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On the Geodesics of Tubular Surfaces in Minkowski 3-Space

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Abstract. In this study, we research geodesics of tubular surfaces which is founded by using two-parameter spatial motion along a curve in Minkowski 3-space. To do this, we solve differential equation $\overline{D}_T T = 0$ of parametric curves on the tubular surface where $\overline{D}$ is the connection of tubular surface and $\overrightarrow{T}$ is the unit vector field of two parametric curves on the tubular surface in particular. It is shown that for fixed s, all of parametric curves of the tubular surface $M = \varphi(s,t)$ are geodesics and for fixed t only the curves $\varphi_{t\,=\,0}(s)$ are geodesics.

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1. Introduction

Let $\mathbb{R}^3 = \{(r_1, r_2, r_3) \mid r_1, r_2, r_3 \in \mathbb{R}\}$ be a 3-dimensional vector space, $r = (r_1, r_2, r_3)$ and $s = (s_1, s_2, s_3)$ be two vectors in $\mathbb{R}^3$. The Lorentz scalar product of the vectors r and s is defined by

$$\langle r, s\rangle_L = -r_1 s_1 + r_2 s_2 + r_3 s_3.$$

The space $\mathbb{R}^3_1 = \left(\mathbb{R}^3, \langle\ ,\ \rangle_L\right)$ is called 3-dimensional Lorentz space, or Minkowski 3-space. The Lorentz vector product of the vectors r and s is defined by

$$r \wedge_L s = \left(r_2 s_3 - r_3 s_2, r_1 s_3 - r_3 s_1, r_2 s_1 - r_1 s_2\right).$$

The vector r in $\mathbb{R}^3_1$ is called a spacelike vector, a lightlike (null) vector or a timelike vector if $\langle r, r\rangle_L > 0$, $\langle r, r\rangle_L = 0$ or $\langle r, r\rangle_L < 0$ respectively. The norm of the

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vector r is defined by $\|r\|_L = \sqrt{|\langle r, r\rangle_L|}$ and r is called a unit vector if $\|r\|_L = 1$ [4]. Semi-orthogonal matrix providing rotation the angle (hyperbolic) t around the vector $\overrightarrow{c}$. Denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha(s)$. Then T, N and B are the tangent, the principal normal and the binormal vector of the curve $\alpha(s)$ respectively.

If $\alpha(s)$ is a spacelike:
Case 1. Let T be spacelike, N spacelike and B timelike. This set of orthogonal unit vectors, known as the Frenet-Serret frame, has the following properties

$$\langle T, T\rangle_L = \langle N, N\rangle_L = 1, \langle B, B\rangle_L = -1,$$
$$\langle T, N\rangle_L = \langle N, B\rangle_L = \langle T, B\rangle_L = 0$$

and

$$T^{'} = \kappa N, N^{'} = -\kappa T + \tau B, B^{'} = \tau B.$$

where κ and τ curvature of the curve $\alpha(s)$ respectively. Given this set of coordinates, let $\alpha(s)$ be a curve parameterized by the arc length (s) and let $T(s)$ be the vector $T(s) = \alpha^{'}(s)$ where the prime indicates differentiation with respect to s. While there might be other canonical parameterizations, only a parameterization by the arc length leads to a normalized vector $T(s)$.

Case 2. Let T be spacelike, N timelike and B spacelike. This set of orthogonal unit vectors, known as the Frenet-Serret frame, has the following properties [4]:

$$\langle T, T\rangle_L = \langle B, B\rangle_L = 1, \langle N, N\rangle_L = -1$$
$$\langle T, N\rangle_L = \langle N, B\rangle_L = \langle T, B\rangle_L = 0$$

and

$$T^{'} = \kappa N, N^{'} = \kappa T + \tau B, B^{'} = \tau B.$$

The shape of the matrix depends on the type of the vector $\overrightarrow{c}$ as the following [2].

