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Analytic Solution of a Free and Forced Convection with Suction and Injection Over a Non-Isothermal Wedge

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Abstract. The Governing Principle of Dissipative Processes (GPDP) formulated by Gyarmati into non-equilibrium thermodynamics is employed to study the effects of suction, injection in laminar, two dimensional, combined free and forced convection flow over a non-isothermal wedge. The velocity and temperature functions inside the boundary layers are approximated by simple third order polynomial and the variational principle is formulated. In addition, the Euler-Lagrange equations of the variational principle are obtained as coupled polynomial equations in terms of momentum and thermal boundary layer thicknesses. Moreover, the effect of buoyancy force on heat transfer and skin friction is analysed for different values of Reynolds, Grashof, Prandtl numbers, wedge parameter and wall temperature exponent. Finally, the obtained analytical solutions are compared with known series solutions and the comparison establishes the fact that the accuracy is remarkable.

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1. Introduction

The effect of buoyancy forces on flow and heat transfer is usually ignored when a forced convection flow over a cooled or heated surface is studied. However, neglecting buoyancy effects cannot be justified, because under certain circumstances, the buoyancy forces influences the flow and temperature functions despite the presence of forced convection flow.

The prime aim of this study deals with the application of the Governing Principle of Dissipative Processes (GPDP) to heat transfer and the buoyancy effects in flow over a non-isothermal wedge with suction and injection, using the recent developments in thermodynamics of irreversible processes, and to obtain analytical solutions with the help of a variational technique based on the GPDP. According to the boundary layer theory, the irreversible processes of momentum and heat transfer

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in flows around bodies occur inside a very thin layer adjacent to the surface of the body. Hence the appropriate way of studying these non-equilibrium processes is by the method of irreversible thermodynamics.

2. Governing equations and boundary conditions

The system of steady, two dimensional, incompressible and laminar fluid flow of constant transport coefficients over a non-isothermal wedge is considered, whose apex angle is $\pi\beta$. The free stream flow velocity U_∞ and the surface temperature T_0 satisfy the power law

$$U_\infty = ax^m \text{ and } T_0 - T_\infty = bx^n, \tag{2.1}$$

where U_∞ is obtained by the inviscid potential flow, a and b are constants, m is the wedge angle parameter and n is the wall temperature exponent. The coordinate axes along x and y directions are considered along and perpendicular to the surface, respectively. On applying the boundary layer approximations if viscous dissipation effect is ignored, the conservation equations of mass, momentum and energy are respectively given by

$$\begin{aligned} u_x + v_y &= 0 \text{ (mass)}, \\ uu_x + vu_y &= \nu u_{yy} + U_\infty U'_\infty + gB(T - T_\infty)\sin\left(\frac{\pi\beta}{2}\right) \\ &\quad + \left[\int_y^\infty gB(T - T_\infty)\cos\left(\frac{\pi\beta}{2}\right) dy\right]_x \text{ (momentum)}, \\ uT_x + vT_y &= \alpha T_{yy} \text{ (energy)}. \end{aligned} \tag{2.2}$$

In the above equations, the subscripts indicate partial differentiation while u, v, T and T_∞ represent the velocity component in the x-direction, velocity component in the y-direction, the temperature inside the boundary layer and the free stream temperature respectively. The symbol ν and α indicate kinematic viscosity and thermal diffusivity respectively.

The initial and boundary conditions of the system are

$$\begin{aligned} &y = 0;\ u = 0,\ v = v_0(x)(\text{constant}),\ T = T_0(x)(\text{uniform}), \\ &y = \infty;\ u = U_\infty,\ T = T_\infty(\text{uniform}), \end{aligned} \tag{2.3}$$

where $v_0(x)$ is suction/injection velocity.

The GPDP has been already applied for various dissipative systems and was established as the most general and exact variational principle of macroscopic continuum physics. For the description of viscous flow systems, Vincze [11] has used the GPDP to derive the equations of thermohydrodynamics. Many other variational principles have been already shown as partial forms of Gyarmati's principle. Gyarmati [4, 5] has established that the local potential principle of Glansdorff and Prigogine is same as the force representation of GPDP. Singh [8] has proved that the lagrangian thermodynamics of Biot is equivalent to the flux representation of GPDP. Recently Antony Raj [1] and Chandrasekar [2, 3] have applied this principle to unsteady boundary layer flow and heat transfer problems.

