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On a Result of Ozawa and Uniqueness of Meromorphic Function

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Abstract. In this paper, we study the problem of meromorphic functions sharing three values with weight and obtain a uniqueness theorem which improves the results given by M. Ozawa, I. Lahiri and others.

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1. Introduction, definitions and main results

By a meromorphic function we shall always mean a meromorphic function in the complex plane. If for some $a \in C \cup \{\infty\}$ the zeros of $f - a$ and $g - a$ coincide in locations and multiplicity we say that f and g share the value a CM and if the zeros coincide in locations only we say that f and g share a IM. It is assumed that the reader is familiar with the notations of Nevanlinna theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $S(r, f)$ and so on, that can be found, for instance, in [1]. We now explain in the following definition the notion of weighted sharing which was introduced by I. Lahiri [2].

Definition 1.1. [2] *For a complex number $a \in C \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f where an a-point with mutiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. For a complex number $a \in C \cup \{\infty\}$, such that $E_k(a, f) = E_k(a, g)$, then we say that f and g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

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Definition 1.2. [2] *Let p be a positive integer and $a \in C \cup \{\infty\}$. We denote by $N_{p)}\left(r, \frac{1}{f-a}\right)$ the counting function of the zeros of $f-a$ (counted with proper multiplicities) whose multiplicities are not greater than p, $N_{(p+1}\left(r, \frac{1}{f-a}\right)$ to denote the counting function of zeros of $f-a$ whose multiplicities are not less than $p+1$, and $\overline{N}_{p)}\left(r, \frac{1}{f-a}\right)$, $\overline{N}_{(p+1}\left(r, \frac{1}{f-a}\right)$ denote their corresponding reduced counting functions (ignoring multiplicities), respectively. Define*

$$\delta_{p)}(a,f) = 1 - \limsup_{r\to\infty} \frac{N_{p)}\left(r, \frac{1}{f-a}\right)}{T(r,f)}, \qquad \delta_2(a,f) = 1 - \limsup_{r\to\infty} \frac{N_2\left(r, \frac{1}{f-a}\right)}{T(r,f)},$$

where $N_2(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right)$. It is obvious that $\delta_{p)}(a,f) \geq \delta(a,f)$ and $0 \leq \delta(a,f) \leq \delta_2(a,f) \leq \Theta(a,f) \leq 1$.

In 1976, M. Ozawa proved the following result.

Theorem 1.1. [7] *Let f and g be two entire functions of finite order such that f and g share 0 and 1 CM. If $\delta(0,f) > \frac{1}{2}$, then $f \equiv g$ or $fg \equiv 1$.*

In 1983, H. Ueda removed the order restriction of f and g in Theorem 1.1 and proved the following theorem.

Theorem 1.2. [8] *Let f and g be two meromorphic functions sharing $0, 1$ and ∞ CM. If*

$$\limsup_{r\to\infty} \frac{N(r, \frac{1}{f}) + N(r,f)}{T(r,f)} < \frac{1}{2}$$

then either $f \equiv g$ or $fg \equiv 1$.

In 1990, H. X. Yi proved the following theorem which is an improvement of Theorem 1.1 and 1.2.

Theorem 1.3. [9] *Let f and g be two meromorphic functions sharing $0, 1$ and ∞ CM. If*

$$N_{1)}(r,f) + N_{1)}\left(r, \frac{1}{f}\right) < (\lambda + o(1))T(r),\ r \in I$$

where λ is a constant such that $\lambda < \frac{1}{2}$, $T(r) = \max\{T(r,f), T(r,g)\}$, and I is a set in $(0,\infty)$ with infinite linear measure, then either $f \equiv g$ or $fg \equiv 1$.

As a corollary of Theorem 1.3, we have:

Theorem 1.4. [9] *Let f and g be two meromorphic functions sharing $0, 1$ and ∞ CM. If*

$$\delta_{1)}(\infty, f) + \delta_{1)}(0,f) > \frac{3}{2}$$

then either $f \equiv g$ *or* $fg \equiv 1$.

In 2002, I. Lahiri improved the above result by the idea of weighted shared values and obtained the following.

Theorem 1.5. [4, 5] *Let* f *and* g *be two meromorphic functions sharing* $(0,1)$, $(1,\infty)$, (∞,∞). *If*

$$A_0 = 2\delta_{1)}(0,f) + 2\delta_{1)}(\infty,f) + \min\left\{\sum_{a\neq 0,1,\infty}\delta_{2)}(a,f), \sum_{a\neq 0,1,\infty}\delta_{2)}(a,g)\right\} > 3$$

then either $f \equiv g$ *or* $fg \equiv 1$.