i. If $\overrightarrow{c(s)}$ is a spacelike vector, then

$$A_1(s,t) = I + C \cdot \sinh t + C^2 \cdot (-1 + \cosh t) \tag{1.1}$$

ii. If $\overrightarrow{c(s)}$ is a timelike vector, then

$$A_2(s,t) = I + C \cdot \sin t + C^2 \cdot (1 - \cos t) \tag{1.2}$$

If C is a semi-skew symmetric matrix, then

$$C(3,1) = \left\{ \begin{array}{c} C \in \mathbb{R}^3_3 \mid, C^T = -\varepsilon C \varepsilon, C = \begin{bmatrix} 0 & c_3 & -c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}, \\ c_i \in \mathbb{R}, \, \varepsilon = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \right\}.$$

Then, let $\overrightarrow{p}$ denote the ground vector and P denote the column matrix form of the point. The equations

$$C \cdot P = \overrightarrow{c} \wedge_L \overrightarrow{p} \tag{1.3}$$

and

$$\overrightarrow{c} \wedge_L (\overrightarrow{c} \wedge_L \overrightarrow{p}) = -\langle \overrightarrow{c}, \overrightarrow{p} \rangle_L + \langle \overrightarrow{c}, \overrightarrow{c} \rangle \overrightarrow{p}_L \tag{1.4}$$

are valid. Therefore, from equation (1.1) and if $\overrightarrow{c(s)}$ is a spacelike vector, then

$$A_1(s,t)P = \left[I + C \cdot \sinh t + C^2 \cdot (-1 + \cosh t)\right] P.$$

From the equation (1.2) and if $\overrightarrow{c(s)}$ is a timelike vector, then

$$A_2(s,t)P = \left[I + C \cdot \sin t + C^2 \cdot (1 - \cos t)\right] P.$$

Using the equations (1.3) and (1.4), we get

$$A_1(s,t)P = \overrightarrow{p} \cosh t + \langle \overrightarrow{c}, \overrightarrow{p} \rangle_L \overrightarrow{c} (1 - \cosh t) + (\overrightarrow{c} \wedge_L \overrightarrow{p}) \sinh t \tag{1.5}$$

and

$$A_2(s,t)P = \overrightarrow{p} \cos t - \langle \overrightarrow{c}, \overrightarrow{p} \rangle_L \overrightarrow{c} (1 - \cos t) + (\overrightarrow{c} \wedge_L \overrightarrow{p}) \sin t. \tag{1.6}$$

Let α be a space curve given by

$$\alpha : I \to \mathbb{R}^3_1, s \to \alpha(s)$$

be differentiable as for $s \in I \subset \mathbb{R}$. In additional, let a vector field defined $C(s)$ along the curve $\alpha(s)$ be given as

$$\begin{aligned} c \quad &: \quad \alpha(I) \to \bigcup_{s \in I} T_{\mathbb{R}^3_1} \\ s \quad &\to \quad c(s) = \left(\alpha(s), \overrightarrow{c(s)}\right) = \overrightarrow{c(s)} \mid_{\alpha(s)} . \end{aligned}$$

Let $C(s)$ be a semi-skew symetric matrix defined by the vector $\overrightarrow{c}$ for all $s \in I$. The matrices $A_1(s,t)$ and $A_2(s,t)$ are semi-orthogonal matrices defined by $C(s)$. The moving Frenet frame defined along the curve $\alpha(I)$ is $\left\{\alpha(s), \overrightarrow{T(s)}, \overrightarrow{N(s)}, \overrightarrow{B(s)}\right\}$ and p is a fixed point according to the frame. With these notations and assumptions, we can give the following definition:

Definition 1.1. *The motion $\varphi(s,t)(P) = A_{1,2}(s,t)P + \alpha(s)$ is called the two parameter motion defined along the curve in Minkowski $3-$space* [1]. *Here $\varphi(s,t)(P)$ indicates a trajectory level.*