3. Variational formulation

Gyarmati [4, 5] has formulated the GPDP which describes the evolution of irreversible processes in space and time, which is given in the energy picture

$$\delta \int_V \left[T\sigma - \Psi^* - \Phi^*\right] dV = 0, \tag{3.1}$$

where the integration is taken over the volume, V, of the system.

Since the balance equations play a fundamental role in the formulations of GPDP based on irreversible thermodynamics, we write the governing equations of motion (2.2) in the following balance forms:

$$\begin{aligned} \nabla \cdot \vec{V} &= 0, \\ \rho\left(\vec{V} \cdot \nabla\right) \vec{V} + \nabla \cdot \bar{\bar{P}} &= gB\rho\left(T - T_\infty\right) \sin\left(\frac{\pi\beta}{2}\right) \\ &+ \left[\int_y^\infty gB\rho\left(T - T_\infty\right) \cos\left(\frac{\pi\beta}{2}\right) dy\right]_x, \\ \rho C_p\left(\vec{V} \cdot \nabla\right) T + \nabla \cdot \vec{J}_q &= 0. \end{aligned} \tag{3.2}$$

Here, the pressure tensor $\bar{\bar{P}}$ can be decomposed as [4]

$$\bar{\bar{P}} = p\,\bar{\bar{\delta}} + \overset{\circ}{\bar{\bar{P}}}{}^{vs}, \tag{3.3}$$

where p is the hydrostatic pressure and $\overset{\circ}{\bar{\bar{P}}}{}^{vs}$ is the symmetrical part of the viscous pressure tensor whose trace is zero. The constitutive relations connecting the independent forces and fluxes for the present two dimensional problem are [6, 7]

$$P_{12} = -L_s\left(\frac{\partial u}{\partial y}\right) \text{ and } J_q = -L_\lambda\left(\frac{\partial lnT}{\partial y}\right), \tag{3.4}$$

where P_{12} is the only component of $\overset{\circ}{\bar{\bar{P}}}{}^{vs}$ and J_q is the energy flux. L_s is equal to the coefficient of viscosity μ while $L_\lambda = \lambda T$, where λ is the conductivity. In the energy picture, the proper state variable is lnT instead of T. The energy dissipation for this problem is

$$T\sigma = -J_q\left(\frac{\partial lnT}{\partial y}\right) - P_{12}\left(\frac{\partial u}{\partial y}\right). \tag{3.5}$$

The dissipation potentials in energy picture are

$$\begin{aligned} \Psi^* &= T\Psi = \left(\frac{1}{2}\right)\left[L_\lambda\left(\frac{\partial lnT}{\partial y}\right)^2 + L_s\left(\frac{\partial u}{\partial y}\right)^2\right], \\ \Phi^* &= T\Phi = \left(\frac{1}{2}\right)\left[R_\lambda J_q^2 + R_s P_{12}^2\right]. \end{aligned} \tag{3.6}$$

Here $L_\lambda = R_\lambda^{-1}$ and $L_s = R_s^{-1}$.

Using equations (3.5) and (3.6), we formulate the principle (3.1) in the form:

$$\begin{aligned} &\delta \int_0^l \int_0^\infty \Big[- J_q \left(\frac{\partial lnT}{\partial y}\right) - P_{12}\left(\frac{\partial u}{\partial y}\right) - \left(\frac{L_\lambda}{2}\right)\left(\frac{\partial lnT}{\partial y}\right)^2 \\ &- \left(\frac{L_s}{2}\right)\left(\frac{\partial u}{\partial y}\right)^2 - \left(\frac{1}{2}\right)\left(\frac{J_q^2}{L_\lambda}\right) - \left(\frac{1}{2}\right)\left(\frac{P_{12}^2}{L_s}\right)\Big] dydx = 0, \end{aligned} \tag{3.7}$$

in which l is the characteristic length of the surface.

