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On Uniqueness of Meromorphic Functions with Shared Four Values in Some Angular Domains

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Abstract. In this paper, we prove the uniqueness of meromorphic functions with shared four values in some angular domains. Our results consider the cases of the finite order and infinite order which improved the theorem of J.H. Zheng.

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1. Introduction

Let f be a meromorphic function defined in the complex plane $\mathbb{C}$. We assumed that the reader is familiar with the notations of Nevanlinna theory (cf.[6]), and the lower order μ, the order ρ and the hyper order ρ_2 are in turn defined as follows:

$$\mu = \mu(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r},$$
$$\rho = \rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$
$$\rho_2 = \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

An $a \in \overline{\mathbb{C}}$ is called an IM (ignoring multiplicities) shared value in $X \subset \mathbb{C}$ of two meromorphic functions $f(z)$ and $g(z)$ if in X, $f(z) = a$ if and only if $g(z) = a$ and a CM (counting multiplicities) shared value in X if $f(z)$ and $g(z)$ assume a at the same points in X with the same multiplicities. J. H. Zheng first consider the uniqueness dealing with shared values in a proper subset of $\mathbb{C}$. (see.[9, 10]) It is an interesting topic to investigate the uniqueness with shared values in the remaining part of complex plane removing an unbounded closed set. In [10], J. H. Zheng proved the following result:

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Theorem 1.1. *Let $f(z)$ and $g(z)$ be both transcendental meromorphic function and let $f(z)$ be of the finite lower order μ and for some $a \in \overline{\mathbb{C}}$, $\delta = \delta(a, f) > 0$.*

Given one angular domain $X = \{z : \alpha \le \arg z \le \beta,\}$ with $0 < \beta - \alpha \le 2\pi$ and

$$\beta - \alpha > \max\left\{\frac{\pi}{\sigma}, 2\pi - \frac{4}{\sigma}\arcsin\sqrt{\frac{\delta}{2}}\right\},$$

where $\mu \le \sigma \le \rho$ and $\sigma < \infty$, we assume that $f(z)$ and $g(z)$ have four distinct IM shared values $a_j (j = 1, 2, 3, 4)$ in X and $a_j \ne a (j = 1, 2, 3, 4)$, then $f(z) \equiv g(z)$.

It is natural to ask: What could we say about Theorem 1.1 for the case that f is of infinite lower order?

In this paper, by using the method in Zheng [11], we can get the following result:

Theorem 1.2. *Let $f(z)$ and $g(z)$ be both transcendental meromorphic function and let $f(z)$ be of the infinite lower order and the finite hyper order, such that for some $a \in \overline{\mathbb{C}}$ and $\delta = \delta(a, f) > 0$.*

Assume that for q radii $\arg z = \alpha_j, (1 \le j \le q)$, satisfying

$$-\pi \le \alpha_1 < \alpha_2 < \cdots < \alpha_q < \pi, \quad \alpha_{q+1} = \alpha_1 + 2\pi,$$

$f(z)$ and $g(z)$ have four distinct IM shared values in $X = \mathbb{C} \setminus \bigcup_{j=1}^{q}\{z : \arg z = \alpha_j\}$. If

$$\max\left\{\frac{\pi}{\alpha_{j+1} - \alpha_j} : 1 \le j \le q\right\} < \rho(f), \tag{1.1}$$

then $f(z) \equiv g(z)$.

Obviously, the order of f can be infinite in Theorem 1.2, but we restrict that the infinite order could not grow too quickly, explicitly to speak, the hyper order of f is finite. In fact, we first get the following result in order to prove Theorem 1.2.

Theorem 1.3. *Let $f(z)$ and $g(z)$ be both transcendental meromorphic function and let $f(z)$ be of the finite lower order μ and such that for some $a \in \overline{\mathbb{C}}$ and $\delta = \delta(a, f) > 0$.*

Assume that for q pair of real numbers $\{\alpha_j, \beta_j\}$ such that

$$-\pi \le \alpha_1 < \beta_1 \le \alpha_2 < \beta_2 \cdots < \alpha_q < \beta_q \le \pi, \quad \alpha_{q+1} = \alpha_1 + 2\pi,$$

and

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j) < \frac{4}{\sigma}\arcsin\sqrt{\frac{\delta}{2}}, \tag{1.2}$$

where $\sigma = \max\{\omega, \mu\}$ and define

$$\omega = \max\left\{\frac{\pi}{\beta_1 - \alpha_1}, \cdots, \frac{\pi}{\beta_q - \alpha_q}\right\}.$$

Assume that $f(z)$ and $g(z)$ have four distinct IM shared values in

$$X = \bigcup_{j=1}^{q}\{z : \alpha_j \le \arg z \le \beta_j\}.$$

If $\omega < \rho(f) < \infty$, then $f(z) \equiv g(z)$.