Let us give some properties of the $\varphi(s,t)(P)$. We will always use the $\left\{\overrightarrow{T},\overrightarrow{N},\overrightarrow{B}\right\}$ instead of the Frenet frame $\left\{\overrightarrow{T(s)},\overrightarrow{N(s)},\overrightarrow{B(s)}\right\}$ for the rest of our work. We will also choose the tangent vector field $\overrightarrow{T}$ instead of the vector field $\overrightarrow{c}$. A trajectory of the point P indicates a surface under the two parameter motion. The equation of this surface is

i. If $\overrightarrow{c}$ is a spacelike vector, then from equations (1.1) and (1.5), we have

$$\varphi_1(s,t)(P) = \overrightarrow{p}\cosh t + \langle\overrightarrow{T},\overrightarrow{p}\rangle_L(1-\cosh t)\overrightarrow{T} + \sinh t\cdot(\overrightarrow{T}\wedge_L\overrightarrow{p}) + \alpha(s). \tag{1.7}$$

ii. If $\overrightarrow{c}$ is a timelike vector, then from equations (1.2) and (1.6), we have

$$\varphi_2(s,t)(P) = \overrightarrow{p}\cos t - \langle\overrightarrow{T},\overrightarrow{p}\rangle_L(1-\cos t)\overrightarrow{T} + \sin t(\overrightarrow{T}\wedge_L\overrightarrow{p}) + \alpha(s). \tag{1.8}$$

2. Helices on tubular surfaces

In this section, we will use frame $\left\{\overrightarrow{T},\overrightarrow{E_1},\overrightarrow{E_2}\right\}$ instead of Frenet frame of the curve and our calculations will be constructed on this case, where E_1 and E_2 are independent from choosing of the curve.

i. If $\alpha(s)$ is a spacelike curve, then tangent $\overrightarrow{T}$ is a spacelike and we have the following cases:
Let's take $\overrightarrow{p}=\lambda\overrightarrow{E_1}, \lambda\in\mathbb{R}$ in two parameter motion (1.7).

(a) $\overrightarrow{T}$ spacelike, $\overrightarrow{E_1}$ timelike and $\overrightarrow{E_2}$ spacelike. We have

$$\varphi_1(s,t)(p) = \lambda\overrightarrow{E_1}\cosh t + \lambda\overrightarrow{E_2}\sinh t + \alpha(s) \tag{2.1}$$

which is parametric equation of tubular surface in Minkowski 3-space defined along the curve $\alpha(s)$. If we take as $\alpha(s)=(0,0,s)$, $\overrightarrow{T}=(0,0,1)$ and another frame $\overrightarrow{E_1}=(1,0,0)$, $\overrightarrow{E_2}=(0,1,0)$, and substitute in equation (2.1), then we have

$$\varphi_1(s,t)(p) = (\lambda\cosh t, \lambda\sinh t, s)$$

which is a Lorentz cylinder.

(b) $\overrightarrow{T}$ spacelike, $\overrightarrow{E_1}$ spacelike and $\overrightarrow{E_2}$ timelike. We have

$$\varphi_1(s,t)(p) = \lambda\overrightarrow{E_1}\cosh t - \lambda\overrightarrow{E_2}\sinh t + \alpha(s) \tag{2.2}$$

which is parametric equation of tubular surface in Minkowski 3-space defined along the curve $\alpha(s)$. If we take as $\alpha(s)=(0,0,s)$, $\overrightarrow{T}=(0,0,1)$, $\overrightarrow{E_1}=(0,1,0)$, $\overrightarrow{E_2}=(1,0,0)$, and substitute in equation (2.2), then we have

$$\varphi_1(s,t)(p) = (-\lambda\sinh t, \lambda\cosh t, s)$$

which is a Lorentz cylinder.

ii. If $\alpha(s)$ is a timelike curve:
Let's take $\overrightarrow{p} = \lambda\overrightarrow{E_1}, \lambda \in \mathbb{R}$ in two parameter motion (1.8). We have

$$\varphi_2(s,t)(p) = \lambda\overrightarrow{E_1}\cos t + \lambda\overrightarrow{E_2}\sin t + \alpha(s) \tag{2.3}$$

which is parametric equation of tubular surface in Minkowski 3-space defined along the curve $\alpha(s)$. If we take as $\alpha(s) = (s,0,0)$, $\overrightarrow{T} = (1,0,0)$, $\overrightarrow{E_1} = (0,1,0)$, $\overrightarrow{E_2} = (0,0,1)$, and substitute in equation (2.3), then we have

$$\varphi_2(s,t)(p) = (s, \lambda\cos t, \lambda\sin t)$$

which is a Lorentz cylinder.