4. Method of solution

We consider a system of two dimensional, laminar, inviscid potential flow past an unlimited wedge placed symmetrically in a stream with apex at the origin and the center line on the positive x-axis. The wedge angle parameter m is connected with the apex angle $\pi\beta$ by the relation

$$m = \frac{\beta}{(2-\beta)} \quad \text{(or)} \quad \beta = \frac{2m}{(m+1)} \tag{4.1}$$

To begin with the thermodynamic analysis, we select the trial functions for velocity and temperature fields inside their respective boundary layers as

$$\begin{aligned} &\frac{u}{U_\infty} = \frac{3y}{d_1} - \frac{3y^2}{{d_1}^2} + \frac{y^3}{{d_1}^3}, (y < d_1); u = U_\infty, (y \geq d_1) \\ &\text{and} \\ &\frac{(T-T_\infty)}{(T_0 - T_\infty)} = 1 - \frac{3y}{2d_2} + \frac{y^3}{2{d_2}^3}, (y < d_2); T = T_\infty, (y \geq d_2) \end{aligned} \tag{4.2}$$

which satisfy the conditions,

$$\begin{aligned} &y = 0;\ u = 0,\ v = v_0(x),\ T = T_0(x),\ \left(\frac{\partial^2 T}{\partial y^2}\right) = 0, \\ &y = d_1;\ u = U_\infty,\ v = v_0(x),\ \left(\frac{\partial u}{\partial y}\right) = 0,\ \left(\frac{\partial^2 u}{\partial y^2}\right) = 0, \\ &y = d_2;\ T = T_\infty,\ \left(\frac{\partial T}{\partial y}\right) = 0,\ \left(\frac{\partial^2 T}{\partial y^2}\right) = 0. \end{aligned} \tag{4.3}$$

The unknown quantities d_1 and d_2 are the extent of the hypothetical hydrodynamical and thermal boundary layer thicknesses respectively. These unknowns are to be determined from our thermodynamic analysis. The transverse velocity component v is obtained from the mass balance equation (2.2) as

$$v = \left(\frac{mU_\infty}{x}\right)\left[\frac{-3y^2}{2d_1} + \frac{y^3}{{d_1}^2} - \frac{y^4}{4{d_1}^3}\right]$$

$$+\,U_\infty\left[\frac{3y^2}{2{d_1}^2}-\frac{2y^3}{{d_1}^3}+\frac{3y^4}{4{d_1}^4}\right]d_1'+v_0(x). \tag{4.4}$$

The velocity and temperature polynomial trial functions (4.2) and the boundary conditions (4.3) are used in the governing equations (3.2) and on integration with respect to y with the help of their corresponding smooth-fit conditions $\left(\frac{\partial u}{\partial y}\right)=0$ and $\left(\frac{\partial T}{\partial y}\right)=0$, the momentum flux P_{12} and energy flux J_q for $Pr\geq 1$ (ie) $d_1\geq d_2$ are obtained as follows:

