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On Legendre Curves in α-Sasakian Manifolds

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Abstract. The torsion of a Legendre curve of an α-Sasakian manifold is obtained. Necessary and sufficient conditions for Legendre curves having parallel mean curvature vector, having proper mean curvature vector, being harmonic and being of type $AW(k)$, $k = 1, 2, 3$ are also obtained.

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1. Introduction

A Riemannian submanifold with vanishing Laplacian of mean curvature vector ΔH is defined as a biharmonic submanifold by B.-Y. Chen [9]. In [11], it was proved that the only biharmonic curves in an Euclidean space are straight lines. In [4], curves satisfying $\Delta^{\perp} H = \lambda H$ in an Euclidean space were classified, where $\Delta^{\perp}$ denotes the Laplacian of the curve in the normal bundle and λ is a real valued function. In [1], the classification of curves satisfying $\Delta H = \lambda H$ and $\Delta^{\perp} H = \lambda H$ in a real space form were given. By looking the Chen's formula (Lemma 4.1, [8]), one sees that the Laplacian in the normal bundle of H, $\Delta^{\perp} H$, is an ingredient of the normal part of ΔH to M and $\Delta^{\perp} H = 0$ is less restrictive than $\Delta H = 0$. However, the condition $\Delta H = \lambda H$ does not imply $\Delta^{\perp} H = \lambda H$. The concepts of submanifolds of type $AW(k)$ are defined in [3]; in particular, curves of type $AW(k)$ were investigated in [2].

On the other hand in [6], Blair and Baikoussis introduced the notion of Legendre curves in a contact metric manifold. A 1-dimensional integral submanifold in the contact subbundle is called a *Legendre curve* [6]. The class of α-Sasakian manifolds [12] include Sasakian manifolds, thus it is a natural motivation for studying Legendre

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curves in α-Sasakian manifolds. The paper is organized as follows. In section 2, it is proved that a Legendre curve in an α-Sasakian manifold is a Frenet curve of order 3 and its torsion is always α. We also give a basic lemma for further use. Section 3 contains main results about Legendre curves having parallel mean curvature vector, having proper mean curvature vector, being harmonic and being of type $AW(k)$, $k = 1, 2, 3$.

2. Legendre curves in α-Sasakian manifolds

Let M be an almost contact metric manifold [7] with an almost contact metric structure (φ, ξ, η, g), that is, φ is a $(1, 1)$ tensor field, ξ is a vector field; η is a 1-form and g is a compatible Riemannian metric such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \varphi Y) = -g(\varphi X, Y), \quad g(X, \xi) = \eta(X) \tag{2.3}$$

for all $X, Y \in TM$.

An almost contact metric structure (φ, ξ, η, g) on M is called an *α-Sasakian structure* [12] if

$$(\nabla_X \varphi) Y = \alpha (g(X, Y)\xi - \eta(Y) X) \tag{2.4}$$

for some nonzero constant α. From (2.4) it follows that

$$\nabla_X \xi = -\alpha\varphi X, \tag{2.5}$$

$$(\nabla_X \eta) Y = -\alpha g(\varphi X, Y). \tag{2.6}$$

If $\alpha = 1$, an α-Sasakian structure reduces to a Sasakian structure.

Let $\gamma(s)$ be a curve in a Riemannian manifold M parameterized by the arc length. The curve γ is called a *Frenet curve of order* r if there exist orthonormal vector fields $E_1, \ldots, E_r$ along γ such that

$$\gamma' = E_1,\ \nabla_{\gamma'} E_1 = \kappa_1 E_2,\ \nabla_{\gamma'} E_2 = -\kappa_1 E_1 + \kappa_2 E_3, \ldots, \nabla_{\gamma'} E_r = -\kappa_{r-1} E_{r-1},$$

where $\kappa_1, \ldots, \kappa_{r-1}$ are positive smooth functions of s, and ∇ is Levi-Civita connection.

A 1-dimensional integral submanifold of a contact manifold is called a *Legendre curve.* It is known from [5] that a 3-dimensional contact metric manifold is Sasakian if and only if the torsion of its Legendre curves is equal to 1. In [5], it was also shown that for a 3-dimensional manifold M endowed with the contact metric structure $(\varphi, \xi, \eta, g, \varepsilon)$, M is Sasakian if and only if the torsion of its Legendre curves is equal to ε. In [14], it was shown that in a Legendre curve $\gamma(s)$ parametrized by the arc length in a Kenmotsu manifold, such that $\nabla_{\dot\gamma}\dot\gamma$ is parallel to the structure vector field ξ, is a circle.
Now, we study a Legendre curve on an α-Sasakian manifold.

