\mathbb{E}\left[\int_t^T |Y_{n-1}(s)|^2\, ds\right] \\ &\qquad + \frac{1}{8}\,\mathbb{E}\left[\int_t^T |Z_n(s)|^2\, ds\right] + 2M^2\,\mathbb{E}\,[\,|\xi|^2\,],\end{aligned} \tag{3.26}$$

from which and from (3.25), we obtain

$$\mathbb{E}\left[\,|Y_n(t)|^2\,\right] \leq C_4 + C_5\,\mathbb{E}\left[\int_t^T |Y_{n-1}(s)|^2\, ds\right], \tag{3.27}$$

$\forall\, n \geq 1$. The constants C_4, C_5 are $\dfrac{C_3}{8k} + 4M^2\,\mathbb{E}\,\left[\,|\xi|^2\,\right]$ and $\dfrac{b}{4k}$ respectively.

Fix now an integer $m \geq 1$. If $1 \leq n \leq m$, we then have

$$\mathbb{E}\,[\,|\,Y_n(t)|^2\,] \leq C_4 + C_5\,\mathbb{E}\left[\int_t^T \sup_{1\leq q\leq m}\,\mathbb{E}\,[\,|Y_q(s)|^2\,]\, ds\right].$$

By Gronwall's inequality we deduce that

$$\sup_{1\leq q\leq m} \mathbb{E}\,[\,|Y_q(t)|^2\,] \;\leq\; C_4\, e^{C_5(T-t)}$$
$$<\; C_4\, e^{C_5 T} =: C_6\,.$$

Since m is arbitrary, we get that

$$\mathbb{E}\,[\,|\,Y_n(t)|^2\,] \leq C_6\,, \tag{3.28}$$

$\forall\, n \geq 1$ and $\forall\, t \in [T-\eta\,, T]$.

On the other hand, by re-writing (3.25) in the following form:

$$\mathbb{E}\left[\int_t^T |Z_n(s)|^2\, ds\right] \leq C_7 + \frac{b}{k}\,\mathbb{E}\left[\int_t^T |Y_{n-1}(s)|^2\, ds\right], \tag{3.29}$$

where $C_7 := \dfrac{C_3}{2k} + 16M^2\,\mathbb{E}\,[\,|\xi|^2\,]$ and using (3.28) we see that

$$\mathbb{E}\left[\int_t^T |Z_n(s)|^2\, ds\right] \leq C_7 + \frac{b}{k}\, C_6 T =: C_8\,. \tag{3.30}$$

Next we assume that $t \in [T-2\eta\,, T-\eta]$ and η satisfies $32M^2 k\,\eta < 1$. Since

$$Y_n(t) = Y_n(T-\eta) + \int_t^{T-\eta} f(s, Y_n(s), Z_n(s))\, ds$$
$$- \int_t^{T-\eta} Z_n(s)\, dW(s),$$

for all $t \in [T-2\eta\,, T-\eta]$, we obtain in the same way in which we derived the inequalities (3.28) and (3.30) the following results.

$$\mathbb{E}\,[\,|Y_n(t)|^2\,] \leq C_6'\,, \tag{3.31}$$

and

$$\mathbb{E}\left[\int_t^{T-\eta} |Z_n(s)|^2\, ds\right] \;\leq\; \frac{C_3}{2k} + 16M^2 C_6 + \frac{b}{k}\, C_6'\,(T-\eta-t)$$
$$<\; C_8'\,, \tag{3.32}$$

$\forall\; t \in [T-2\;\eta\,, T-\eta]$ and $\forall\; n \geq 1$, where $C_6' := \left(\dfrac{C_3}{8k} + 4M^2 C_6\right) e^{C_5 T}$ and $C_8' := \dfrac{C_3}{2k} + 16M^2 C_6 + \dfrac{b\,T}{k}\, C_6'\,.$

The rest of the proof can now be achieved by repeating this procedure for all tiny intervals $[\,(\,T-(l+1)\,\eta\,)\vee 0\,, T-l\,\eta\,]$ where $0\leq l\leq q,\ l$ is integer of length at most η (with $32M^2 k\,\eta < 1$) and q is the largest integer such that $q < \frac{T}{\eta}$. This completes the proof. ∎

Lemma 3.3. *If the hypotheses (H1), (H2) hold, then there exist two constants* $C_9 > 0$ *and* $C_{10} > 0$ *such that* $\forall\, 0\leq t\leq T$ *and* $\forall\, n, m \geq 1$, *we have*

$$\mathbb{E}\,[\,|Y_{n+m}(t) - Y_n(t)|^2\,] \leq C_9 \int_t^T c(\,\mathbb{E}\,[\,|Y_{n+m-1}(s) - Y_{n-1}(s)|^2\,]\,)\, ds \tag{3.33}$$

and

$$\mathbb{E}\left[\int_t^T |Z_{n+m}(s) - Z_n(s)|^2\, ds\right] \leq$$

$$C_{10} \int_t^T c(\, \mathbb{E}\,[\, |Y_{n+m-1}(s) - Y_{n-1}(s)|^2\,]\,)\, ds. \tag{3.34}$$

Proof. Note that

$$\begin{aligned} Y_{n+m}(t) - Y_n(t) = \\ &\int_t^T U(s,t)[f(s, Y_{n+m-1}(s), Z_{n+m}(s)) - f(s, Y_{n-1}(s), Z_n(s))]\, ds \\ &- \int_t^T U(s,t)[Z_{n+m}(s) - Z_n(s)]\, dW(s). \end{aligned} \tag{3.35}$$