Let's take $t = s$. Then a curve on the tubular surface is obtained. The equation of this curve from equation (2.1) is

$$\beta(s) = \lambda\overrightarrow{E_1}\cosh s + \lambda\overrightarrow{E_2}\sinh s + \alpha(s). \tag{2.4}$$

From equation (2.2), it is

$$\beta(s) = \lambda\overrightarrow{E_1}\cosh s - \lambda\overrightarrow{E_2}\sinh s + \alpha(s). \tag{2.5}$$

From equation (2.3), it is

$$\beta(s) = \lambda\overrightarrow{E_1}\cosh s - \lambda\overrightarrow{E_2}\sinh s + \alpha(s). \tag{2.6}$$

These curves are helix curves on the tubular surfaces in Minkowski 3-space.

If $\alpha(s) = (0,0,s)$ is a spacelike curve, then

$$\beta(s) = (\cosh s, \sinh s, s) \tag{2.7}$$

is obtained from equation (2.4), where $\overrightarrow{T} = (0,0,1)$, $\overrightarrow{E_1} = (1,0,0)$, $\overrightarrow{E_2} = (0,1,0)$ and $\lambda = 1$.

$$\beta(s) = (-\sinh s, \cosh s, s) \tag{2.8}$$

is obtained from equation (2.5), where $\overrightarrow{T} = (0,0,1)$, $\overrightarrow{E_1} = (0,1,0)$, $\overrightarrow{E_2} = (1,0,0)$ and $\lambda = 1$.

If $\alpha(s) = (s,0,0)$ is a timelike curve, then

$$\beta(s) = (s, \cos s, \sin s) \tag{2.9}$$

is obtained from equation (2.6), where $\overrightarrow{T} = (1,0,0)$, $\overrightarrow{E_1} = (0,1,0)$, $\overrightarrow{E_2} = (0,0,1)$ and $\lambda = 1$.

These curves are helix curves on the cylinder in Minkowski 3-space with z-axis or x-axis.

3. Tubular surfaces defined by $\beta(s)$

In this section, we investigate tubular surfaces by using $\beta(s)$ curves in equations (2.4), (2.5) and (2.6). Furthermore, we can use Frenet frame of the curve. For the equations of tubular surfaces from equation (1.7),

$$\varphi^*(s,t)(P^*) = \lambda\overrightarrow{N^*(s)}\cosh t + \lambda\overrightarrow{B^*(s)}\sinh t + \beta(s) \tag{3.1}$$

and

$$\varphi^*(s,t)(P^*) = \lambda\overrightarrow{N^*(s)}\cosh t - \lambda\overrightarrow{B^*(s)}\sinh t + \beta(s) \tag{3.2}$$

are obtained, where $\overrightarrow{p^*} = \lambda \overrightarrow{N^*}$. Furthermore, from equation (1.8)

$$\varphi^*(s,t)(p^*) = \lambda \overrightarrow{N^*(s)} \cos t + \lambda \overrightarrow{B^*(s)} \sin t + \beta(s) \tag{3.3}$$

is also obtained. We can find the equations of tubular surfaces with the helping of $\beta(s)$ curves on the surfaces of the equations (2.1), (2.2) and (2.3).

(a) Tubular surface defined by $\beta(s) = (\cosh s, \sinh s, s)$.
By using the curve $\beta(s)$, then

$$\begin{aligned} \overrightarrow{T^*} &= \frac{1}{\sqrt{2}}(\sinh s, \cosh s, 1) \\ \overrightarrow{N^*} &= (-\cosh s, -\sinh s, 0) \\ \overrightarrow{B^*} &= \frac{1}{\sqrt{2}}(\sinh s, \cosh s, -1) \end{aligned}$$

