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A Truncated Bivariate Cauchy Distribution

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Abstract. A truncated version of the bivariate Cauchy distribution is introduced. Explicit expressions for its moments and estimation procedures are derived. Unlike the Cauchy distribution, this possesses finite moments of all orders and could therefore be a better model for certain practical situations. An application with real data is discussed to show one such situation.

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1. Introduction

The main weakness of Cauchy distributions is that they have no moments. Nadarajah and Kotz [3] suggested a truncated univariate Cauchy distribution that overcomes this weakness. In this note, we study the corresponding bivariate version. The bivariate Cauchy distribution is given by the joint probability density function (pdf):

$$g(x, y) = \frac{1}{2\pi}\left(1 + x^2 + y^2\right)^{-3/2} \tag{1.1}$$

for $-\infty < x < \infty$ and $-\infty < y < \infty$. Let G denote the corresponding joint cumulative distribution function (cdf) given by:

$$G(x, y) = \frac{1}{4} + \frac{1}{2\pi}\left(\arctan x + \arctan y + \arctan \frac{xy}{\sqrt{1 + x^2 + y^2}}\right). \tag{1.2}$$

Then the truncated version of (1.1) is given by the joint pdf:

$$f(x, y) = \frac{1}{2\pi\Omega}\left(1 + x^2 + y^2\right)^{-3/2} \tag{1.3}$$

for $-\infty < B \leq x \leq A < \infty$ and $-\infty < D \leq y \leq C < \infty$, where $\Omega = G(A, C) - G(A, D) - G(B, C) + G(B, D)$. The cdf associated with (1.3) is:

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$$F(x,y) = \frac{1}{\Omega}\left\{G(x,y) - G(x,D) - G(B,y) + G(B,D)\right\}. \tag{1.4}$$

The corresponding marginal cdfs are:

$$F_X(x) = \frac{1}{\Omega}\left\{G(x,C) - G(x,D) - G(B,C) + G(B,D)\right\}$$

and

$$F_Y(y) = \frac{1}{\Omega}\left\{G(A,y) - G(A,D) - G(B,y) + G(B,D)\right\}.$$

The corresponding marginal pdfs are:

$$f_X(x) = \begin{cases} \frac{\mid C \mid}{2\Omega\pi}\left(1+x^2\right)^{3/2}\left\{\frac{\left(2C^*+5\right)\sqrt{C^*+1}}{8} + \frac{3\text{arcsinh}\left(\sqrt{C^*}\right)}{8\sqrt{C^*}}\right. \\ \qquad \left. -\frac{\left(2D^*+5\right)\sqrt{D^*+1}}{8} - \frac{3\text{arcsinh}\left(\sqrt{D^*}\right)}{8\sqrt{D^*}}\right\}, \ (D > 0), \\ \frac{\mid C \mid}{2\Omega\pi}\left(1+x^2\right)^{3/2}\left\{\frac{\left(2C^*+5\right)\sqrt{C^*+1}}{8} + \frac{3\text{arcsinh}\left(\sqrt{C^*}\right)}{8\sqrt{C^*}}\right. \\ \qquad \left. +\frac{\left(2D^*+5\right)\sqrt{D^*+1}}{8} + \frac{3\text{arcsinh}\left(\sqrt{D^*}\right)}{8\sqrt{D^*}}\right\}, \ (D < 0, C > 0), \\ \frac{\mid C \mid}{2\Omega\pi}\left(1+x^2\right)^{3/2}\left\{\frac{\left(2D^*+5\right)\sqrt{D^*+1}}{8} + \frac{3\text{arcsinh}\left(\sqrt{D^*}\right)}{8\sqrt{D^*}}\right. \\ \qquad \left. -\frac{\left(2C^*+5\right)\sqrt{C^*+1}}{8} + \frac{3\text{arcsinh}\left(\sqrt{C^*}\right)}{8\sqrt{C^*}}\right\}, \ (C < 0) \end{cases}$$