In 2003, I. Lahiri further obtained the following theorem.

Theorem 1.6. [6] *Let* f *and* g *be two meromorphic functions sharing* $(0,1)$, $(1,m)$, (∞,k) *where* m, k *are positive integers satisfying* $(m-1)(km-1) > (1+m)^2$. *If*

$$A_0 = 2\delta_{1)}(0,f) + 2\delta_{1)}(\infty,f) + \min\left\{\sum_{a\neq 0,1,\infty}\delta_{2)}(a,f), \sum_{a\neq 0,1,\infty}\delta_{2)}(a,g)\right\} > 3$$

then either $f \equiv g$ *or* $fg \equiv 1$.

In 2006, Q. C. Zhang proved the following theorem which is an improvement of the above results.

Theorem 1.7. [10] *Let* f *and* g *be two meromorphic functions sharing* (a_1,k_1), (a_2,k_2) *and* (a_3,k_3), *where* $\{a_1,a_2,a_3\} = \{0,1,\infty\}$ *and* k_1, k_2, k_3 *are three positive integers satisfying* $k_1k_2k_3 - k_1 - k_2 - k_3 - 2 > 0$. *If*

$$2\delta_{1)}(0,f) + 2\delta_{1)}(\infty,f) + \sum_{a\neq 0,1,\infty}\delta_{2)}(a,f) > 3$$

or

$$2\delta_{1)}(0,g) + 2\delta_{1)}(\infty,g) + \sum_{a\neq 0,1,\infty}\delta_{2)}(a,g) > 3$$

then either $f \equiv g$ *or* $fg \equiv 1$.

From the above theorem, we see that the weights of sharing values are positive integers or ∞. We may ask the following question: What can be said if the weight of one of the three shared values is relaxed to 0?

In this article, we settle the problem and prove the following theorem.

Theorem 1.8. *Let* f *and* g *be two nonconstant meromorphic functions sharing* $(0,m), (\infty,0)$ *and* $(1,1)$, *where* $m \geq 2$. *If*

$$2\delta_2(0,f) + \frac{4m}{m-1}\delta_2(\infty,f) + \min\left\{\sum_{a\neq 0,1,\infty}\delta_2(a,f), \sum_{a\neq 0,1,\infty}\delta_2(a,g)\right\} > \frac{5m-1}{m-1}$$

then either $f \equiv g$ *or* $fg \equiv 1$.

Corollary 1.1. *Let f and g be two nonconstant meromorphic functions sharing $(0,2),(\infty,0)$ and $(1,1)$. If*

$$2\delta_2(0,f)+8\delta_2(\infty,f)+\min\left\{\sum_{a\neq 0,1,\infty}\delta_2(a,f),\sum_{a\neq 0,1,\infty}\delta_2(a,g)\right\}>9$$

then either $f\equiv g$ or $fg\equiv 1$.

Corollary 1.2. *Let f and g be two nonconstant meromorphic functions sharing $(0,3),(\infty,0)$ and $(1,1)$. If*

$$2\delta_2(0,f)+6\delta_2(\infty,f)+\min\left\{\sum_{a\neq 0,1,\infty}\delta_2(a,f),\sum_{a\neq 0,1,\infty}\delta_2(a,g)\right\}>7$$

then either $f\equiv g$ or $fg\equiv 1$.

2. Some lemmas

The following Lemmas are needed in the proof of Theorem 1.8.

Lemma 2.1. [4] *If f, g share $(0,0)$, $(\infty,0)$, $(1,0)$. Then*

$$T(r,f)\leq 3T(r,g)+S(r,f)\,,$$

$$T(r,g)\leq 3T(r,f)+S(r,g)\,.$$

Lemma 2.1 shows that $S(r,f)=S(r,g)$ and we denote them by $S(r)$. We shall denote by H a meromorphic function defined by

$$H=\left(\frac{f''}{f'}-2\frac{f'}{f-1}\right)-\left(\frac{g''}{g'}-2\frac{g'}{g-1}\right)\,. \tag{2.1}$$