are obtained. If we substitute these values in equation (3.1), where

$$\overrightarrow{p^*} = \lambda \overrightarrow{N^*} = \lambda(-\cosh s, -\sinh s, 0), \lambda \in \mathbb{R},$$

then

$$\varphi^*(s,t)(p^*) = \begin{pmatrix} -\lambda \cosh s \cosh t + \dfrac{\lambda \sinh s \sinh t}{\sqrt{2}} + \cosh s, \\ -\lambda \sinh s \cosh t + \dfrac{\lambda \cosh s \sinh t}{\sqrt{2}} + \sinh s, \\ -\dfrac{\lambda \sinh t}{\sqrt{2}} + s \end{pmatrix} \tag{3.4}$$

is obtained. This shows the characterization of tubular surface.

(b) Tubular surface defined by $\beta(s) = (-\sinh s, \cosh s, s)$.
Since $\overrightarrow{T^*} = \beta'(s) = (-\cosh s, \sinh s, 1)$, the curve is a null curve. Let's choose binormal vector field of the curve such that

$$\overrightarrow{B^*} = W_1 = \beta''(s) = (-\sinh s, \cosh s, 0).$$

We need to find the vector field $\overrightarrow{N^*}$ such that $\langle \overrightarrow{N^*}, \overrightarrow{N^*} \rangle_L = 0$ and $\langle \overrightarrow{T^*}, \overrightarrow{N^*} \rangle_L = 1$. For that reason, we can find V vector such that $\langle \overrightarrow{T^*}, V \rangle_L \neq 0$. If we take $V = (1,0,0)$, then $\langle \overrightarrow{T^*}, V \rangle_L \neq 0$. If we substitute $\overrightarrow{T^*}$ and V in equation

$$\overrightarrow{N^*} = \frac{1}{\langle \overrightarrow{T^*}, V \rangle_L} \left(V - \frac{\langle V, V \rangle_L}{2\langle \overrightarrow{T^*}, V \rangle_L} \overrightarrow{T^*} \right),$$

then

$$\overrightarrow{N^*} = \left(\frac{1}{2\cosh s}, \frac{\sinh s}{2\cosh^2 s}, \frac{1}{2\cosh^2 s} \right)$$

is obtained. So, Frenet frame of null curve in Minkowski 3-space is obtained. If we substitute these values in equation (3.2), then

$$\varphi^*(s,t)(p^*) = \begin{pmatrix} \dfrac{\lambda\cosh t}{2\cosh s} + \lambda\sinh s\sinh t - \sinh s, \\ \\ \dfrac{\lambda\sinh s\cosh t}{2\cosh^2 s} - \lambda\cosh s\sinh t + \cosh s, \\ \\ \dfrac{\lambda\cosh t}{2\cosh^2 s} + s \end{pmatrix} \tag{3.5}$$

is obtained. This shows the characterization of tubular surface.

(c) Tubular surface defined by $\beta(s) = (s, \cos s, \sin s)$.
Since $\overrightarrow{T^*} = \beta'(s) = (1, -\sin s, \cos s)$,the curve is a null curve. Let's choose binormal vector field of the curve such that

$$\overrightarrow{B^*} = W_1 = \beta''(s) = (0, -\cos s, -\sin s).$$

We need to find the vector field $\overrightarrow{N^*}$ such that $\langle\overrightarrow{N^*}, \overrightarrow{N^*}\rangle_L = 0$ and $\langle\overrightarrow{T^*}, \overrightarrow{N^*}\rangle_L = 1$. For that reason, we can find V vector such that $\langle\overrightarrow{T^*}, V\rangle_L \neq 0$. If we take $V = (1,0,0)$, then $\langle\overrightarrow{T^*}, V\rangle_L \neq 0$. If we substitute $\overrightarrow{T^*}$ and V in equation

$$\overrightarrow{N^*} = \frac{1}{\langle\overrightarrow{T^*}, V\rangle_L}\left(V - \frac{\langle V, V\rangle_L}{2\langle\overrightarrow{T^*}, V\rangle_L}\overrightarrow{T^*}\right),$$

then

$$\overrightarrow{N^*} = \left(-\frac{1}{2}, -\frac{\sin s}{2}, \frac{\cos s}{2}\right)$$