Let $\gamma(s)$ be a Legendre curve in an α-Sasakian manifold. Then its associated Frenet frame is $\{\gamma', \varphi\gamma', \xi\}$, so that we have the following equations:

$$\nabla_{\gamma'}\gamma' = k\varphi\gamma', \tag{2.7}$$

$$\nabla_{\gamma'}\varphi\gamma' = -k\gamma' + \alpha\xi, \tag{2.8}$$

$$\nabla_{\gamma'}\xi = -\alpha\varphi\gamma'. \tag{2.9}$$

Hence, we conclude the following:

Proposition 2.1. *In an α-Sasakian manifold, a Legendre curve is a Frenet curve of order 3 and its torsion is always α.*

In view of (2.7), (2.8) and (2.9) we can state the following:

Lemma 2.1. *Let $\gamma(s)$ be a Legendre curve in an α-Sasakian manifold. Then*

$$\nabla_{\gamma'}\nabla_{\gamma'}\gamma' = -k^2\gamma' + k'\varphi\gamma' + \alpha k\xi, \tag{2.10}$$

$$\nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\gamma' = -3kk'\gamma' + \left(k'' - k\left(k^2 + \alpha^2\right)\right)\varphi\gamma' + 2\alpha k'\xi. \tag{2.11}$$

3. Main results

Consider a curve γ in a 3-dimensional Riemannian manifold. Chen [8] proved the following identity:

$$\Delta H = \Delta H = -\nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\gamma',$$

where H is the mean curvature vector. Moreover, the Laplacian of the mean curvature in the normal bundle (see [13]) is defined by

$$\Delta^{\perp} H = -\nabla^{\perp}_{\gamma'}\nabla^{\perp}_{\gamma'}\nabla^{\perp}_{\gamma'}\gamma',$$

where $\nabla^{\perp}$ denotes the normal connection in the normal bundle.

A curve $\gamma(s)$ in a Riemannian manifold M is called a *curve with proper mean curvature vector field* [10] if $\Delta H = \lambda H$, where λ is a function. In particular, if $\Delta H = 0$ then it becomes a *biharmonic curve* [9].

A curve $\gamma(s)$ is known to be a *curve with proper mean curvature vector field* in the normal bundle [4] if $\Delta^{\perp} H = \lambda H$, where $\Delta^{\perp} H$ is the Laplacian of the mean curvature in the normal bundle and λ is a function. In particular, if $\Delta^{\perp} H = 0$ then it reduces to a *curve with harmonic mean curvature vector field* in the normal bundle [4].

Theorem 3.1. *Let $\gamma(s)$ be a Legendre curve in an α-Sasakian manifold. Then γ has parallel mean curvature vector field if and only if $k = 0$.*

Proof. The proof is obvious from (2.10). ∎

Theorem 3.2. *Let $\gamma(s)$ be a Legendre curve in an α-Sasakian manifold. Then γ is a curve with proper mean curvature vector field if and only if either $k = 0$ or λ is a constant equal to $\alpha^2 + k^2$.*

Proof. We note that

$$\Delta H = -\nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\gamma'.$$

In view of (2.11), the condition $\Delta H = \lambda H$ gives

$$3kk'\gamma' - \left(k'' - k\left(k^2+\alpha^2\right)\right)\varphi\gamma' - 2\alpha k'\xi = \lambda k\varphi\gamma', \tag{3.1}$$

which implies that

(1) $kk' = 0$,
(2) $k'' - k\left(k^2+\alpha^2-\lambda\right) = 0$ and
(3) $\alpha k' = 0$.

From (3) we have $k = c$, where c is a constant. Then in view of (2), we find that either $c = 0$ or $\lambda = c^2 + \alpha^2$. The converse is straightforward. ∎

As a corollary, we have the following result:

Corollary 3.1. *A Legendre curve in an α-Sasakian manifold is biharmonic if and only if its curvature is zero.*

Next, we prove the following:

Theorem 3.3. *Let $\gamma(s)$ be a Legendre curve in an α-Sasakian manifold. Then γ is a curve with proper mean curvature vector field in the normal bundle if and only if either $k = 0$ or k is a nonzero constant and $\lambda = \alpha^2$.*

Proof. From (2.10), we have

$$\left(\nabla_{\gamma'}H\right)^{\perp} = k'\varphi\gamma' + \alpha k\xi. \tag{3.2}$$

From the above equation, we obtain the following equation.

$$\nabla_{\gamma'}\left(\left(\nabla_{\gamma'}H\right)^{\perp}\right) = -kk'\gamma' + \left(k'' - \alpha^2 k\right)\varphi\gamma' + 2\alpha k'\xi,$$

which gives

$$\Delta^{\perp}H = -\left(k'' - \alpha^2 k\right)\varphi\gamma' - 2\alpha k'\xi. \tag{3.3}$$

Now if $\Delta^{\perp}H = 0$ then from (3.3), we get

(1) $k'' - \alpha^2 k + \lambda k = 0$ and
(2) $k' = 0$.