$$\begin{aligned}
\frac{-P_{12}}{L_s} = \left(\frac{\partial u^*}{\partial y}\right) = {} & \left(\frac{U_\infty}{d_1}\right)+\left(\frac{mU_\infty^2}{\nu x}\right)\left[\frac{53d_1}{160}-y+\frac{3y^3}{2d_1^2}-\frac{3y^4}{2d_1^3}+\frac{3y^5}{4d_1^4}\right.\\
& \left.-\frac{y^6}{4d_1^5}+\frac{y^7}{28d_1^6}\right]+\left(U_\infty^2\frac{d_1'}{\nu}\right)\left[\frac{9}{160}-\frac{3y^3}{2d_1^3}+\frac{3y^4}{d_1^4}-\frac{9y^5}{4d_1^5}\right.\\
& \left.+\frac{3y^6}{4d_1^6}-\frac{3y^7}{28d_1^7}\right]+\left[v_0(x)\frac{U_\infty}{\nu}\right]\left[\frac{-3}{4}+\frac{3y}{d_1}-\frac{3y^2}{d_1^2}+\frac{y^3}{d_1^3}\right]\\
& +\left[\left(\frac{gBd_1}{\nu}\right)(T_0-T_\infty)\right]\left\{\left[\sin\left(\frac{\pi\beta}{2}\right)\left(\frac{1}{2}-\frac{d_1}{4d_2}+\frac{d_1^3}{40d_2^3}\right.\right.\right.\\
& \left.\left.-\frac{y}{d_1}+\frac{3y^2}{4d_1d_2}-\frac{y^4}{8d_1d_2^3}\right)\right]+\left(\frac{nd_2}{x}\right)\left[\cos\left(\frac{\pi\beta}{2}\right)\left(\frac{3}{16}-\frac{d_1}{6d_2}\right.\right.\\
& \left.\left.+\frac{d_1^2}{16d_2^2}-\frac{d_1^4}{240d_2^4}-\frac{3y}{8d_1}+\frac{y^2}{2d_1d_2}-\frac{y^3}{4d_1d_2^2}+\frac{y^5}{40d_1d_2^4}\right)\right]\\
& +d_2'\left[\cos\left(\frac{\pi\beta}{2}\right)\left(\frac{3}{16}-\frac{d_1^2}{16d_2^2}+\frac{d_1^4}{80d_2^4}-\frac{3y}{8d_1}+\frac{y^3}{4d_1d_2^2}\right.\right.\\
& \left.\left.\left.-\frac{3y^5}{40d_1d_2^4}\right)\right]\right\}(d_2\leq y\leq d_1)
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
\frac{-J_q}{L_\lambda} = \left(\frac{\partial T^*}{\partial y}\right) = {} & \left(\frac{U_\infty}{\alpha}\right)(T_0-T_\infty)\left\{d_1'\left[\frac{3d_2^2}{10d_1^2}-\frac{3d_2^3}{12d_1^3}+\frac{9d_2^4}{140d_1^4}-\frac{3y^3}{4d_1^2d_2}\right.\right.\\
& \left.+\frac{3y^4}{4d_1^3d_2}+\frac{9y^5}{20d_1^2d_2^3}-\frac{9y^5}{40d_1^4d_2}-\frac{3y^6}{6d_1^3d_2^3}+\frac{9y^7}{56d_1^4d_2^3}\right]\\
& +d_2'\left[\frac{-3d_2}{5d_1}+\frac{3d_2^2}{8d_1^2}-\frac{3d_2^3}{35d_1^3}+\frac{3y^3}{2d_1d_2^2}-\frac{9y^4}{8d_1^2d_2^2}+\frac{3y^5}{10d_1^3d_2^2}\right.\\
& \left.-\frac{9y^5}{10d_1d_2^4}+\frac{3y^6}{4d_1^2d_2^4}-\frac{3y^7}{14d_1^3d_2^4}\right]+\left(\frac{nd_2}{x}\right)\left[\frac{-3d_2}{10d_1}+\frac{d_2^2}{8d_1^2}\right.\\
& -\frac{3d_2^3}{140d_1^3}+\frac{3y^2}{2d_1d_2}-\frac{3y^3}{2d_1d_2^2}-\frac{y^3}{d_1^2d_2}+\frac{9y^4}{8d_1^2d_2^2}+\frac{y^4}{4d_1^3d_2}\\
& \left.+\frac{3y^5}{10d_1d_2^4}-\frac{3y^5}{10d_1^3d_2^2}-\frac{y^6}{4d_1^2d_2^4}+\frac{y^7}{14d_1^3d_2^4}\right]+\left(\frac{md_2}{x}\right)\left[\frac{-3d_2}{10d_1}\right.\\
& +\frac{3d_2^2}{24d_1^2}-\frac{3d_2^3}{140d_1^3}+\frac{3y^3}{4d_1d_2^2}-\frac{3y^4}{8d_1^2d_2^2}+\frac{3y^5}{40d_1^3d_2^2}-\frac{9y^5}{20d_1d_2^4}
\end{aligned}$$

$$+\frac{3y^6}{12d_1^2d_2^4}-\frac{3y^7}{56d_1^3d_2^4}\Bigg]\Bigg\}+\left[v_0(x)\frac{(T_0-T_\infty)}{\alpha}\right]\left[1-\frac{3y}{2d_2}\right.$$

$$+\left.\frac{3y^3}{6d_2^3}\right](0\leq y\leq d_2). \tag{4.6}$$

Here the prime indicates partial differentiation with respect to x. Using the expressions of momentum flux P_{12} and thermal flux J_q along with the velocity and temperature functions (4.2) the variational principle (3.7) is formulated. After carrying out the integration with respect to y, the variational principle (3.7) is obtained as

$$\delta\int_0^l L(d_1,d_2,d_1',d_2')dx=0, \tag{4.7}$$

where L is the Lagrangian density of the principle. The boundary layer thicknesses d_1 and d_2 are the independent parameters which are to be varied and the Euler-Lagrange equations corresponding to these variational parameters are

$$\left(\frac{d}{dx}\right)\left(\frac{\partial L}{\partial d_i'}\right)-\left(\frac{\partial L}{\partial d_i}\right)=0,\ i=1,2. \tag{4.8}$$