and

$$f_Y(y) = \begin{cases} \frac{\mid A \mid}{2\Omega\pi}\left(1+y^2\right)^{3/2}\left\{\frac{\left(2A^*+5\right)\sqrt{A^*+1}}{8} + \frac{3\text{arcsinh}\left(\sqrt{A^*}\right)}{8\sqrt{A^*}}\right. \\ \qquad \left. -\frac{\left(2B^*+5\right)\sqrt{B^*+1}}{8} - \frac{3\text{arcsinh}\left(\sqrt{B^*}\right)}{8\sqrt{B^*}}\right\}, \ (B > 0), \\ \frac{\mid A \mid}{2\Omega\pi}\left(1+y^2\right)^{3/2}\left\{\frac{\left(2A^*+5\right)\sqrt{A^*+1}}{8} + \frac{3\text{arcsinh}\left(\sqrt{A^*}\right)}{8\sqrt{A^*}}\right. \\ \qquad \left. +\frac{\left(2B^*+5\right)\sqrt{B^*+1}}{8} + \frac{3\text{arcsinh}\left(\sqrt{B^*}\right)}{8\sqrt{B^*}}\right\}, \ (B < 0, A > 0), \\ \frac{\mid A \mid}{2\Omega\pi}\left(1+y^2\right)^{3/2}\left\{\frac{\left(2B^*+5\right)\sqrt{B^*+1}}{8} + \frac{3\text{arcsinh}\left(\sqrt{B^*}\right)}{8\sqrt{B^*}}\right. \\ \qquad \left. -\frac{\left(2A^*+5\right)\sqrt{A^*+1}}{8} + \frac{3\text{arcsinh}\left(\sqrt{A^*}\right)}{8\sqrt{A^*}}\right\}, \ (A < 0), \end{cases}$$

where $A^* = A^2/(1+y^2)$, $B^* = B^2/(1+y^2)$, $C^* = C^2/(1+x^2)$, $D^* = D^2/(1+x^2)$ and ${}_2F_1$ denotes the Gauss hypergeometric function defined by

$$ {}_2F_1\left(a,b;c;x\right) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k} \frac{x^k}{k!}, $$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial, see Prudnikov et al. [5] and Gradshteyn and Ryzhik [1]. We refer to (1.3) as the truncated bivariate Cauchy distribution. Because (1.3) is defined over a finite interval, the truncated bivariate Cauchy distribution has all it moments. Thus, (1.3) may prove to be a better model for certain practical situations than one based on just the bivariate Cauchy distribution, see Section 4.

The rest of this paper is organized as follows: explicit expressions for the moments of (1.3) are derived in Section 2, estimation issues are discussed in Section 3, and an application to consumer price indices data is illustrated in Section 4.

2. Moments

We derive three representations for the product moment $E(X^\alpha Y^\beta)$. Theorems 2.1 and 2.2 provide general representations while Theorem 2.3 considers the particular case that β is an odd integer.

Theorem 2.1. *If* (X,Y) *has the joint pdf* (1.3) *then*

(2.1)