From (2), it follows that k is some constant c. Then from (1), we get $c\left(\lambda - \alpha^2\right) = 0$ which implies that either $c = 0$ or $c \neq 0$ and $\lambda = \alpha^2$. The converse follows easily. ∎

In particular, we can state the following:

Corollary 3.2. *A Legendre curve in an α-Sasakian manifold is with harmonic mean curvature vector field in the normal bundle if and only if $k = 0$.*

Definition 3.1. *A Frenet curve $\gamma(s)$ is said to be* [2]

(i) *of type $AW(1)$ if $N_3(s) = 0$,*
(ii) *of type $AW(2)$ if*

$$\left\|N_2(s)\right\|^2 N_3(s) = \left\langle N_3(s), N_2(s)\right\rangle N_2(s), \tag{3.4}$$

(iii) *of type $AW(3)$ if*

$$\left\|N_1(s)\right\|^2 N_3(s) = \left\langle N_3(s), N_1(s)\right\rangle N_1(s), \tag{3.5}$$

where

$$N_1(s) = (\gamma'')^{\perp}(s), \quad N_2(s) = (\gamma''')^{\perp}(s), \quad N_3(s) = \left(\gamma^{(iv)}\right)^{\perp}(s).$$

For general case, we refer to [3].

Let $\gamma(s)$ be a Legendre curve in an α-Sasakian manifold. Then from (2.7), (2.10), (2.11) we get

$$N_1(s) = k\varphi\gamma', \tag{3.6}$$

$$N_2(s) = k'\varphi\gamma' + \alpha k\xi, \tag{3.7}$$

$$N_3(s) = \left(k'' - k\left(k^2+\alpha^2\right)\right)\varphi\gamma' + 2\alpha k'\xi, \tag{3.8}$$

respectively.

Theorem 3.4. *A Legendre curve in an α-Sasakian manifold is of type $AW(1)$ if and only if $k = 0$.*

Proof. If a Legendre curve $\gamma(s)$ in an α-Sasakian manifold is of type $AW(1)$ then from (3.8) we have

(1) $k'' - k\left(k^2+\alpha^2\right) = 0$ and
(2) $k' = 0$.

The statement (2) implies that k is a constant, which in view of (1) becomes zero. The converse is easily verified. ∎

Theorem 3.5. *A Legendre curve in an α-Sasakian manifold is of type $AW(2)$ if and only if either $k = 0$ or k satisfies the differential equation*

$$2\alpha\left(k'\right)^2 - \alpha k\left(k'' - k\left(k^2+\alpha^2\right)\right) = 0.$$

Proof. Putting the values from (3.7) and (3.8) in (3.4), we get

$$\left\{2\alpha^2 kk' + k'\left(k'' - k\left(k^2+\alpha^2\right)\right)\right\}\alpha k = 2\alpha k'\left(\alpha^2k^2 + \left(k'\right)^2\right) \tag{3.9}$$

$$\left\{2\alpha^2 kk' + k'\left(k'' - k\left(k^2+\alpha^2\right)\right)\right\}k' = \left(\alpha^2k^2 + \left(k'\right)^2\right)\left(k'' - k\left(k^2+\alpha^2\right)\right). \tag{3.10}$$

If $k = 0$, then in view of (3.9) and (3.10), the Legendre curve becomes of type $AW(2)$. If $k \neq 0$ and the Legendre curve is of type $AW(2)$, then from (3.9) and (3.10) we obtain

$$\left(\alpha^2k^2 + \left(k'\right)^2\right)\left\{2\alpha\left(k'\right)^2 - \alpha k\left(k'' - k\left(k^2+\alpha^2\right)\right)\right\} = 0. \tag{3.11}$$

Since $k \neq 0$ so $\left(\alpha^2k^2 + \left(k'\right)^2\right)$ cannot vanish. Therefore, we have

$$2\alpha\left(k'\right)^2 - \alpha k\left(k'' - k\left(k^2+\alpha^2\right)\right) = 0,$$

which proves the theorem. ∎

Theorem 3.6. *A Legendre curve in an α-Sasakian manifold is of type $AW(3)$ if and only if k is a constant.*

Proof. In view of (3.6), (3.8) and (3.5), the condition for a Legendre curve $\gamma(s)$ in an α-Sasakian manifold to be of type $AW(3)$ is equivalent to the following relation

$$k^2\left(\left(k'' - k\left(k^2+\alpha^2\right)\right)\varphi\gamma' + 2\alpha k'\xi\right) = k^2\left(k'' - k\left(k^2+\alpha^2\right)\right)\varphi\gamma',$$

which is equivalent to $k' = 0$. ∎

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