Thus by using Lemma 3.1 we deduce the following two inequalities:

$$\begin{aligned} \mathbb{E}\,[\, |Y_{n+m}(t) - Y_n(t)|^2\,] \leq \\ &M^2(T-t)\, \mathbb{E}\left[\int_t^T c(\, |Y_{n+m-1}(s) - Y_{n-1}(s)|^2)\, ds\right] \\ &+ M^2(T-t)\, k\, \mathbb{E}\left[\int_t^T |Z_{n+m}(s) - Z_n(s)|^2\, ds\right], \end{aligned} \tag{3.36}$$

$$\begin{aligned} \mathbb{E}\left[\int_t^T |Z_{n+m}(s) - Z_n(s)|^2\, ds\right] \leq \\ &8\, M^2(T-t)\, \mathbb{E}\left[\int_t^T c(\, |Y_{n+m-1}(s) - Y_{n-1}(s)|^2)\, ds\right] \\ &+ 8\, M^2(T-t)\, k\, \mathbb{E}\left[\int_t^T |Z_{n+m}(s) - Z_n(s)|^2\, ds\right]. \end{aligned} \tag{3.37}$$

As done in the proof of Lemma 3.2 we divide the interval $[0, T]$ into tiny sub-intervals of length η which satisfies $0 \leq \eta \leq T$ and $16 M^2 \eta k < 1$. Let $t \in [T - \eta\,, T]$. Then (3.37) becomes

$$\mathbb{E}\left[\int_t^T |Z_{n+m}(s) - Z_n(s)|^2\, ds\right] \leq$$

$$\frac{1}{k}\, \mathbb{E}\left[\int_t^T c(|Y_{n+m-1}(s) - Y_{n-1}(s)|^2)\, ds\right]. \tag{3.38}$$

Consequently by using (3.36) and (3.38) we get

$$\mathbb{E}\,[\, |Y_{n+m}(t) - Y_n(t)|^2\,] \leq C_{11} \int_t^T \mathbb{E}\,[\, c(\, |Y_{n+m-1}(s) - Y_{n-1}(s)|^2)]\, ds, \tag{3.39}$$

with $C_{11} := \dfrac{9}{16\,k}$. Jensen's inequality gives that

$$\mathbb{E}\,[\,|Y_{n+m}(t) - Y_n(t)|^2\,] \leq C_{11} \int_t^T c(\,\mathbb{E}\,[\,|Y_{n+m-1}(s) - Y_{n-1}(s)|^2\,])\; ds, \tag{3.40}$$

for all $t \in [T-\eta\,, T]$ and $n, m \geq 1$.

Next assume that $t \in [T - 2\eta\,, T-\eta]$ and rewrite (3.35) as follows:

$$\begin{aligned} Y_{n+m}(t) - Y_n(t) = \\ & U(T-\eta, t)(Y_{n+m}(T-\eta) - Y_n(T-\eta)) \\ & + \int_t^{T-\eta} U(s,t)[f(s, Y_{n+m-1}(s), Z_{n+m}(s)) - f(s, Y_{n-1}(s), Z_n(s))]\; ds \\ & - \int_t^{T-\eta} U(s,t)[Z_{n+m}(s) - Z_n(s)]\; dW(s). \end{aligned} \tag{3.41}$$

Applying Lemma 3.1 to the equation (3.41) and using the assumption $(16M^2\eta k < 1)$, the inequality (3.40) and Jensen's inequality imply that

$$\begin{aligned} \mathbb{E}\; \left[\int_t^{T-\eta} |Z_{n+m}(s) - Z_n(s)|^2\; ds \right] \leq \\ & \left(\frac{1}{2k}\right) \int_t^{T-\eta} c(\mathbb{E}\,[\,|Y_{n+m-1}(s) - Y_{n-1}(s)|^2\,])\; ds \\ & + 8M^2\, C_{11} \int_{T-\eta}^T c(\mathbb{E}\,[\,|Y_{n+m-1}(s) - Y_{n-1}(s)|^2\,])\; ds \\ & \leq \left(\frac{1}{2k} + 8M^2\, C_{11}\right) \int_t^T c(\mathbb{E}\,[\,|Y_{n+m-1}(s) - Y_{n-1}(s)|^2\,])\; ds, \end{aligned} \tag{3.42}$$

$\forall\, t \in [T-2\eta\,, T-\eta]$ and $\forall\, n, m \geq 1$.

Again by using Lemma 3.1[2] together with (3.40), (3.42) and Jensen's inequality we deduce that

$$\mathbb{E}\,[\,|Y_{n+m}(t) - Y_n(t)|^2\,] \leq C_{12} \int_t^T c(\mathbb{E}\,[|Y_{n+m-1}(s) - Y_{n-1}(s)|^2\,])\, ds, \tag{3.43}$$

$\forall\, t \in [T-2\eta\,, T-\eta]$ and $\forall\, n, m \geq 1$, where $C_{12} := \dfrac{3}{16k} + 3M^2 C_{11}$. Therefore (3.33) holds for all $t \in [T-2\eta\,, T]$.

On the other hand, note that (3.38) and (3.42) prove (3.34) for the case where $t \in [T-2\eta\,, T]$.

Finally by considering the argument stated at the end of the preceding lemma, we conclude that the two inequalities (3.33) and (3.34) follow for each $t \in [0, T]$. ∎

Lemma 3.4. *Under the hypotheses (H1) and (H2) there exists a constant $C_{13} > 0$ such that for all $t \in [0, T]$ and $n, m \geq 1$,*

$$\mathbb{E}\,[\,|Y_{n+m}(t) - Y_n(t)|^2\,] \leq C_{13}\; (T-t).$$

[2] or simply by taking conditional expectation on (3.41) and using Jensen's inequality.