is obtained. So, Frenet frame of null curve in Minkowski 3-space is obtained. If we substitute these values in equation (3.3), then

$$\varphi^*(s,t)(p^*) = \begin{pmatrix} -\dfrac{\lambda\cos t}{2} + s, \\ \\ -\dfrac{\lambda\sin s\cos t}{2} - \lambda\cos s\sin t, \\ \\ \dfrac{\lambda\cos s\cos t}{2} - \lambda\sin s\sin t \end{pmatrix} \tag{3.6}$$

is obtained. This shows the characterization of tubular surface.

4. Geodesic curves of tubular surfaces

Definition 4.1. *Let $\alpha : I \to M$ be a curve such that $\overline{D}_T T = 0$, where $\overline{D}$ and T are connection of M and unit tangent vector field of α respectively. Then α is called a geodesics curve on M* [3].

Now we are ready to state the following theorem.

Theorem 4.1. *Let $\gamma : I \to M$ be a curve. Then $\overline{D}_T T = 0$ if and only if γ one of the following curves.*

i. *For any fixed s, the corresponding curve.*
ii. *The curves $\eta_{t=0}(s)$ which corresponds to $t = 0$ where M is tubular surface given by equation (3.4).*

Proof. If we denote the connections of $\mathbb{R}^3_1$ and M, by D and $\overline{D}$ respectively, then we can write the Gauss equation

$$\overline{D}_X Y = D_X Y + \varepsilon \langle S(X), Y\rangle_L U \tag{4.1}$$

$X, Y \in \chi(M)$, $\langle U, U\rangle_L = \varepsilon, \varepsilon = \mp 1$ [3]. Where $\chi(M)$ is vector space of tangential vector fields on M and U is the unit normal vector field of M and S denotes the Weingarten map of M. If we have $X = Y = T$, then (4.1) reduces to

$$\overline{D}_T T = D_T T + \varepsilon \langle S(T), T\rangle_L U \tag{4.2}$$

on the geodesics curves, $\overline{D}_T T = 0$, for this kind of curve we have

$$D_T T + \varepsilon \langle S(T), T\rangle_L U = 0. \tag{4.3}$$

On the other hand, since

$$\langle T, U\rangle_L = 0$$

or

$$\langle D_T T, U\rangle_L + \langle T, D_T U\rangle_L = 0$$

or

$$D_T U = S(T),$$

the last equation gives us

$$\langle T, S(T)\rangle_L = \langle S(T), T\rangle_L = -\langle D_T T, U\rangle_L$$

and equation (4.3) reduces to

$$D_T T - \varepsilon \langle D_T T, U\rangle_L U = 0. \tag{4.4}$$

i. In equation (3.4), choose those curves such that $s =$ constant. Then for these curves we have

$$\gamma(t) = \varphi_s(t) = \begin{pmatrix} -\lambda \cosh s \cosh t + \dfrac{\lambda \sinh s \sinh t}{\sqrt{2}} + \cosh s, \\[2ex] -\lambda \sinh s \cosh t + \dfrac{\lambda \cosh s \sinh t}{\sqrt{2}} + \sinh s, \\[2ex] -\dfrac{\lambda \sinh t}{\sqrt{2}} + s \end{pmatrix}, \tag{4.5}$$

$$T = \frac{\frac{d\gamma}{dt}}{\left\| \frac{d\gamma}{dt} \right\|_L}$$

$$T = \left(-\cosh s \sinh t + \frac{\cosh t \sinh s}{\sqrt{2}}, -\sinh t \sinh s + \frac{\cosh t \cosh s}{\sqrt{2}}, -\frac{\cosh t}{\sqrt{2}} \right)$$

and

(4.6)
$$D_T T = \left(-\cosh t\cosh s + \frac{\sinh s\sinh t}{\sqrt{2}}, -\cosh t\sinh s + \frac{\sinh t\cosh s}{\sqrt{2}}, -\frac{\sinh t}{\sqrt{2}}\right).$$