Remark 1.1. Obviously, it is a very important question: Can the four shared values $a_j (j = 1, 2, 3, 4)$ be replaced by four shared small functions in our theorems?

2. Lemmas

Lemma 2.1. [4, 7] *Let $f(z)$ be a transcendent and meromorphic function in the plane with finite lower order $0 \le \mu < \infty$ and the order $0 < \rho \le \infty$. Then for arbitrary positive number σ satisfying $\mu \le \sigma \le \rho$ and a set E with finite linear measure, there exist a sequence of positive numbers $\{r_n\}$ such that*

(i) $r_n \notin E$, $\lim_{n\to\infty} \dfrac{r_n}{n} = \infty$;

(ii) $\liminf_{n\to\infty} \dfrac{\log T(r_n, f)}{\log r_n} \ge \sigma$;

(iii) $T(t, f) \le (1 + o(1)) \left(\dfrac{t}{r_n}\right)^{\sigma} T(r_n, f), t \in \left[\dfrac{r_n}{n}, nr_n\right]$.

A sequence of increasing of real numbers $\{r_n\}$ satisfying (i), (ii) and (iii) is called a Pólya peak of order σ outside E in this paper. For $r > 0$ and $a \in \mathbb{C}$ define

$$D(r, a) := \left\{\theta \in [-\pi, \pi) : \log^+ \frac{1}{|f(re^{i\theta}) - a|} > \frac{1}{\log r} T(r, f)\right\} \tag{2.1}$$

and

$$D(r, \infty) := \left\{\theta \in [-\pi, \pi) : \log^+ |f(re^{i\theta})| > \frac{1}{\log r} T(r, f)\right\}.$$

The following result is a special version of the main result of Baernstein [1]. It is enough to prove our theorem.

Lemma 2.2. *Let $f(z)$ be a meromorphic transcendent and function in the plane with finite lower order $\mu > 0$ and the order $0 < \rho \le \infty$ and for $a \in \mathbb{C}$, $\delta(a, f) > 0$. Then for arbitary Pólya peak $\{r_n\}$ of order σ, $\mu \le \sigma \le \rho$, we have*

$$\liminf_{n\to\infty} mesD(r_n, a) \ge \min\left\{2\pi, \frac{4}{\sigma} \arcsin\sqrt{\frac{\delta}{2}}\right\}.$$

In order to prove our theorems, we need Nevanlinna theory on an angular domain. Let $f(z)$ be a meromorphic function on the angular domain

$$\overline{\Omega}(\alpha, \beta) = \{z; \alpha \le \arg z \le \beta, \},$$

where $0 < \beta - \alpha \le 2\pi$. Following Nevanlinna (see [5]) define

$$A_{\alpha,\beta}(r, f) = \frac{\omega}{\pi} \int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right) \{\log^+ |f(te^{i\alpha})| + \log^+ |f(te^{i\beta})|\} \frac{dt}{t}, \tag{2.2}$$

$$B_{\alpha,\beta}(r, f) = \frac{2\omega}{\pi r^\omega} \int_\alpha^\beta \log^+ |f(re^{i\theta})| \sin \omega(\theta - \alpha) d\theta, \tag{2.3}$$

$$C_{\alpha,\beta}(r, f) = 2 \sum_{1<|b_n|<r} \left(\frac{1}{|b_n|^\omega} - \frac{|b_n|^\omega}{r^{2\omega}}\right) \sin \omega(\theta_n - \alpha), \tag{2.4}$$

where $\omega = \dfrac{\pi}{\beta-\alpha}$ and $b_n = |b_n|e^{i\theta_n}$ are the poles of $f(z)$ on $\overline{\Omega}(\alpha,\beta)$ appearing according to their multiplicities. $C_{\alpha,\beta}(r,f)$ is called the angular counting function of the poles of f on $\overline{\Omega}(\alpha,\beta)$ and Nevanlinna's angular characteristic is defined as follows:

$$S_{\alpha,\beta}(r,f) = A_{\alpha,\beta}(r,f) + B_{\alpha,\beta}(r,f) + C_{\alpha,\beta}(r,f).$$

Throughout, we denote by $R_{\alpha,\beta}(r,*)$ a quantity satisfying

$$R_{\alpha,\beta}(r,*) = O\{\log(rS_{\alpha,\beta}(r,*))\}, \quad r \notin E,$$

where E denotes a set of positive real numbers with finite linear measure. It is not necessarily the same for every occurrence in the context.