The equations (4.8) are non-linear second order ordinary differential equations in terms of d_1 and d_2 whose coefficients are functions of Re, Pr, Gr, K, L and n,

where

$$\begin{aligned}
Re &= \frac{U_\infty x}{\nu}\ \text{(Reynolds number)},\\
Pr &= \frac{\mu C_p}{K}\ \text{(Prandtl number)},\\
Gr &= \frac{gB(T_0-T_\infty)x^3}{\nu^2}\ \text{(Grashof number)},\\
K &= \frac{Gr}{(Re^{\frac{5}{2}})}\ \text{(Buoyancy parameter)},\\
L &= \frac{\pi m}{(m+1)},\\
\text{and } n &= \text{wall temperature exponent.}
\end{aligned}$$

Although the equations (4.8) can be solved directly by using a numerical method, we can obtain a simple solution for the considered problem by employing the following transformations

$$d_i=d_i^*\sqrt{\frac{\nu x}{U_\infty}},\ i=1,2. \tag{4.9}$$

in the variational principle (4.7). Thus the Euler-Lagrange equations of the transformed principle assume the simple forms as

$$\left(\frac{\partial L}{\partial d_i^*}\right)=0,\ i=1,2. \tag{4.10}$$

The equations (4.10) are observed as coupled polynomial equations in non-dimensional boundary layer thicknesses d_1^* and d_2^* and the coefficients of these equations depend on the independent parameter Pr, wedge angle parameter m and H, where H is the non dimensional suction/injection speed and is given by

$$H = \left[\frac{v_0(x)}{\nu}\right]\sqrt{\frac{\nu x}{U_\infty}}. \tag{4.11}$$

Suction and injection are represented by $H < 0$ and $H > 0$ respectively. Equations (4.10) can be solved for any value of Pr, suction/injection speed H, wedge angle parameter m and the wall temperature exponent n and from the present variational procedure the non-linear partial differential equations governing the boundary layer flow are transformed into simple coupled polynomial equations which are of much useful for engineering applications.

5. Results and discussions

The main results of engineering interest are skin friction (shear stress) and heat transfer values (Nusselt number) and hence we analyze these two important characteristics for the present considered problem. After obtaining the simultaneous solution of d_1^* and d_2^* for the the given combination of Pr, H, m and n we can calculate the skin friction values and the local Nusselt number on using the following expressions respectively

$$\begin{aligned} \tau_\omega^* &= \sqrt{\frac{\nu x}{U_\infty^3}}\left(\frac{-P_{12}}{L_s}\right)_{y=0}, \\ Nu_l &= \sqrt{\nu x/U_\infty(T_0 - T_\infty)^2}\left(\frac{-J_q}{L_\lambda}\right)_{y=0}. \end{aligned} \tag{5.1}$$

It is known that the Grashof number (Gr) and Reynolds number (Re) are the controlling parameters of free and forced convection flows, respectively. Thus, the buoyancy parameter K is for both free and forced convection flows. For very small values of K, the forced convection predominates and the free convection becomes negligible. For large values of K, the flow is controlled by the free convection, and the forced convection becomes decrease. When the surface is heated $[(T_0 - T_\infty) > 0]$ then the buoyancy parameter becomes $K > 0$, and the flow has favourable pressure gradient. The adjacent boundary layer is accelerated, and with the increasing value of K, the velocity and thermal boundary layer thicknesses d_1^* and d_2^* become decrease, while the skin friction and heat transfer values increase. When the surface is cooled, then the buoyancy parameter becomes $K < 0$ and the boundary layer is decelerated. Since the free and forced convections are in the opposite directions a flow separation occurs in the present study. This means that, the condition for the boundary layer separation from the wedge surface is occurred, when the flow becomes reverse at the interface. This phenomenon occurs when $\left(\frac{\partial u^*}{\partial y}\right)_{y=0} = 0$ at a certain negative buoyancy parameter K, for the case of $H = 0$.