$$ E\left(X^\alpha Y^\beta\right) = \begin{cases} H(A,C)-H(B,C)-H(A,D)+H(B,D), \\ \qquad\qquad \text{if } B>0 \text{ and } D>0, \\ (-1)^\alpha\left\{H(-B,C)-H(-A,C)-H(-B,D)+H(-A,D)\right\}, \\ \qquad\qquad \text{if } A<0 \text{ and } D>0, \\ (-1)^{\alpha+\beta}\left\{H(-B,-D)-H(-B,-C)-H(-A,-D)\right. \\ \quad \left.+H(-A,-C)\right\}, \qquad \text{if } A<0 \text{ and } C<0, \\ (-1)^\beta\left\{H(A,-D)-H(A,-C)-H(B,-D)+H(-B,-C)\right\}, \\ \qquad\qquad \text{if } B>0 \text{ and } C<0, \\ H(A,C)+(-1)^\alpha H(-B,C)+(-1)^\beta H(A,-D) \\ \quad +(-1)^{\alpha+\beta}H(-B,-D), \text{if } B<0,\ A>0,\ D<0 \text{ and } C>0, \\ H(A,C)-H(A,D)+(-1)^\alpha\left\{H(-B,C)-H(-B,D)\right\}, \\ \qquad\qquad \text{if } B<0,\ A>0 \text{ and } D>0, \\ H(A,C)-H(B,C)+(-1)^\beta\left\{H(A,-D)-H(B,-D)\right\}, \\ \qquad\qquad \text{if } B>0,\ D<0 \text{ and } C>0, \\ (-1)^\alpha\left\{H(-B,C)-H(-A,C)\right\}+(-1)^{\alpha+\beta}\left\{H(-B,-D)\right. \\ \quad \left.-H(-A,-D)\right\}, \qquad \text{if } A<0,\ D<0 \text{ and } C>0, \\ (-1)^\beta\left\{H(A,-D)-H(A,-C)\right\}+(-1)^{\alpha+\beta}\left\{H(-B,-D)\right. \\ \quad \left.-H(-B,-C)\right\}, \qquad \text{if } B<0,\ A>0 \text{ and } C<0, \end{cases} $$

for $\alpha \geq 1$ *and* $\beta \geq 1$*, where*

(2.2)
$$ H(P,Q) = \frac{1}{2\pi}\int_0^P \int_0^Q \frac{x^\alpha y^\beta}{\left(1+x^2+y^2\right)^{3/2}} dydx. $$

Proof. The result of the theorem is obvious. For example, if $B > 0$ then $D > 0$ one can write

$$\begin{aligned}
E\left(X^\alpha Y^\beta\right) &= \frac{1}{2\pi}\int_B^A \int_D^C x^\alpha y^\beta \left(1+x^2+y^2\right)^{-3/2} dydx \\
&= \frac{1}{2\pi}\left\{ \int_0^A \int_0^C x^\alpha y^\beta \left(1+x^2+y^2\right)^{-3/2} dydx \right. \\
&\qquad - \int_0^A \int_0^D x^\alpha y^\beta \left(1+x^2+y^2\right)^{-3/2} dydx \\
&\qquad - \int_0^B \int_0^C x^\alpha y^\beta \left(1+x^2+y^2\right)^{-3/2} dydx \\
&\qquad \left. + \int_0^B \int_0^D x^\alpha y^\beta \left(1+x^2+y^2\right)^{-3/2} dydx \right\} \\
&= H(A,C) - H(B,C) - H(A,D) + H(B,D).
\end{aligned}$$

The remaining cases can be established similarly. ∎

Theorem 2.2. *If* (X, Y) *has the joint pdf* (1.3) *then* (2.2) *can be expressed in the form*

$$\begin{aligned}
H(P,Q) = \frac{P^{\alpha+1}Q^{1+2\beta}}{2\pi(\alpha+1)(1+2\beta)} \sum_{k=0}^{\infty} \frac{(-1)^k(\beta+1/2)_k(3/2)_kQ^{2k}}{(\beta+3/2)_k k!} \\
\times {}_2F_1\left(\frac{\alpha+1}{2}, \frac{3}{2}+k; \frac{\alpha+3}{2}; -P^2\right).
\end{aligned}$$

Proof. Setting $z = 1 + x^2 + y^2$, (2.2) can be reexpressed as

$$H(P,Q) = \frac{1}{4\pi}\int_0^P \int_{1+x^2}^{1+x^2+Q^2} x^\alpha \left(z-1-x^2\right)^{(\beta-1)/2} z^{-3/2} dzdx. \tag{2.3}$$