Proof. From Lemma 3.3 and Lemma 3.2 it follows that

$$\begin{aligned}\mathbb{E}\,[\,|Y_{n+m}(t)-Y_n(t)|^2\,] &\leq C_9\int_t^T c(4C_1)\;ds\\ &= C_{13}\;(T-t),\end{aligned} \tag{3.44}$$

where $C_{13} := C_9\, c(4\, C_1)$. ∎

Let us now state the following inequality due to Bihari's, which we will need in our proof below. We refer the reader to [6, p. 83] for its proof.

Proposition 3.3 (Bihari's inequality)**.** *Let u and v be two positive continuous functions defined on $[0, T]$. Assume that $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative nondecreasing continuous function. If for some positive constants C and D,*

$$u(t) \leq C + D\int_0^t v(s)\;c(u(s))\;ds,\ \ 0\leq t\leq T,$$

then

$$u(t) \leq G^{-1}(G(C) + D\int_0^t v(s)\;ds),$$

for all such $t \in [0, T]$ that $G(C) + \int_0^t v(s)\;ds \in Dom(G^{-1})$, where

$$G(r) := \int_1^r \frac{ds}{c(s)}\;,\ \ r > 0,$$

and G^{-1} is the inverse function of G.

In particular if the constant $C = 0$ and $\lim_{r\to 0^+} G(r) = -\infty$, then $u(t) = 0, 0 \leq t \leq T$.

We are now ready to establish the proof of Theorem 3.1. We shall argue mainly like [13].

Proof of Theorem 3.1. Existence: Recall the sequence $\{Y_n\}$ which is defined recursively in the equation (3.18). Recall also the two constants C_9 and C_{13} from Lemma 3.3 and Lemma 3.4. We claim first that

$$\sup_{t\in[0,T]} \mathbb{E}\,[\,|Y_{n+m}(t)-Y_n(t)|^2\,] \to 0,\ \ \text{as}\ \ n, m\to\infty. \tag{3.45}$$

We need to prepare some results in order to be able to derive this claim. Let $\bar{c}$ be the function $C_9\, c$, defined on $\mathbb{R}_+$ and $\zeta_1 \in [0, T]$ be such that $\bar{c}(C_{13}\;(T-t)) \leq C_{13}$ for all $\zeta_1 \leq t \leq T$. Define recursively the following deterministic sequence: for $0 \leq t \leq T$,

$$\begin{aligned}\varphi_1(t) &= C_{13}\,(T-t),\\ \varphi_{n+1}(t) &= \int_t^T \bar{c}(\varphi_n(s))\;ds,\ \ n=1,2,\ldots.\end{aligned}$$

For a fixed $k \geq 1$ we have $\forall\, t \in [\zeta_1, T]$ and $\forall\, n \geq 2$,

$$\mathbb{E}\,[\,|Y_{n+k}(t)-Y_n(t)|^2\,] \leq \varphi_{n-1}(t) \leq \cdots \leq \varphi_1(t). \tag{3.46}$$

This is actually proved by induction using Lemma 3.4 and Lemma 3.3 as follows. The case $n = 2$ follows directly from Lemma 3.4. If (3.46) holds for some fixed

$n > 2$, then

$$\begin{aligned}\mathbb{E}\,[\,|Y_{n+1+k} - Y_{n+1}(t)|^2\,] &\leq \int_t^T \bar{c}(\,\mathbb{E}\,[\,|Y_{n+k}(s) - Y_n(s)|^2\,]\,)\,ds \\ &\leq \int_t^T \bar{c}(\varphi_{n-1}(s))\,ds = \varphi_n(t) \\ &\leq \int_t^T \bar{c}(\varphi_{n-2}(s))\,ds = \varphi_{n-1}(t).\end{aligned}$$

Thus (3.46) holds for all $n \geq 2$.

On the other hand, by the monotonicity of $\varphi_n(t)$ the sequence $\{\varphi_n(t) : t \in [\zeta_1, T], n \geq 1\}$ attains a limit $\{\varphi(t) : t \in [\zeta_1, T]\}$. Moreover from the definition of $\varphi_n(t)$ and the dominated convergence theorem we obtain

$$\varphi(t) = \lim_{n\to\infty} \int_t^T \bar{c}(\varphi_n(s))\,ds = \int_t^T \bar{c}(\varphi(s))\,ds,\ t \in [\zeta_1, T],$$

which shows that φ is continuous on $[\zeta_1, T]$. Now applying Bihari's inequality yields $\varphi(t) = 0$ for all $t \in [\zeta_1, T]$. It follows that

$$\lim_{n,m\to\infty}\ \sup_{t\in[\zeta_1,T]} \mathbb{E}\,[\,|Y_{n+m}(t) - Y_n(t)|^2\,] \leq \lim_{n\to\infty}\ \sup_{t\in[\zeta_1,T]} \varphi_n(t) = \varphi(\zeta_1) = 0. \tag{3.47}$$