Lemma 2.2. [3] *If f, g share $(1,1)$ and $H\not\equiv 0$. Then*

$$N_{1)}\left(r,\frac{1}{f-1}\right)\leq N(r,H)+S(r,f)+S(r,g)\,.$$

Lemma 2.3. *Let f and g share $(0,m)$, $(\infty,0)$ and $(1,1)$, where $m\geq 2$. Then*

$$\begin{aligned}N(r,H)\leq&\overline{N}_{(m+1}\left(r,\frac{1}{f}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-1}\right)+\overline{N}(r,f)+\overline{N}_0\left(r,\frac{1}{f'}\right)+\overline{N}_0\left(r,\frac{1}{g'}\right)\\&+\sum_{i=1}^{n}\overline{N}_{(2}\left(r,\frac{1}{f-a_i}\right)+\sum_{i=1}^{n}\overline{N}_{(2}\left(r,\frac{1}{g-a_i}\right)+S(r)\,,\end{aligned}$$

where $\overline{N}_0\left(r,\frac{1}{f'}\right)$ is the reduced counting function of the zeros of f' which are not the zeros of $f(f-1)\prod_{i=1}^{n}(f-a_i)$ where $a_i\neq 0,1,\infty(i=1,2,...,n)$.

Proof. The possible poles of H occur at

(i) multiple zeros of f and g,
(ii) multiple zeros of $f-1$ and $g-1$,
(iii) multiple poles of f and g,

(iv) zeros of f' and g' which are not the zeros of $f(f-1)\prod_{i=1}^{n}(f-a_i)$ and $g(g-1)\prod_{i=1}^{n}(g-a_i)$ respectively,

(v) multiple zeros of $f-a_i$, $g-a_i$ $(i=1,2,...,n)$.

Since f and g share $(0,m)$, $(\infty,0)$ and $(1,1)$, where $m\geq 2$, and all the poles of H are simple, we obtain the conclusion. ∎

Lemma 2.4. *Let f and g share $(0,m),(\infty,0)$ and $(1,1)$ and $f\not\equiv g$, where $m\geq 2$. Then*

$$\begin{aligned}\overline{N}_{(2}\left(r,\frac{1}{f-1}\right) &\leq \frac{m+1}{m-1}\overline{N}(r,f)+S(r)\,,\\ \overline{N}_{(m+1}\left(r,\frac{1}{f}\right) &\leq \frac{2}{m-1}\overline{N}(r,f)+S(r)\,.\end{aligned}$$

Proof. Let $\phi=\dfrac{f'}{f-1}-\dfrac{g'}{g-1}$ and $\psi=\dfrac{f'}{f}-\dfrac{g'}{g}$. Suppose that $\overline{N}\left(r,\dfrac{1}{f-a}\right)\neq S(r)$ for $a=0,1$ because otherwise the lemma is trivial. Since $f\not\equiv g$, it follows that $\phi\not\equiv 0$ and $\psi\not\equiv 0$. Now

$$\begin{aligned}\overline{N}_{(2}\left(r,\frac{1}{f-1}\right)\leq N\left(r,\frac{1}{\psi}\right) &\leq T(r,\psi)+O(1)=N(r,\psi)+S(r)\\ &\leq \overline{N}_{(m+1}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r)\,,\end{aligned}$$

and

$$\begin{aligned}m\overline{N}_{(m+1}\left(r,\frac{1}{f}\right)\leq N\left(r,\frac{1}{\phi}\right) &\leq T(r,\phi)+O(1)=N(r,\phi)+S(r)\\ &\leq \overline{N}_{(2}\left(r,\frac{1}{f-1}\right)+\overline{N}(r,f)+S(r)\,.\end{aligned}$$

From above, we get

$$\overline{N}_{(2}\left(r,\frac{1}{f-1}\right)\leq \frac{1}{m}\overline{N}_{(2}\left(r,\frac{1}{f-1}\right)+\frac{1}{m}\overline{N}(r,f)+\overline{N}(r,f)+S(r)\,.$$

So

$$\overline{N}_{(2}\left(r,\frac{1}{f-1}\right)\leq \frac{m+1}{m-1}\overline{N}(r,f)+S(r)\,,$$

and

$$\begin{aligned}&\overline{N}_{(m+1}\left(r,\frac{1}{f}\right)\leq \frac{1}{m}\overline{N}_{(2}\left(r,\frac{1}{f-1}\right)+\frac{1}{m}\overline{N}(r,f)+S(r)\\ &\leq \frac{1}{m}\left(\frac{m+1}{m-1}+1\right)\overline{N}(r,f)+S(r)=\frac{2}{m-1}\overline{N}(r,f)+S(r)\,.\end{aligned}$$