When $\beta = 0.5$, $H = 0$, $n = 0$ and $Re = 100$ the boundary layer flow is separated for $Pr = 0.7, 1.0$ and 3.0 at $K = -0.0952, -0.1007$ and -0.1322, respectively. It reveals that the separation is delayed with the increasing Prandtl number (Pr). It is also observed that, the flow separation is delayed with the increase of wall temperature exponent n. When $\beta = 0.5$, $H = 0$, $Re = 100$ and $Pr = 1$ the separation occurs at $K = -0.1007$ and -0.1039 when $n = 0$ and $n = 0.3333$, respectively.

Figures (1) and (2) represent the skin friction values for the cases of constant surface temperature and constant heat flux as a function of K, for three values of β when $Pr = 0.7$, $H = 0$ and $Re = 100$, respectively. From these two figures it is observed that the vanishing of skin friction values is delayed by the increase of β. In Figures (3) and (4) the effect of K for high Prandtl number, $Pr = 100$, $H = 0$ and $Re = 100$ is analyzed for constant surface temperature and constant surface heat flux respectively. It is demonstrated that the skin friction values increase with the value of K when the surface is heated. Figures (5) and (6) represent the local Nusselt number for $Pr = 0.7$ and $Pr = 100$ respectively, when the surface temperature becomes uniform.

The constant heat flux along the wedge is presented in Figures (7) and (8). When K becomes negative the effect of buoyancy is to decrease the skin friction and heat transfer values. It is evident that the vanishing of local heat flux occurs beyond the point of zero skin friction. The skin friction is more strongly affected by buoyancy forces than in the heat transfer. It is noted that lower Pr fluids are more sensitive to buoyancy effects. From the present analysis when $\beta = 0, 0.5, 1.0$ and 1.6 the surface heat transfer vanishes at $n = -0.5, -0.6666, -1.0$ and -2.5, respectively. This phenomenon occurs for any combination of the Pr and the buoyancy parameter K when $H = 0$.

When a new mathematical technique (present technique) is applied to a problem, it is customary to compare the obtained results with the available solution in order to establish the accuracy of the results in the present analysis. Accordingly, we compare the present results with series solutions of Sparrow and Minkowycz [9] and Saeid [10] when $\beta = 0$, $H = 0$ and $Pr = 0.7$ which are given in Table 1.

Table 1. Comparison of present results with series solution ($Pr = 0.7, H = 0$ and $\beta = 0$)

K	Skin friction values			Heat transfer values		
	Present method	Series solution method [9]	Saeid method [10]	Present method	Series solution method [9]	Saeid method [10]
0.01	0.3562	0.3493	-	0.3042	0.2963	-
0.05	0.4268	0.4182	-	0.3116	0.3106	-
0.1	0.5049	0.5043	-	0.3204	0.3284	-
0.2	0.5569	-	0.5502	0.3426	-	0.3305
0.3	0.7036	-	0.6925	0.3489	-	0.3425
0.4	0.7398	-	0.7351	0.3638	-	0.3561
0.5	0.8215	-	0.8185	0.3809	-	0.3716

From Table 1, it proves that the present values of skin friction and heat transfer are well comparable with the known series solution. One can also note that the order of accuracy remains same for very large and small Pr for various H and β.

This paper presents an analytical result of free and forced convection with the effects of suction and injection over a non-isothermal wedge. The governing partial differential equations are reduced to coupled polynomial equations, the coefficients of which are functions of independent parameters Pr, H and m. The great advantage involved in the present technique is that the results are obtained with remarkable accuracy and the cost of calculation is considerably less than that of numerical procedure. Hence, it is concluded that this variational technique is a unique approximate method based on sound physical reasoning as a powerful tool for solving heat transfer and boundary layer problems.