This proves the claim (3.45) in the case where $t \in [\zeta_1, T]$. In particular $\{Y_n\}_{n\geq 1}$ is a Cauchy sequence in $L^2_{\mathcal{F}}(\zeta_1, T; K)$. Call its limit $\{Y(t),\ \zeta_1 \leq t \leq T\}$. Moreover from (3.34) and (3.47) we find that

$$\begin{aligned}&\mathbb{E}\left[\int_{\zeta_1}^T |Z_{n+m}(s) - Z_n(s)|^2\,ds\right] \\ &\leq\ C_{10} \int_{\zeta_1}^T c(\,\mathbb{E}\,[\,|Y_{n+m-1}(s) - Y_{n-1}(s)|^2\,]\,)\,ds \ \to\ 0, \text{ as } n, m \to \infty.\end{aligned}$$

So $\{Z_n\}_{n\geq 1}$ is also a Cauchy sequence in $L^2_{\mathcal{F}}(\zeta_1, T; L_2(H; K))$. Denote its limit by $\{Z(t),\ \zeta_1 \leq t \leq T\}$. Therefore (3.18), (H2) and the convergence of Y_n to Y show that

$$\begin{aligned}&\mathbb{E}\left[\left|\int_t^T U(s,t)\,Z_n(s)\,dW(s) - \int_t^T U(s,t)\,Z(s)\,dW(s)\right|^2\right] = \\ &\mathbb{E}\left[\left|Y_n(t) - Y(t) + \int_t^T U(s,t)\,(\,f(s, Y_n(s), Z_n(s)) - f(s, Y(s), Z(s))\,)\,ds\right|^2\right] \\ &\leq\ 2\,\mathbb{E}\,[\,|Y_n(t) - Y(t)|^2\,] + 2\,M^2\,T\,\mathbb{E}\left[\int_t^T c(|Y_n(s) - Y(s)|^2)\,ds\right] \\ &+ 2\,M^2\,T\,k\,\mathbb{E}\left[\int_t^T |Z_n(s) - Z(s)|^2\,ds\right] \to 0,\end{aligned}$$

as $n \to \infty$, for all $t \in [\zeta_1\,, T]$. Note that the convergence of the second term follows from Jensen's inequality and the continuity of the function c. Now we can pass to the limit as $n \to \infty$ in (3.18) to see particularly that (Y, Z) solves our original BSEE (1.1) on the interval $[\zeta_1, T]$.

The second step is to try to extend this solution to make it defined on the whole interval $[0, T]$. For this we define

$$\zeta_2 := \inf \left\{ \zeta \in [0, T] : \sup_{\zeta \le t \le T} \mathbb{E}\,[\,|Y_{n+m}(t) - Y_n(t)|^2\,] \to 0, \text{ as } n, m \to \infty \right\}.$$

We claim that

$$\sup_{\zeta_2 \le t \le T} \mathbb{E}\,[\,|Y_{n+m}(t) - Y_n(t)|^2\,] \to 0, \text{ as } n, m \to \infty, \ \forall m \ge 1. \tag{3.48}$$

Let us now prove this claim. It is clear from (3.47) that $0 \le \zeta_2 \le \zeta_1 < T$. Let $\varepsilon > 0$ and choose $\lambda \in (0, T - \zeta_2)$ so that $C_{13}\ \lambda < \dfrac{\varepsilon}{2}$. Since $\bar{c}(0) = 0$, $\exists\ \theta \in (0, \varepsilon)$ such that $T\ \bar{c}(\theta) < \dfrac{\varepsilon}{2}$. Let $N \ge 1$ be sufficiently large so that if $\zeta_2 + \lambda \le t \le T$, then $\mathbb{E}\ [\,|Y_{n+m}(t) - Y_n(t)|^2\,] < \theta$, $\forall\ n \ge N$ and $\forall\ m \ge 1$. By using Lemma 3.3 and Lemma 3.2 and noting that $C_{13} = \bar{c}(4\,C_1)$ we see that, if $n \ge N + 1$, $m \ge 1$ and $t \in [\zeta_2, \zeta_2 + \lambda]$, then

$$\begin{aligned} \mathbb{E}\,[\,|Y_{n+m}(t) - Y_n(t)|^2\,] &\le \int_{\zeta_2}^{\zeta_2+\lambda} \bar{c}(\,\mathbb{E}\,[\,|Y_{n+m-1}(s) - Y_{n-1}(s)|^2\,]\,)\, ds \\ &+ \int_{\zeta_2+\lambda}^{T} \bar{c}(\,\mathbb{E}\,[\,|Y_{n+m-1}(s) - Y_{n-1}(s)|^2\,]\,)\, ds \\ &\le \lambda\, C_{13} + (T - \zeta_2 - \lambda)\, \bar{c}(\theta) \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore we obtain

$$\sup_{\zeta_2 \le t \le T} \mathbb{E}\,[\,|Y_{n+m}(t) - Y_n(t)|^2\,] < \varepsilon, \quad \forall\, n \ge N + 1 \ \text{ and } \ m \ge 1.$$

Hence the claim (3.48) follows.