∎

Lemma 2.5. [3] *Let a_1, a_2,..., a_n be pairwise distinct complex numbers such that $a_i \neq 0, 1, \infty (i = 1, 2, ..., n)$. Then*

$$\overline{N}_0\left(r, \frac{1}{f'}\right) + \sum_{i=1}^{n} \overline{N}_{(2}\left(r, \frac{1}{f - a_i}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-1}\right) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, f),$$

where $\overline{N}_0\left(r, \frac{1}{f'}\right)$ is the reduced counting function of the zeros of f' which are not the zeros of $f(f-1)\prod_{i=1}^{n}(f - a_i)$.

3. Proof of Theorem 1.8

Let $f \not\equiv g$. We shall show that $fg \equiv 1$. Suppose that $H \not\equiv 0$. Let a_1, a_2, ... ,a_n be pairwise distinct complex numbers such that $a_i \neq 0, 1, \infty (i = 1, 2, ..., n)$. By the second fundamental theorem, we get

$$\begin{aligned}(n+1)T(r, f) \leq{}& \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f-1}\right) \\ &+ \sum_{i=1}^{n} \overline{N}\left(r, \frac{1}{f - a_i}\right) - N_0\left(r, \frac{1}{f'}\right) + S(r, f). \end{aligned} \tag{3.1}$$

Here, $N_0\left(r, \frac{1}{f'}\right)$ is the counting function of those zeros of f' which are not the zeros of $f(f-1)\prod_{i=1}^{n}(f - a_i)$. By Lemma 2.2 and Lemma 2.3, we obtain

$$\begin{aligned}\overline{N}\left(r, \frac{1}{f-1}\right) &= N_{1)}\left(r, \frac{1}{f-1}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-1}\right) \\ &\leq N(r, H) + \overline{N}_{(2}\left(r, \frac{1}{f-1}\right) + S(r, f) \\ &\leq \overline{N}_{(m+1}\left(r, \frac{1}{f}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-1}\right) + \overline{N}(r, f) \\ &+ \overline{N}_0\left(r, \frac{1}{f'}\right) + \overline{N}_0\left(r, \frac{1}{g'}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-1}\right) \\ &+ \sum_{i=1}^{n} \overline{N}_{(2}\left(r, \frac{1}{f - a_i}\right) + \sum_{i=1}^{n} \overline{N}_{(2}\left(r, \frac{1}{g - a_i}\right) + S(r, f). \end{aligned} \tag{3.2}$$

From (3.1) and Lemma 2.4, Lemma 2.5 we get

$$\begin{aligned}(n+1)T(r, f) \leq{}& \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + \overline{N}_{(m+1}\left(r, \frac{1}{f}\right) \\ &+ \overline{N}_{(2}\left(r, \frac{1}{f-1}\right) + \overline{N}(r, f) + \overline{N}_0\left(r, \frac{1}{g'}\right) \\ &+ \overline{N}_{(2}\left(r, \frac{1}{f-1}\right) + \sum_{i=1}^{n} \overline{N}_{(2}\left(r, \frac{1}{g - a_i}\right) \end{aligned}$$

$$+\sum_{i=1}^{n}\overline{N}_{(2}\left(r,\frac{1}{f-a_i}\right)+\sum_{i=1}^{n}\overline{N}\left(r,\frac{1}{f-a_i}\right)+S(r,f)$$

$$\leq 2\overline{N}\left(r,\frac{1}{f}\right)+3\overline{N}(r,f)+\sum_{i=1}^{n}N_2\left(r,\frac{1}{f-a_i}\right)$$

$$+\frac{m+1}{m-1}\overline{N}(r,f)+\frac{2}{m-1}\overline{N}(r,f)+S(r)$$

$$\leq 2\overline{N}\left(r,\frac{1}{f}\right)+\frac{4m}{m-1}\overline{N}(r,f)+\sum_{i=1}^{n}N_2(r,\frac{1}{f-a_i})+S(r,f)\,. \tag{3.3}$$

Similarly we have

$$(n+1)T(r,g)\leq 2\overline{N}\left(r,\frac{1}{g}\right)+\frac{4m}{m-1}\overline{N}(r,g)+\sum_{i=1}^{n}N_2\left(r,\frac{1}{g-a_i}\right)+S(r,g)\,. \tag{3.4}$$