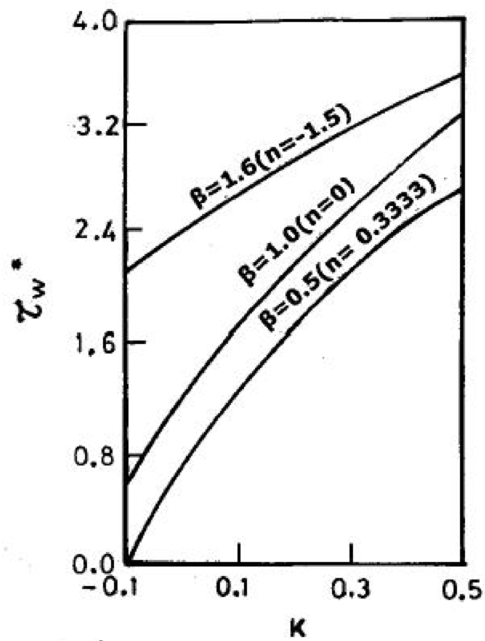

Figure 1. Skin friction values for various K when $Pr = 0.7$, $H = 0$ and $Re = 100$. (Constant surface temperature)

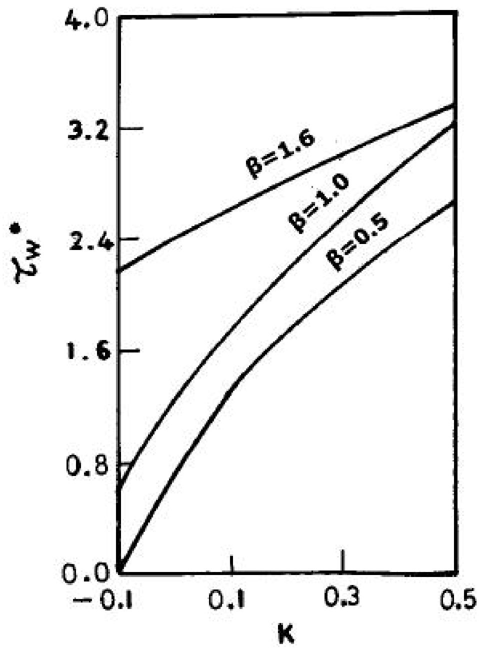

Figure 2. Skin friction values for various K when $Pr = 0.7, H = 0$ and $Re = 100$. (Constant surface heat flux)

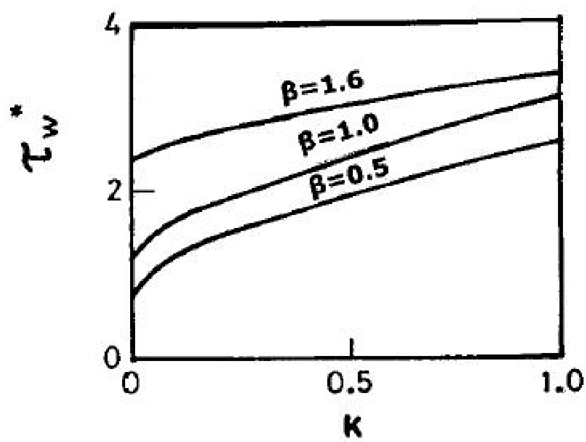

Figure 3. Skin friction values for various K when $Pr = 100, H = 0$ and $Re = 100$. (Constant surface temperature)

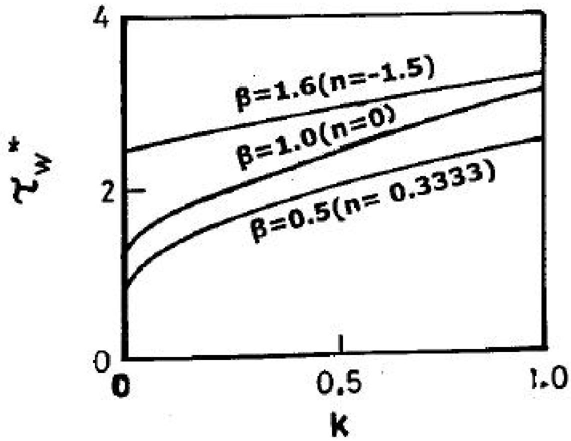

Figure 4. Skin friction values for various K when $Pr = 100, H = 0$ and $Re = 100$. (Constant surface heat flux)