Thirdly, remark that by proving that $\zeta_2 = 0$ the proof of the claim (3.45) finishes and so the theorem as shown earlier. So the rest of the proof is devoted to proving this fact. Let us suppose otherwise that $\zeta_2 > 0$. Then by using the claim (3.48) we can choose a sequence of decreasing numbers $\{a_n\}_{n \ge 1}$ such that $a_n \to 0$ as $n \to \infty$ and

$$\sup_{\zeta_2 \le t \le T} \mathbb{E}\,[\,|Y_{n+m}(t) - Y_n(t)|^2\,] < a_n\,, \ \ \forall\, n \ge 1. \tag{3.49}$$

If $0 \le t \le \zeta_2$ and $n \ge 1$, then by Lemma 3.3 and (3.49) we derive that

$$\begin{aligned} \mathbb{E}\,[\,|Y_{n+m}(t) - Y_n(t)|^2\,] &\le \int_t^T \bar{c}(\,\mathbb{E}\,[\,|Y_{n+m-1}(s) - Y_{n-1}(s)|^2\,]\,)\, ds \\ &\le \left(\int_t^{\zeta_2} + \int_{\zeta_2}^T \right) \bar{c}(\,\mathbb{E}\,[\,|Y_{n+m-1}(s) - Y_{n-1}(s)|^2\,]\,)\, ds \\ &\le \int_t^{\zeta_2} \bar{c}(\,\mathbb{E}\,[\,|Y_{n+m-1}(s) - Y_{n-1}(s)|^2\,]\,)\, ds + T\,\bar{c}(a_{n-1}) \end{aligned}$$

$$\leq \quad (\zeta_2 - t)\, C_{13} + T\, \bar{c}(a_{n-1}). \tag{3.50}$$

Pick now $\delta \in (0, \zeta_2)$ and $j \geq 1$ so that

$$C_{13}\, \delta + T\, \bar{c}(a_j) \leq 4\, C_1 \tag{3.51}$$

and define a sequence of functions $\{\psi_k(t),\ \zeta_2 - \delta \leq t \leq \zeta_2\,,\ k \geq 1\}$ by

$$\psi_1(t) = C_{13}\, \delta + T\, \bar{c}(a_j) \leq 4\, C_1\,,$$

$$\psi_{k+1}(t) = T\, \bar{c}(a_{j+1}) + \int_t^{\zeta_2} \bar{c}(\psi_k(s))\, ds, \quad k \geq 1.$$

We claim that for fixed $l \geq 1$ if $\zeta_2 - \delta \leq t \leq \zeta_2$ and $k \geq 1$, then

$$\mathbb{E}\,[\,|Y_{l+j+k}(t) - Y_{j+k}(t)|^2\,] \leq \psi_k(t) \leq \cdots \leq \psi_1(t). \tag{3.52}$$

This claim is obvious when $k = 1$ since

$$\begin{aligned} \mathbb{E}\,[\,|Y_{l+j+1}(t) - Y_{j+1}(t)|^2\,] \quad &\leq \quad (\zeta_2 - t)\, C_{13} + T\, \bar{c}(a_j) \\ &= \quad \psi_1(t). \end{aligned}$$

Also as we did in (3.50) we find by using Lemma 3.3, (3.51) and the definition of $C_{13} = \bar{c}(4\, C_1)$ that, if $k = 2$, then

$$\begin{aligned} &\mathbb{E}\,[\,|Y_{l+j+2}(t) - Y_{j+2}(t)|^2\,] \\ &\quad \leq T\, \bar{c}(a_{j+1}) + \int_t^{\zeta_2} \bar{c}(\,\mathbb{E}\,[\,|Y_{l+j+1}(s) - Y_{j+1}(s)|^2\,]\,)\, ds \\ &\quad \leq T\, \bar{c}(a_{j+1}) + \int_t^{\zeta_2} \bar{c}(\psi_1(s))\, ds \ = \psi_2(t) \\ &\quad \leq T\, \bar{c}(a_j) + C_{13}\, (\zeta_2 - t) \ = \psi_1(t). \end{aligned}$$

The last inequality here comes from $a_{j+1} \leq a_j$ and the monotonicity of $\bar{c}$. Thus (3.52) holds for $k = 1, 2$. Assume now that (3.52) is true for some $k > 2$. Again by using Lemma 3.3 and (3.49) we get similarly that

$$\begin{aligned} &\mathbb{E}\,[\,|Y_{l+j+k+1}(t) - Y_{j+k+1}(t)|^2\,] \\ &\quad \leq T\, \bar{c}(a_{j+k}) + \int_t^{\zeta_2} \bar{c}(\,\mathbb{E}\,[\,|Y_{l+j+k}(s) - Y_{j+k}(s)|^2\,]\,)\, ds \\ &\quad = T\, \bar{c}(a_{j+k}) + \int_t^{\zeta_2} \bar{c}(\psi_k(s))\, ds \ = \psi_{k+1}(t) \\ &\quad \leq T\, \bar{c}(a_{j+k-1}) + \int_t^{\zeta_2} \bar{c}(\psi_{k-1}(s))\, ds \ = \psi_k(t). \end{aligned}$$

Hence (3.52) holds for $k + 1$ as well, and so (3.52) holds for every $k \geq 1$.

Finally we are ready to show that $\zeta_2 = 0$. Note that it follows from (3.52) that for each $t \in [\zeta_2 - \delta\,, \zeta_2]$, the sequence $\psi_k(t)$ attains a limit defined on $[\zeta_2 - \delta\,, \zeta_2]$, say $\psi(t)$. Now since

$$\psi(t) = \lim_{k \to \infty} \psi_{k+1}(t) = \lim_{k \to \infty} \left[T\, \bar{c}(a_{j+k}) + \int_t^{\zeta_2} \bar{c}(\psi_k(s))\, ds \right]$$

$$= \int_t^{\zeta_2} \bar{c}(\psi(s))\, ds,$$

for all $t \in [\zeta_2 - \delta\,, \zeta_2]$, it follows from Bihari's inequality that $\psi(t) = 0$ on $\zeta_2 - \delta \le t \le \zeta_2$. Thus (3.52) again implies that

$$\sup_{\zeta_2 - \delta \le t \le \zeta_2} \mathbb{E}\left[\, |Y_{l+j+k}(t) - Y_{j+k}(t)|^2 \,\right] \le \psi_k(\zeta_2 - \delta) \ \to 0,$$

as $j, k \to \infty$. But $l \ge 1$ is arbitrary. Thus

$$\sup_{\zeta_2 - \delta \le t \le \zeta_2} \mathbb{E}\left[\, |Y_{n+m}(t) - Y_n(t)|^2 \,\right] \ \to 0,$$

as $n, m \to \infty$. This together with (3.48) gives

$$\sup_{\zeta_2 - \delta \le t \le T} \mathbb{E}\left[\, |Y_{n+m}(t) - Y_n(t)|^2 \,\right] \ \to 0,$$

contradicting the definition of ζ_2. Hence $\zeta_2 = 0$ as required.