Now, an application of equation (2.2.6.1) in Prudnikov et al. [5] to calculate the inner integral in (2.3) shows that

$$\begin{aligned}
H(P,Q) &= \frac{Q^{1+2\beta}}{2\pi(1+2\beta)}\int_0^P x^\alpha \left(1+x^2\right)^{-3/2} {}_2F_1\left(\beta+\frac{1}{2}, \frac{3}{2}; \beta+\frac{3}{2}; -\frac{Q^2}{1+x^2}\right) dx \\
&= \frac{Q^{1+2\beta}}{2\pi(1+2\beta)}\int_0^P x^\alpha \left(1+x^2\right)^{-3/2} \sum_{k=0}^{\infty} \frac{(\beta+1/2)_k(3/2)_k}{(\beta+3/2)_k k!}\left(-\frac{Q^2}{1+x^2}\right)^k dx \\
&= \frac{Q^{1+2\beta}}{2\pi(1+2\beta)} \sum_{k=0}^{\infty} \frac{(\beta+1/2)_k(3/2)_k(-1)^kQ^{2k}}{(\beta+3/2)_k k!} \int_0^P x^\alpha \left(1+x^2\right)^{-3/2-k} dx, \qquad (2.4)
\end{aligned}$$

where we have used the definition of the Gauss hypergeometric function. The result of the theorem follows by applying equation (2.2.6.1) in Prudnikov et al. [5] again to calculate the integral in (2.4). ∎

Theorem 2.3. *If (X, Y) has the joint pdf (1.3) and if $(\beta - 1)/2 \geq 1$ is an integer then (2.2) can be expressed in the form*

$$H(P,Q) = \frac{1}{4\pi}\sum_{k=0}^{(\beta-1)/2}\binom{(\beta-1)/2}{k}(-1)^{(\beta-1)/2-k}\left\{S(k,Q)-S(k,0)\right\},$$

where

$$\begin{aligned} S(k,Q) &= \sum_{l=0}^{(\beta-1)/2-k}\binom{(\beta-1)/2-k}{l}\frac{P^{\alpha+1+2l}\left(1+Q^2\right)^{k-1/2}}{(k-1/2)(\alpha+1+2l)} \\ &\quad \times {}_2F_1\left(\frac{\alpha+1}{2}+l, \frac{1}{2}-k; \frac{\alpha+3}{2}+l; -\frac{P^2}{1+Q^2}\right). \end{aligned}$$

Proof. If β is an odd integer then one can write (2.3) as

$$\begin{aligned} H(P,Q) &= \frac{1}{4\pi}\sum k = 0^{(\beta-1)/2}\binom{(\beta-1)/2}{k} \\ &\quad \times \int_0^P\int_{1+x^2}^{1+x^2+Q^2} x^\alpha\left(-1-x^2\right)^{(\beta-1)/2-k} z^{k-3/2}dzdx \\ &= \frac{1}{4\pi}\sum_{k=0}^{(\beta-1)/2}\binom{(\beta-1)/2}{k}(-1)^{(\beta-1)/2-k}\int_0^P x^\alpha\left(1+x^2\right)^{(\beta-1)/2-k} \\ &\quad \times\left\{\frac{\left(1+Q^2+x^2\right)^{k-1/2}}{k-1/2}-\frac{\left(1+x^2\right)^{k-1/2}}{k-1/2}\right\}dx \\ &= \frac{1}{4\pi}\sum_{k=0}^{(\beta-1)/2}\binom{(\beta-1)/2}{k}\left\{S(k,Q)-S(k,0)\right\}, \end{aligned}$$

where

$$S(k,Q) = \frac{1}{k-1/2}\int_0^P x^\alpha\left(1+x^2\right)^{(\beta-1)/2-k}\left(1+Q^2+x^2\right)^{k-1/2}dx. \tag{2.5}$$