Combining (3.3) and (3.4) and using Lemma 2.1 we get

$$(n+1)T(r)\leq 2\overline{N}\left(r,\frac{1}{f}\right)+\frac{4m}{m-1}\overline{N}(r,f)$$

$$+\max\left\{\sum_{i=1}^{n}N_2\left(r,\frac{1}{f-a_i}\right),\sum_{i=1}^{n}N_2\left(r,\frac{1}{g-a_i}\right)\right\}+S(r) \tag{3.5}$$

Let $S=\{a: a\in C, a\neq 0,1,\infty$ and $\delta_2(a,f)+\delta_2(a,g)>0\}$. Since S is countable, suppose that $S=\{a_i: i\in N_+\}$ where N_+ is a set of positive integers.

If $\sum_{a\neq 0,1,\infty}\delta_2(a,f)<\sum_{a\neq 0,1,\infty}\delta_2(a,g)$, then there exists a positive integer n_0 such that

$$\sum_{i=1}^{n_0}\delta_2(a_i,f)\leq\sum_{i=1}^{n_0}\delta_2(a_i,g) \tag{3.6}$$

and

$$\sum_{i=1}^{n_0}\delta_2(a_i,f)>\sum_{a\neq 0,1,\infty}\delta_2(a,f)-\epsilon. \tag{3.7}$$

Then, from (3.5) we get

$$n_0+1<2+\frac{4m}{m-1}+n_0-2\Theta(0,f)-\frac{4m}{m-1}\Theta(\infty,f)-\sum_{a\neq 0,1,\infty}\delta_2(a,f)+\epsilon\,. \tag{3.8}$$

Since $0\leq\delta(a,f)\leq\delta_2(a,f)\leq\Theta(a,f)\leq 1$, from (3.8) we get

$$2\delta_2(0,f)+\frac{4m}{m-1}\delta_2(\infty,f)+\sum_{a\neq 0,1,\infty}\delta_2(a,f)<\frac{5m-1}{m-1}+\epsilon\,. \tag{3.9}$$

Since $\epsilon(>0)$ is arbitrary, it follows that

$$2\delta_2(0,f)+\frac{4m}{m-1}\delta_2(\infty,f)+\sum_{a\neq 0,1,\infty}\delta_2(a,f)\leq\frac{5m-1}{m-1}. \tag{3.10}$$

If $\sum_{a\neq 0,1,\infty}\delta_2(a,g)<\sum_{a\neq 0,1,\infty}\delta_2(a,f)$, similarly we can prove that

$$2\delta_2(0,f)+\frac{4m}{m-1}\delta_2(\infty,f)+\sum_{a\neq 0,1,\infty}\delta_2(a,g)\leq\frac{5m-1}{m-1}. \tag{3.11}$$

If $\sum_{a\neq 0,1,\infty}\delta_2(a,g)=\sum_{a\neq 0,1,\infty}\delta_2(a,f)$, then from (3.3) we obtain (3.10). Now (3.10) and (3.11) contradict the given condition. Therefore $H\equiv 0$ and so

$$f\equiv\frac{ag+b}{cg+d} \tag{3.12}$$

where a, b, c, d are constants and $ad-bc\neq 0$.
If $c=0$, then from (3.12) we get

$$f=Ag+B \tag{3.13}$$

where $A=\frac{a}{d}$, $B=\frac{b}{d}$ and $ad\neq 0$.

Let 0, ∞ be Picard values of f and g. From (3.13) we see that B is also Picard value of f which is impossible unless $B=0$. So from (3.13), we have $f\equiv Ag$. Since $f\not\equiv g$, it follows that $A\neq 1$ and 1 becomes a Picard value of f because f and g share $(1,1)$. This is again impossible.

Let ∞ be a Picard value of f and g but 0 be not a Picard value of f and g. Since f, g share $(0,m)$, from (3.13) we get $B=0$ and so $f\equiv Ag$. Since $f\not\equiv g$, $A\neq 1$ and so 1 becomes Picard value of f and g. Hence $\sum\limits_{t\neq 1,\infty}\delta_2(t,f)=0$. This contradicts the given condition.