Uniqueness: Suppose that (Y, Z) and (Y', Z') are two solutions of (1.1). Then

$$\begin{aligned} Y(t) - Y'(t) \quad &= \quad \int_t^T U(s,t)\, [f(s, Y(s), Z(s)) - f(s, Y'(s), Z'(s))]\, ds \\ &\quad - \int_t^T U(s,t)\, [Z(s) - Z'(s)]\, dW(s), \end{aligned} \tag{3.53}$$

where $t \in [0,T]$. Denote $\tilde{Y}(t) = Y(t) - Y'(t)$, $\tilde{Z}(t) = Z(t) - Z'(t)$ and $\tilde{f}(t) = f(t, Y(t), Z(t)) - f(t, Y'(t), Z'(t))$, $t \in [0,T]$. The hypotheses (H1) and (H2) and Jensen's inequality give:

$$\mathbb{E}\left[\int_0^T |\tilde{f}(s)|^2\, ds\right] \le c(\, \mathbb{E}\left[\int_0^T |\tilde{Y}(s)|^2\, ds\right]) + \mathbb{E}\left[\int_0^T k\, |\tilde{Z}(s)|^2\, ds\right] < \infty,$$

as $c(x) \le a\ x + b$, for some $a, b > 0$. The equation (3.53) is then similar to the equation (3.8) in Lemma 3.1. Consequently we obtain

$$\begin{aligned} \mathbb{E}\left[|\tilde{Y}(t)|^2\right] \quad &\le \quad 2M^2\, (T-t)\, \mathbb{E}\left[\int_t^T |\tilde{f}(s)|^2\, ds\right] \\ &\le \quad 2\, M^2\, (T-t)\, \mathbb{E}\left[\int_t^T c(\, |\tilde{Y}(s)|^2)\, ds\right] \\ &+ \quad 2M^2\, (T-t)\, k\, \mathbb{E}\left[\int_t^T |\tilde{Z}(s)|^2\, ds\right] \end{aligned} \tag{3.54}$$

and

$$\begin{aligned} \mathbb{E}\left[\int_t^T |\tilde{Z}(s)|^2\, ds\right] \quad &\le \quad 8\, M^2\, (T-t)\, \mathbb{E}\left[\int_t^T |\tilde{f}(s)|^2\, ds\right] \\ &\le \quad 8\, M^2\, (T-t)\, \mathbb{E}\left[\int_t^T c(|\tilde{Y}(s)|^2)\, ds\right] \end{aligned}$$

$$+ \quad 8\, M^2\, (T-t)\, k\, \mathbb{E}\, \left[\int_t^T |\tilde{Z}(s)|^2\, ds \right]. \tag{3.55}$$

As we did in the proof of Lemma 3.2 we can also apply the trick of partitioning to the two inequalities (3.54) and (3.55) by using a fixed small scale $\eta \in (0,T)$ with $16\, M^2\, \eta\, k < 1$. Using this we find eventually that if $(T-(l+1)\,\eta) \vee 0 \ \le t \le T - l\,\eta$ for some $0 \le l \le q$, then $\exists\ C(l) > 0$, a constant that possibly depends on l, such that

$$\mathbb{E}\,[\,|\tilde{Y}(t)|^2\,] \le C(l) \int_t^T c(\,\mathbb{E}\,[\,|\tilde{Y}(s)|^2\,]\,)\, ds \tag{3.56}$$

and

$$\mathbb{E}\, \left[\int_0^T |\tilde{Z}(s)|^2\, ds \right] \le C(l) \int_0^T c(\,\mathbb{E}\,[\,|\tilde{Y}(s)|^2\,]\,)\, ds. \tag{3.57}$$

Recall that q is the largest integer such that $q < \frac{T}{\eta}$. Hence summing over $0 \le l \le q$ in (3.56) and applying Bihari's inequality afterwards imply that $\tilde{Y}(t) = 0\ a.s.\ \ \forall\, t \in [0,T]$. From this and from (3.57) we conclude that $\mathbb{E}\, \int_0^T |\tilde{Z}(s)|^2\, ds = 0$. This completes the proof. ∎

We close this section by giving the following example.

Example 3.3. Let $K = L^2(\mathbb{R}^d;\mathbb{R})$. Assume that a is a bounded nonnegative definite $d \times d$ real matrix defined on $[0,T] \times \mathbb{R}^d$, and that $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded function, for which there exist numbers $\alpha > 0$ and $0 < \gamma \le 1$ and $C < \infty$ such that

(i) $< \theta, a(t,x)\theta >_{\mathbb{R}^d} \ \ge \alpha\, |\theta|^2_{\mathbb{R}^d}$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\theta \in \mathbb{R}^d$,

(ii) $|a(t,x) - a(s,x)|_{L_2(\mathbb{R}^d)} + |b(t,x) - b(s,y)|_{\mathbb{R}^d} \le C\,(|x-y|^\gamma_{\mathbb{R}^d} + |t-s|^\gamma)$ for all $(t,x), (s,y) \in [0,T] \times \mathbb{R}^d$.