Note that (2.5) can be expanded as

(2.6)

$$\begin{aligned} S(k,Q) &= \frac{1}{k-1/2}\sum_{l=0}^{(\beta-1)/2-k}\binom{(\beta-1)/2-k}{l}\int_0^P x^{\alpha+2l}\left(1+Q^2+x^2\right)^{k-1/2}dx \\ &= \frac{1}{2k-1}\sum_{l=0}^{(\beta-1)/2-k}\binom{(\beta-1)/2-k}{l}\int_0^{P^2} y^{(\alpha-1)/2+l}\left(1+Q^2+y\right)^{k-1/2}dy. \end{aligned}$$

The result of the theorem follows by using equation (2.2.6.1) in Prudnikov et al. [5] to calculate the integral in (2.6). ∎

3. Estimation

Here, we consider estimation by the method of maximum likelihood and the method of moments. Suppose $\{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ is a random sample from (1.3). It is easy to show that the maximum likelihood estimators of A, B, C and D are $\widehat{A} = \max x_i$, $\widehat{B} = \min x_i$, $\widehat{C} = \max y_i$ and $\widehat{D} = \min y_i$, respectively. The method of moments estimators can be determined by equating, say

$$E(XY) = \frac{1}{n} \sum_{i=1}^{n} x_i y_i,$$

$$E\left(X^2 Y\right) = \frac{1}{n} \sum_{i=1}^{n} x_i^2 y_i,$$

$$E\left(X Y^2\right) = \frac{1}{n} \sum_{i=1}^{n} x_i y_i^2$$

and

$$E\left(X^2 Y^2\right) = \frac{1}{n} \sum_{i=1}^{n} x_i^2 y_i^2,$$

where the expectations can be calculated by using the results of Theorems 2.1 to 2.3.

4. Application

The bivariate Cauchy distribution has received applications in many areas, including biological analyses, clinical trials, stochastic modelling of decreasing failure rate life components, study of labour turnover, queueing theory, and reliability (see, for example, Nayak [4] and Lee and Gross [2]). For data from these areas, there is no reason to believe that empirical moments of any order should be infinite. Thus, the choice of the bivariate Cauchy distribution as a model is unrealistic since its product moments $E(X^\alpha Y^\beta)$ are not finite for all α and β. The alternative given by (1.3) will be a more appropriate model for the kind of data mentioned. The choice of the limits A, B, C and D could be based on historical records or the methods discussed in Section 3. As an example, consider modelling the dependence between the consumer price indices of Australia and the United States. The data for the years from 1901 to 2005 extracted from the website www.globalfindata.com/ are shown in Table 1. We fitted the models given by (1.1) and (1.3) to this data set. The method of maximum likelihood was used. Prior to fitting, a probability integral transformation was used to standardize the data to have Cauchy marginals. The two models yielded the following maximized log likelihoods: -494.7993 and -446.8655. It follows by the likelihood ratio test that the truncated model is a significant improvement. The fitted joint densities for the two models are shown in Figures 1 and 2.

Table 1. Consumer price index data for Australia and the United States for the years 1901–2005.