On the other hand, for the unit vector field U we have

(4.7)
$$U = \frac{\frac{d\varphi}{ds}\wedge_L \frac{d\varphi}{dt}}{\left\|\frac{d\varphi}{ds}\wedge_L \frac{d\varphi}{dt}\right\|_L} = \frac{1}{\sqrt{|\sinh^2 t - \cosh^2 t|}}\begin{pmatrix} -\cosh t\cosh s + \frac{\sinh s\sinh t}{\sqrt{2}}, \\ -\cosh t\sinh s + \frac{\sinh t\cosh s}{\sqrt{2}}, \\ -\frac{\sinh t}{\sqrt{2}} \end{pmatrix}$$

or

$$D_T T = U\Big|_{\gamma(t)}$$

or

$$\langle D_T T, U\rangle_L = \langle U, U\rangle_L = \varepsilon = -1$$

and from (4.4) we have

$$D_T T - (-1)\langle D_T T, U\rangle_L U = D_T T + \langle D_T T, U\rangle_L U = 0$$

and so, the equation (4.2) give us $\overline{D}_T T = 0$ which implies that each of $s =$ constant parameter curve, lies on M is a geodesic.

ii. Now we take the curves η such that $t =$ constant on M then we can obtain the parametric representation of η from (3.4) as

$$\eta(s) = \varphi_t(s) = \begin{pmatrix} -\lambda\cosh s\cosh t + \frac{\lambda\sinh s\sinh t}{\sqrt{2}} + \cosh s, \\ -\lambda\sinh s\cosh t + \frac{\lambda\cosh s\sinh t}{\sqrt{2}} + \sinh s, \\ -\frac{\lambda\sinh t}{\sqrt{2}} + s \end{pmatrix}. \tag{4.8}$$

In this case, the unit tangent vector field T of η is

$$T = \frac{\frac{d\eta}{ds}}{\left\|\frac{d\eta}{ds}\right\|_L}$$

$$T = \frac{1}{\sqrt{1 + (1-\lambda\cosh t)^2 - \frac{\lambda^2\sinh^2 t}{2}}}\begin{pmatrix} -\lambda\sinh s\cosh t + \frac{\lambda\sinh t\cosh s}{\sqrt{2}} + \sinh s, \\ -\lambda\cosh t\cosh s + \frac{\lambda\sinh t\sinh s}{\sqrt{2}} + \cosh s, \\ 1 \end{pmatrix}$$

and

(4.9)

$$D_T T = \frac{1}{1+(1-\lambda\cosh t)^2 - \frac{\lambda^2\sinh^2 t}{2}} \begin{pmatrix} -\lambda\cosh s\cosh t + \frac{\lambda\sinh t\sinh s}{\sqrt{2}} + \cosh s, \\ -\lambda\cosh t\sinh s + \frac{\lambda\sinh t\cosh s}{\sqrt{2}} + \sinh s, \\ 0 \end{pmatrix},$$

$$\langle D_T T, U\rangle_L = \frac{1}{1+(1-\lambda\cosh t)^2 - \frac{\lambda^2\sinh^2 t}{2}} \left(-\lambda\cosh^2 t + \frac{\lambda\sinh^2 t}{2} + \cosh t\right).$$

If we write the equation (4.7) and (4.9) in equation,

$$D_T T - (-1)\langle D_T T, U\rangle_L U = D_T T + \langle D_T T, U\rangle_L U = 0,$$

we obtain three algebric differential equations and since one of these equation in being

$$-\frac{1}{\sqrt{2}}\sinh t = 0,$$

the solution of this equation are $t = 0$. If we put this value of t in (3.4), we have

$$\eta(s) = \varphi_{t=0}(s) = ((1-\lambda)\cosh s, (1-\lambda)\sinh s, s)$$

which implies that only geodesic curves among the parameter curves $t =$ constant are parametric curve which corresponds to $t = 0$. The converse of the proof of this theorem in obvious. Since parameter curves in the equations (3.5) and (3.6) of the surfaces are not getting the equation $\overline{D}_T T = 0$. Parameter curves are not geodesic curves. This completes the proof. ∎

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