Lemma 2.3. *Let $f(z)$ be meromorphic on $\overline{\Omega}(\alpha,\beta)$. Then for arbitrary complex number a, we have*

$$S_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) = S_{\alpha,\beta}(r,f) + O(1) \tag{2.5}$$

and for an integer $p \geq 0$,

$$A_{\alpha,\beta}\left(r, \frac{f^{(p)}}{f}\right) + B_{\alpha,\beta}\left(r, \frac{f^{(p)}}{f}\right) = R_{\alpha,\beta}(r,f)$$

and $R_{\alpha,\beta}(r,f^{(p)}) = R_{\alpha,\beta}(r,f)$. But in general, we do not know if $R_{\alpha,\beta}(r,f) = R_{\alpha,\beta}(r,f^{(p)})$.

Lemma 2.4. *Let $f(z)$ be meromorphic on $\overline{\Omega}(\alpha,\beta)$. Then for arbitrary q distinct $a_j \in \overline{\mathbb{C}}(1 \leq j \leq q)$, we have*

$$(q-2)S_{\alpha,\beta}(r,f) \leq \sum_{j=1}^{q} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + R_{\alpha,\beta}(r,f),$$

where the term $\overline{C}_{\alpha,\beta}\left(r, \dfrac{1}{f-a_j}\right)$ will be replaced by $\overline{C}_{\alpha,\beta}(r,f)$ when some $a_j = \infty$.

Lemma 2.5. [4] *Let $f(z)$ be meromorphic function with $\delta = \delta(\infty, f) > 0$. Then for given $\varepsilon > 0$, we have*

$$mesE(r,f) > \frac{1}{T^{\varepsilon}(r,f)[\log r]^{1+\varepsilon}}, \quad r \notin F,$$

where

$$E(r,f) = \left\{\theta \in [-\pi,\pi) : \log^{+}|f(re^{i\theta})| > \frac{\delta}{4}T(r,f)\right\}$$

and F is a set of positive real numbers with finite logarithmic measure depending on ε.

3. The proof of theorems

Proof of Theorem 1.3. Suppose $f(z) \not\equiv g(z)$. Let $a_j \in \overline{\mathbb{C}}(1 \le j \le 4)$ be four distinct IM shared values in X of $f(z)$ and $g(z)$. For the convenience, below we omit the subscript of all the notations, such as $S(r,*)$, $C(r,*)$. As in the proof of Theorem 1.1, first we estimate $B\left(r, \frac{1}{(f-a)}\right)$,

$$B\left(r, \frac{1}{f-a}\right) = O(\log(rT(r,f))), \quad r \notin E. \tag{3.1}$$

For completeness, we give the proof of equation (3.1). By applying Lemma 2.4 to f and equation (2.5), we have

$$\begin{aligned} 2S(r,f) &\le \sum_{j=1}^{4} \overline{C}\left(r, \frac{1}{f-a_j}\right) + R(r,f) \\ &\le C\left(r, \frac{1}{f-a_j}\right) + R(r,f) \le S(r,f-g) + R(r,f) \\ &\le S(r,f) + S(r,g) + R(r,f), \end{aligned} \tag{3.2}$$

so that

$$S(r,f) - R(r,f) \le S(r,g). \tag{3.3}$$

The same argument shows that

$$S(r,g) - R(r,g) \le S(r,f). \tag{3.4}$$

This implies that $R(r,g) = R(r,f)$. We assume that $a \in \mathbb{C}$. By the same argument we can show Theorem 1.3 for the case when $a = \infty$. Using Lemma 2.4 again and combining equation (3.2) together with equation (3.3) and equation (3.4), we deduce

$$\begin{aligned} 3S(r,f) &\le \sum_{j=1}^{4} \overline{C}\left(r, \frac{1}{f-a_j}\right) + \overline{C}\left(r, \frac{1}{f-a}\right) + R(r,f), \\ &\le 2S(r,f) + C\left(r, \frac{1}{f-a}\right) + R(r,f). \end{aligned} \tag{3.5}$$

Thus

$$B\left(r, \frac{1}{f-a}\right) = O(\log(rT(r,f))), \quad r \notin E. \tag{3.6}$$

The following method comes from [11]. Note that $\rho(f) > \omega$. We need to consider two cases:

Case 1. $\rho(f) > \mu$. Then $\rho(f) > \sigma \geq \mu$. By (1.2), we can take a real number $\varepsilon > 0$ such that

$$\sum_{j=1}^{q}(\alpha_{j+1} - \beta_j + 2\varepsilon) < \frac{4}{\sigma + 2\varepsilon} \arcsin \sqrt{\frac{\delta}{2}}, \tag{3.7}$$

where $\alpha_{q+1} = \alpha_1 + 2\pi$, and

$$\rho(f) > \sigma + 2\varepsilon > \mu.$$

Applying Lemma 2.1 to $f(z)$ gives the existence to the Pólya peak $\{r_n\}$ of order $\sigma + 2\varepsilon$ of f such that $r_n \notin E$, and then from Lemma 2.2 for sufficiently large n we have

$$mesD(r_n, a) > \frac{4}{\sigma + 2\varepsilon} \arcsin \sqrt{\frac{\delta}{2}} - \varepsilon, \tag{3.8}$$

since $\sigma + 2\varepsilon > \dfrac{1}{2}$. We can assume for all the n, equation(3.8) holds. Set

$$K := mes\left(D(r_n, a) \cap \bigcup_{j=1}^{q} (\alpha_j + \varepsilon, \beta_j - \varepsilon) \right).$$

Then from equations(3.7) and (3.8), it follows that

$$\begin{aligned} K &\geq mes\big(D(r_n, a)\big) - mes\left([0, 2\pi) \setminus \bigcup_{j=1}^{q} (\alpha_j + \varepsilon, \beta_j - \varepsilon) \right) \\ &\geq mes\big(D(r_n, a)\big) - mes\left(\bigcup_{j=1}^{q} (\beta_j - \varepsilon, \alpha_{j+1} + \varepsilon) \right) \\ &\geq mes\big(D(r_n, a)\big) - \sum_{j=1}^{q} (\alpha_{j+1} - \beta_j + 2\varepsilon) > \varepsilon > 0. \end{aligned}$$

It is easy to see that there exists a j_0 such that for infinitely many n, we have

$$mes\big(D(r_n, a) \cap (\alpha_{j_0} + \varepsilon, \beta_{j_0} - \varepsilon)\big) > \frac{K}{q}. \tag{3.9}$$

We can assume for all the n, (3.9) holds. Set $E_n = D(r_n, a) \cap (\alpha_{j_0} + \varepsilon, \beta_{j_0} - \varepsilon)$. Thus from the definition (2.1) of $D(r, a)$, it follows that

$$\begin{aligned} \int_{\alpha_{j_0}+\varepsilon}^{\beta_{j_0}-\varepsilon} \log^+ \frac{1}{|f(r_n e^{i\theta}) - a|} d\theta &\geq \int_{E_n} \log^+ \frac{1}{|f(r_n e^{i\theta}) - a|} d\theta \\ &\geq mes(E_n) \frac{T(r_n, f)}{\log r_n} \\ &\geq \frac{K}{q} \frac{T(r_n, f)}{\log r_n} \end{aligned} \tag{3.10}$$

On the other hand, by the definition (2.3) of $B_{\alpha,\beta}(r,*)$ and (3.1), we have

$$\begin{aligned}(3.11)\qquad \int_{\alpha_{j_0}+\varepsilon}^{\beta_{j_0}-\varepsilon} \log^+ \frac{1}{|f(r_n e^{i\theta})-a|}d\theta &< \frac{\pi}{2\omega_{j_0}\sin(\varepsilon\omega_{j_0})} r_n^{\omega_{j_0}} B_{\alpha_{j_0},\beta_{j_0}}\left(r_n, \frac{1}{f-a}\right)\\ &< K_{j_0} r_n^{\omega_{j_0}} O(\log(r_n T(r_n,f))), \quad r \notin E,\end{aligned}$$

Combining (3.10) with (3.11) gives

$$T(r_n,f) \le \frac{qK_{j_0}}{K} r_n^{\omega_{j_0}} O(\log(r_n T(r_n,f))).$$

Thus from (ii) in Lemma 2.1 for $\sigma+2\varepsilon$, we have

$$\sigma+2\varepsilon \le \limsup_{r\to\infty} \frac{\log T(r_n,f)}{\log r_n} \le \omega_{j_0} \le \sigma+\varepsilon.$$

This is impossible.