Year	OZ CPI	US CPI	Year	OZ CPI	US CPI	Year	OZ CPI	US CPI
1901	2.4282	7.644	1936	3.9644	14	1971	19.2357	41.1
1902	2.5663	7.84	1937	4.059	14.4	1972	20.0812	42.5
1903	2.5088	7.84	1938	4.1583	14	1973	22.7587	46.2
1904	2.3707	7.938	1939	4.2577	14	1974	26.4226	51.9
1905	2.4858	8.134	1940	4.532	14.1	1975	30.157	55.5
1906	2.4858	8.526	1941	4.7355	15.5	1976	34.4551	58.2
1907	2.4743	8.82	1942	5.1518	16.9	1977	37.6963	62.1
1908	2.6239	8.82	1943	5.166	17.4	1978	40.5852	67.7
1909	2.6239	9.31	1944	5.1944	17.8	1979	44.6719	76.7
1910	2.6814	9.31	1945	5.2369	18.2	1980	48.829	86.3
1911	2.762	9.506	1946	5.6201	21.5	1981	54.3249	94
1912	3.1187	9.8	1947	5.9607	23.4	1982	60.3141	97.6
1913	3.0266	10	1948	6.5568	24.1	1983	65.5281	101.3
1914	3.1509	10.1	1949	7.1529	23.6	1984	67.2192	105.3
1915	3.6825	10.3	1950	8.0044	25	1985	72.7151	109.3
1916	3.5905	11.6	1951	10.001	26.5	1986	79.8316	110.5
1917	3.6825	13.7	1952	10.9504	26.7	1987	85.5389	115.4
1918	3.8322	16.5	1953	11.3302	26.9	1988	92.0212	120.5
1919	4.3846	18.9	1954	11.3935	26.7	1989	99.2082	126.1
1920	4.807	19.4	1955	11.8394	26.8	1990	106	133.8
1921	4.3326	17.3	1956	12.6457	27.6	1991	107.6	137.9
1922	4.4275	16.9	1957	12.9286	28.4	1992	107.9	141.9
1923	4.554	17.3	1958	13.169	28.9	1993	110	145.8
1924	4.4275	17.3	1959	13.3246	29.4	1994	112.8	149.7
1925	4.5856	17.9	1960	13.7914	29.8	1995	118.5	153.5
1926	4.6172	17.7	1961	14.0177	30	1996	120.3	158.6
1927	4.6488	17.3	1962	14.0885	30.4	1997	120	161.3
1928	4.6172	17.1	1963	14.1733	30.9	1998	121.9	163.9
1929	4.7121	17.2	1964	14.6118	31.2	1999	124.1	168.3
1930	4.2719	16.1	1965	15.0645	31.8	2000	131.3	174
1931	3.8981	14.6	1966	15.5454	32.9	2001	135.4	176.7
1932	3.7373	13.1	1967	15.9698	33.9	2002	139.5	180.9
1933	3.7136	13.2	1968	16.479	35.5	2003	142.8	184.3
1934	3.7799	13.4	1969	17.0306	37.7	2004	146.5	190.3
1935	3.8603	13.8	1970	17.8793	39.8	2005	150.6	196.8

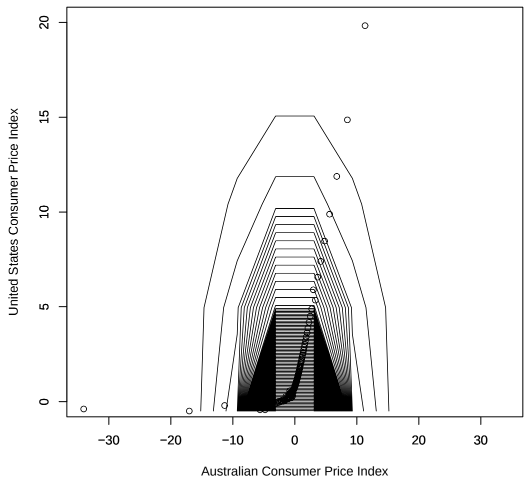

Figure 1. Fitted joint density of (1) for the data on consumer price indices of Australia and the United States

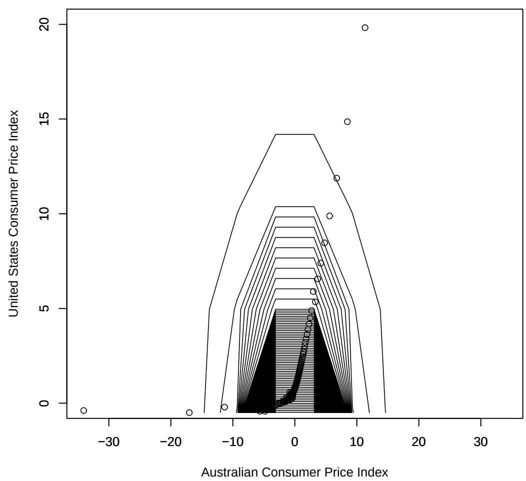

Figure 2. Fitted joint density of (3) for the data on consumer price indices of Australia and the United States

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