Let 0 be a Picard value of f and g but ∞ be not a Picard value of f and g. If 1 is a Picard value of f then $\sum\limits_{t\neq 0,1}\delta_2(t,f)=0$ which contradicts the given condition. Hence there is z_0 such that $f(z_0)=g(z_0)=1$ and so from (3.13) we get $A+B=1$ and so

$$f\equiv Ag+1-A \tag{3.14}$$

Since f and g share $(0,1)$ and 0 is a Picard value of f, it follows from (3.14) that $1-A$ is a Picard value of f and $(A-1)/A$ is a Picard value of g. Since $f\not\equiv g$, from (3.14) we see that $A\neq 1$ and it follows that $\delta_2(\infty,f)=0$ and $\sum\limits_{t\neq 0,1,\infty}\delta_2(t,f)=1$, which contradicts the given condition.

Let 0, ∞ are not Picard values of f and so of g, then from (3.13) we get $f\equiv Ag$ because f, g share $(0,1)$. Since $f\not\equiv g$, it follows that $A\neq 1$ and 1 becomes a Picard value of f and g. Then we get

$$\delta_2(0,f)+\delta_2(\infty,f)+\sum_{t\neq 0,1,\infty}\delta_2(t,f)\leq 1\,. \tag{3.15}$$

So

$$\begin{aligned}2\delta_2(0,f)+\frac{4m}{m-1}\delta_2(\infty,f)+\sum_{t\neq 0,1,\infty}\delta_2(t,f) &\leq 1+\delta_2(0,f)+\frac{3m+1}{m-1}\delta_2(\infty,f)\\ &\leq \frac{5m-1}{m-1}\end{aligned} \tag{3.16}$$

which is a contradiction to the given condition.

If $c\neq 0$, then from (3.12) we get

$$f-\frac{a}{c}\equiv\frac{b-\dfrac{ad}{c}}{cg+d}. \tag{3.17}$$

Since f, g share $(\infty,0)$, it follows from (3.17) that $\dfrac{a}{c}$, ∞ are Picard values of f and $-\frac{d}{c}$, ∞ are Picard values of g.

If $a=0$, then from (3.17) we get

$$f\equiv\frac{1}{\alpha g+\beta} \tag{3.18}$$

where $\alpha=\dfrac{c}{b}$, $\beta=\dfrac{d}{b}$ and $b\neq 0$. Since 0, ∞ are Picard values of f and f, g share $(1,1)$, it follows that there exists z_0 such that $f(z_0)=g(z_0)=1$. So from (3.18) we get $\alpha+\beta=1$ and hence

$$f\equiv\frac{1}{\alpha g+1-\alpha}. \tag{3.19}$$

Since f and g share $(0,m)$, $(\infty,0)$ and 0, ∞ are Picard values of f, it follows from (3.19) that 0, ∞, $\dfrac{\alpha-1}{\alpha}$ are Picard values of f which is impossible unless $\alpha=1$, then from (3.19) we get $fg\equiv 1$.

Let $a\neq 0$. Since $\dfrac{a}{c}$ and ∞ are Picard values of f, it follows that $\delta_2(0,f)=0$ and $\sum\limits_{t\neq 0,1,\infty}\delta_2(t,f)\leq 1$ which contradicts the given condition. This proves the theorem.

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References

[1] W. K. Hayman, *Meromorphic Functions*, Clarendon, Oxford, 1964.
[2] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, *Nagoya Math. J.* **161**(2001), 193–206.
[3] I. Lahiri, Weighted sharing and a result of Ozawa, *Hokkaido Math. J.* **30**(2001), 679–688.
[4] I. Lahiri, On a result of Ozawa concerning uniqueness of meromorphic functions, *J. Math. Anal. Appl.* **271**(2002), 206–216.
[5] I. Lahiri, Corrigendum to "On a result of Ozawa concerning uniqueness of meromorphic functions, [J. Math. Anal. Appl. 271(2002), 206–216.]", *J. Math. Anal. Appl.* **287**(2003), 320.
[6] I. Lahiri, On a result of Ozawa concerning uniqueness of meromorphic functions II, *J. Math. Anal. Appl.* **283**(2003), 66–76.
[7] M. Ozawa, Unicity theorems for entire function, *J. D'Anal. Math.* **30**(1976), 411–420.
[8] H. Ueda, Unicity theorems for meromorphic or entire function II, *Kodai Math. J.* **6**(1983), 26–36.
[9] H. X. Yi, Meromorphic functions that share two or three values, *Kodai Math. J.* **13**(1990), 363–372.
[10] Q. C. Zhang, On the results of Lahiri concerning uniqueness of meromorphic functions, *J. Math. Anal. Appl.* **318**(2006), 707–725.