Case 2. $\rho(f)=\mu$. Then $\sigma=\mu=\rho(f)$. By the same argument as in Case 1 with all the $\sigma+2\varepsilon$ replaced by $\sigma=\mu$, we can derive

$$\max\{\omega,\mu\} = \sigma \le \omega < \rho(f).$$

This is impossible. The proof is completed. ∎

Proof of Theorem 1.2. As in the proof of Theorem 1.3, we have for each j,

$$(3.12)\qquad B_{\alpha_j,\alpha_{j+1}}\left(r, \frac{1}{f-a}\right) = O(\log(rT(r,f)), \quad r \notin E.$$

Applying Lemma 2.5 to $f(z)$ implies the existence of a sequence $\{r_n\}$ of positive numbers such that $\{r_n\} \to \infty (n\to\infty)$ and $r_n \notin E$ and

$$(3.13)\qquad mesE\left(r_n, \frac{1}{f-a}\right) > \frac{1}{T^{\varepsilon}(r_n,f)[\log r_n]^{1+\varepsilon}}.$$

Set

$$\varepsilon_n = \frac{1}{2q+1}\cdot\frac{1}{T^{\varepsilon}(r,f)[\log r]^{1+\varepsilon}}.$$

Then for (3.13), it follows that

$$\begin{aligned} &mes\left(E\left(r_n, \frac{1}{f-a}\right) \bigcap \bigcup_{j=1}^{q} (\alpha_j+\varepsilon_n, \alpha_{j+1}-\varepsilon_n)\right)\\ &\ge mesE(r_n, \frac{1}{f-a}) - mes\left(\bigcup_{j=1}^{q}(\alpha_j+\varepsilon_n, \alpha_{j+1}-\varepsilon_n)\right)\\ &\ge (2q+1)\varepsilon_n - 2q\varepsilon_n = \varepsilon_n > 0;\end{aligned}$$

so there exists a j such that for infinitely many n, we have

$$(3.14)\qquad mesE_n \ge \frac{\varepsilon_n}{q},$$

where $E_n = E(r_n, \frac{1}{f-a})\bigcap(\alpha_j+\varepsilon_n, \alpha_{j+1}-\varepsilon_n)$. We can assume that (3.14) holds for all the n. Thus from the definition of $E(r,f)$, it follows that

$$\begin{aligned}\int_{\alpha_j+\varepsilon_n}^{\alpha_{j+1}-\varepsilon_n}\log^+\frac{1}{|f(r_ne^{i\theta})-a|}d\theta &\geq \int_{E_n}\log^+\frac{1}{|f(r_ne^{i\theta})-a|}d\theta\\ &\geq mes(E_n)\frac{\delta}{4}T(r_n,f)\\ &\geq \frac{\delta\varepsilon_n}{4q}T(r_n,f).\end{aligned}\tag{3.15}$$

On the other hand, by the definition of $B_{\alpha,\beta}(r,*)$ and (3.12), we have

$$\begin{aligned}\int_{\alpha_j+\varepsilon_n}^{\alpha_{j+1}-\varepsilon_n}\log^+\frac{1}{|f(r_ne^{i\theta})-a|}d\theta &< \frac{\pi}{2\omega_j\sin(\varepsilon_n\omega_j)}r_n^{\omega_j}B_{\alpha_j,\alpha_{j+1}}\left(r_n,\frac{1}{f-a}\right)\\ &< \frac{\pi^2}{4\omega_j^2\varepsilon_n}O\bigg(r_n^{\omega_j}\log(r_nT(r_n,f))\bigg), \quad r\notin E,\end{aligned}\tag{3.16}$$

where $\omega_j = \dfrac{\pi}{\alpha_{j+1}-\alpha_j}$.

Combining (3.15) and (3.16) gives

$$\varepsilon_n^2T(r_n,f)\leq O\bigg(r_n^{\omega_j}\log(r_nT(r_n,f))\bigg),$$

so that

$$T^{1-2\varepsilon}(r_n,f)\leq O\bigg(r_n^{\omega_j}\log(r_nT(r_n,f))\bigg).$$

Note that the definition of hyper order of $f(z)$, thus $\mu(f)\leq\dfrac{\omega}{(1-2\varepsilon)}<\infty$. It is a contradiction. We complete the proof of the theorem. ∎

Remark 3.1. If the condition $\delta(a,f)>0$ reduces to $\delta(a,f^{(p)})>0$, where p is integer and $p\geq 0$, we need to add the restriction that $\overline{C}(r,f=a)=R(r,f)$. In fact, from (3.5) we know $S(r,f)\leq\overline{C}(r,f=a)+R(r,f)$. Thus,

$$\begin{aligned}B(r,f^{(p)}=a)&\leq S(r,f^{(p)})+R(r,f)\\ &=(A+B)\left(r,\frac{f^{(p)}}{f}\right)+(A+B)(r,f)+p\overline{C}(r,f)+C(r,f)+R(r,f)\\ &=(p+1)S(r,f)+R(r,f)=O(\log(rT(r,f))).\end{aligned}\tag{3.17}$$

If we use (3.17) instead of (3.6), then the above claim holds.

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