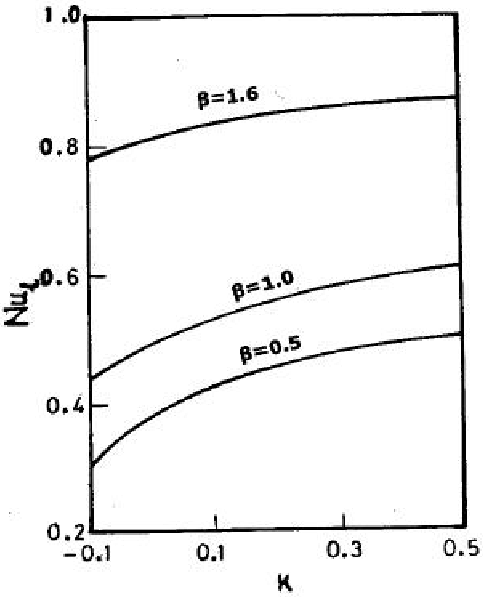

Figure 5. Heat transfer values for various K ($Pr = 0.7, n = 0, H = 0$ and $Re = 100$).

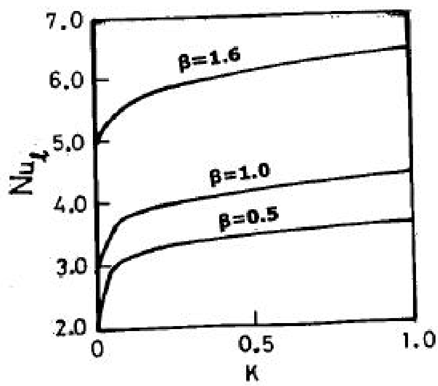

Figure 6. Heat transfer values for various K $(Pr = 100, n = 0, H = 0$ and $Re = 100)$.

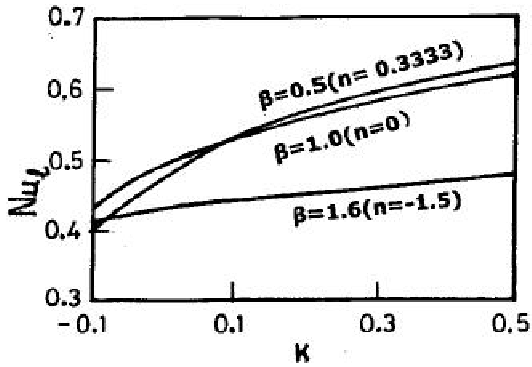

Figure 7. Buoyancy effect on heat transfer when $Pr = 0.7$, $H = 0$ and $Re = 100$. (Constant surface heat flux)

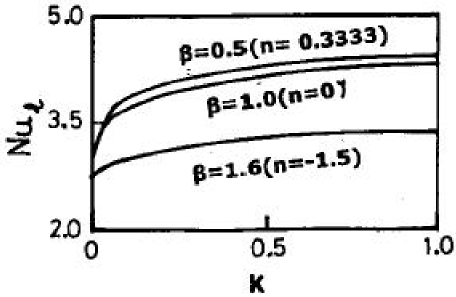

Figure 8. Buoyancy effect on heat transfer when $Pr = 100$, $H = 0$ and $Re = 100$. (Constant surface heat flux)

Nomenclature

x–coordinate measuring distance along the plate.
y–coordinate measuring distance normal to plate.
u–velocity component in the x-direction.
v–velocity component in the y-direction.
v_0–suction and injection velocity.
T–temperature of fluid.
T_0–temperature of plate.
T_∞–temperature of ambient fluid.
d_1–hydrodynamical boundary layer thickness.
d_2–thermal boundary layer thickness.
P_{12}–momentum flux.
J_q–thermal flux.
L–Lagrangian function.
L_s, L_λ–conductivities.
ν–kinematic viscosity.
d_1^*, d_2^*–non dimensional boundary layer thicknesses.
α–thermal diffusivity.
g–acceleration of gravity.
B–coefficient of thermal expansion.
H–non dimensional suction and injection speed.
Re–Reynolds number.
Pr–Prandtl number.
Gr–Grashof number.
K–Buoyancy parameter.
δ–symbol for variation.
σ–entropy production.
Ψ^*, Φ^*–local dissipation potentials in energy picture.
τ_ω^*–non-dimensional skin friction.
Nu_l–Nusselt number.

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