Define

$$A(t) = \frac{1}{2} \sum_{i,j=1}^d a^{ij}(t,x) \frac{\partial^2}{\partial\, x_i \partial\, x_j} + \sum_{i=1}^d b^i(t,x) \frac{\partial}{\partial\, x_i}\,.$$

Then there exists a unique positive function $p(t,x;s,y),\ 0 \le t < s$ and $x,y \in \mathbb{R}^d$, which is continuous jointly with respect to all its variables and has the property that if $\phi \in C_0^\infty(\mathbb{R}^d)$, then for each $s > 0$ the function

$$f(t,x) = \int p(t,x;s,y)\phi(y)\, dy\ (\,=: (U(s,t)\phi)(x)\,)$$

is in $C_b^{1,2}([0,T] \times \mathbb{R}^d)$ and satisfies

$$\begin{cases} \frac{\partial f}{\partial t} + A(t) f = 0, & 0 \le t < s, \\ f(s,\cdot) = \phi\,. \end{cases}$$

We refer the reader to Theorem 3.2.1 and Corollary 3.1.2 in [23] for more details.

As it can be seen from the definition of $U(s,t)$ given above it is shown in [23, Corollary 3.1.2] that $U(s,t)$ is an evolution operator or a time-inhomogenous semigroup in the language of [23], which is denoted by $T_{t,s}$.

Note that the condition (i) implies actually that $A(t)$ satisfies the coercivity property; cf. [5].

Now let H be a separable real Hilbert space (e.g. $L^2(\mathbb{R}^n;\mathbb{R})$) and consider the following problem:

$$
(3.58)\qquad \begin{cases} -\,dY(t,x) = \Bigg[\dfrac{1}{2}\left(\displaystyle\sum_{i,j=1}^{d} a^{ij}(t,x)\,\frac{\partial^2}{\partial x_i \partial x_j}\,Y(t,x)\right) \\ \qquad\qquad +\displaystyle\sum_{i=1}^{d}\left(b^i(t,x)\,\frac{\partial}{\partial x_i}\,Y(t,x)\right) \\ \qquad\qquad +f(Y(t),Z(t))(x)\Big]\Bigg]\,dt - \ (Z(t)\,dW(t))(x), \\ Y(T,x) = \phi(x), \end{cases}
$$

where $\phi \in L^2(\Omega,\mathcal{F}_T,\mathbb{P};K)$ and $f : K \times L_2(H;K) \to K$ is the mapping:

$$
f(y,z)(x) = (\,h(|y|) + \sqrt{k}\,|z|\,)\,v(x),
$$

for some fixed element v of K (e.g. $v(x) = e^{-\frac{1}{2}|x|^2}$, $x \in \mathbb{R}^d$), and for some positive constant k. The mapping h is the one defined in (3.2).[3]

From (3.4), the triangle inequality and the monotonicity of c_3, which is defined in (3.3), we conclude that f satisfies the condition (H2) with c being the function c_3. Thus, in particular, the equation (3.58) can be considered as a BSEE of the type of the equation (1.1). It follows then from Theorem 3.1 that there exists a unique solution (Y,Z) of (3.58) in $L^2_{\mathcal{F}}(0,T;K) \times L^2_{\mathcal{F}}(0,T;L_2(H;K))$.

4. Regularity properties of the solutions of BSEEs

We note that if, for each s, $A(s)$ (or A for the BSEE (3.5)) is bounded, then $\{U(s,0)\}_{s\geq 0}$ is a group, and so from (3.1) the solution Y of the BSEE (1.1) is a semimartingale that has a continuous version. In this section we shall discuss in general the continuity in t of the solution Y of the BSEE (1.1) and provide some apriori estimates for $\mathbb{E}\,[\sup_{t\in[0,T]}\ |Y(t)|^2\,]$ and for $\mathbb{E}\,[\sup_{0\leq t\leq T}\ |\ \int_t^T U(s,t)\ Z(s)\ dW(s)\ |^2\,]$.

Proposition 4.1. *Let $f \in L^2_{\mathcal{F}}(0,T;K)$ and $\xi \in L^2(\Omega,\mathcal{F}_T,\mathbb{P};K)$. Then the solution Y of the following BSEE:*

$$
(4.1)\qquad \begin{cases} -\,dY(t) = A\,Y(t)\,dt + f(t)\,dt - Z(t)\,dW(t) \\ \quad Y(T) = \xi \end{cases}
$$

has a version which is continuous almost surely as a process in K.

Proof. Note first that the solution of (4.1) is given by

$$
(4.2)\qquad Y(t) = U(T,t)\,\xi + \int_t^T U(s,t)\ f(s)\ ds - \int_t^T U(s,t)\ Z(s)\ dW(s).
$$

For convenience we shall show that each term of (4.2) is continuous in t.

Since $\{\mathcal{F}_t\,,\ 0 \leq t \leq T\}$ is the Wiener filtration, then according to Theorem 2.1, $\mathbb{E}\,[\,\xi\,|\,\mathcal{F}_t]$ is continuous in t for each $t \in [0,T]$. Consequently $\mathbb{E}\,[\,U(T,t)\,\xi|\,\mathcal{F}_t] =$

[3]It is also possible to let f depend on ω and on t.

$U(T,t)\,\mathbb{E}\,[\,\xi \,|\, \mathcal{F}_t]$ is also continuous in t. For the same reason $\mathbb{E}\,[\,U(s,t)\,f(s)|\,\mathcal{F}_t]$ is continuous in t for each $t \leq s \leq T$. Moreover, since

$$\mathbb{E}\,\left[\int_t^T U(s,t)\,f(s)\;ds \,|\, \mathcal{F}_t\right] = \int_t^T (U(s,t)\,\mathbb{E}\,[f(s)|\,\mathcal{F}_t])\;ds,$$

then $\mathbb{E}\,\left[\,\int_t^T U(s,t)\,f(s)\,ds \,|\, \mathcal{F}_t\right]$ is continuous in t.

But

$$Y(t) = \mathbb{E}\,[\;U(T,t)\,\xi \,|\, \mathcal{F}_t] + \mathbb{E}\,\left[\int_t^T U(s,t)\,f(s)\,ds \,|\, \mathcal{F}_t\right].$$

Hence Y is continuous. ∎

As a result from this proof and from (4.2) we obtain also the continuity of $\int_t^T U(s,t)\,Z(s)\,dW(s)$. Thus we conclude immediately the following continuity property of the solution of the BSEE (1.1).

Corollary 4.1. *The solution $Y(t)\,,\ 0 \leq t \leq T$, of the BSEE (1.1), which is given by the form (3.1), and the integral $\int_t^T U(s,t)\,Z(s)\,dW(s)$ are almost surely continuous in t.*

The following proposition provides some estimates for the solution (Y, Z) of the BSEE (1.1).

Proposition 4.2. *Under the same conditions as in Theorem 3.1 the solution (Y, Z) of (1.1) satisfies*

$$\mathbb{E}\,\left[\sup_{0\leq t\leq T} |Y(t)|^2\right] < \infty, \tag{4.3}$$

and

$$\mathbb{E}\,\left[\sup_{0\leq t\leq T} \left|\int_t^T U(s,t)\;Z(s)\;dW(s)\right|^2\right] < \infty. \tag{4.4}$$

Proof. Since Y is given in the following form:

$$\begin{aligned} Y(t) = U(T,t)\,\xi + \int_t^T U(s,t)\;f(s,Y(s),Z(s))\;ds \\ - \int_t^T U(s,t)\,Z(s)\,dW(s), \end{aligned}$$

then we have a.s.

$$\begin{aligned} |Y(t)| &= \left|\mathbb{E}\,[\,U(T,t)\,\xi + \int_t^T U(s,t)\;f(s,Y(s),Z(s))\;ds \,|\, \mathcal{F}_t\,]\right| \\ &\leq \mathbb{E}\,\left[M\,|\xi| + M\,\int_t^T |f(s,Y(s),Z(s))|\;ds \,|\, \mathcal{F}_t\right] \\ &\leq M\,\mathbb{E}\,\left[|\xi| + \int_0^T |f(s,Y(s),Z(s))|\;ds \,|\, \mathcal{F}_t\right], \end{aligned}$$

for all $t \in [0, T]$. The right hand side of this last inequality is a continuous martingale. It follows by using Doob's inequality for martingales that

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y(t)|^2\right] \le$$

$$2\,M^2\,\mathbb{E}\left[\sup_{0\le t\le T}\left(\mathbb{E}\left[\,|\xi|^2 + T\int_0^T |f(s,Y(s),Z(s))|^2\,ds\,|\,\mathcal{F}_t\,|\right)\right]$$

$$\le 8\,M^2\left(\mathbb{E}\left[\,|\xi|^2\,\right] + T\,\mathbb{E}\left[\int_0^T |f(s,Y(s),Z(s))|^2\,ds\right]\right).$$

But (H1), (H2) and Jensen's inequality imply that

$$\mathbb{E}\left[\int_0^T |f(s,Y(s),Z(s))|^2\,ds\right] \le 2\,c\left(\mathbb{E}\left[\int_0^T |Y(s)|^2\,ds\right]\right)$$

$$+\,2\,k\,\mathbb{E}\left[\int_0^T |Z(s)|^2\,ds\right] + 2\,\mathbb{E}\left[\int_0^T |f(s,0,0)|^2\,ds\right] < \infty.$$

Thus we derive (4.3).

Finally since for all $t \in [0, T]$

$$\int_t^T U(s,t)\;Z(s)\;dW(s) = U(T,t)\,\xi + \int_t^T U(s,t)\;f(s,Y(s),Z(s))\;ds - Y(t),$$

then

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_t^T U(s,t)\;Z(s)\;dW(s)\right|^2\right] \le$$

$$3\,M^2\,\mathbb{E}\left[\,|\xi|^2\,\right] + 3\,T\,M^2\,\mathbb{E}\left[\int_0^T |f(s)|^2\,ds\right] + \;\mathbb{E}\left[\sup_{0\le t\le T}|Y(t)|^2\right] < \infty.$$

The proof is complete. ∎

Remark 4.1. In Proposition 4.2 we estimated only the second order moments of the solution (Y, Z). However, as seen from its proof, one can easily obtain the same result for higher order moments. In particular, for $2 < p < \infty$ if $\mathbb{E}\,[\,|\xi|^p] + \mathbb{E}\,[\int_0^T |Y(s)|^p\,ds] + \mathbb{E}\,[\int_0^T |Z(s)|^p\,ds]\; < \infty$, then

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y(t)|^p\right] < \infty,$$

and

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_t^T U(s,t)\;Z(s)\;dW(s)\right|^p\right] < \infty.$$

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