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Uniqueness of Meromorphic Functions Sharing Three Weighted Values

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Abstract. In this paper, we study the uniqueness of meromorphic functions sharing three values and improve some previous results.

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1. Introduction, definitions and results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \{\infty\} \cup \mathbb{C}$ we say that f and g share the value a CM (counting multiplicities) if f and g have the same a-points with the same multiplicities. If the multiplicities are not taken into account, we say that f and g share the value a IM (ignoring multiplicities). For standard definitions and notations of the value distribution theory, we refer to [2]. We denote by E a set of non-negative real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function f, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \longrightarrow \infty (r \notin E)$. We use $N_0(r)(\overline{N}_0(r))$ to denote the counting function (reduced counting function) of those zeros of $f - g$ which are not the zeros of $g(g-1)$, $\frac{1}{g}$ and $N_0^*(r)(\overline{N}_0^*(r))$ to denote the counting function (reduced counting function) of those zeros of $f-g$ which are not the zeros of $g(g-1)$.

In 1999, Q. C. Zhang proved the following results:

Theorem 1.1. [10] *Let f and g be two non-constant meromorphic functions sharing 0, 1, ∞ CM. If*

$$\limsup_{r\to\infty, r\notin E} \frac{N_0(r)}{T(r,f)} > \frac{1}{2},$$

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then f is a bilinear transformation of g and one of the following relations holds:

(i) $f \equiv g$,
(ii) $f+g \equiv 1$,
(iii) $(f-1)(g-1) \equiv 1$, *and*
(iv) $fg \equiv 1$.

Theorem 1.2. [10] *Let f and g be two distinct non-constant meromorphic functions sharing $0, 1, \infty$ CM. If*

$$0 < \limsup_{r\to\infty, r\notin E} \frac{N_0(r)}{T(r,f)} \leq \frac{1}{2},$$

then $N_0(r) = \frac{1}{k}T(r,f) + s(r,f)$ and f is not any fractional linear transformation of g and assume one of the following forms:

(i) $f = \dfrac{e^{s\gamma}-1}{e^{(k+1)\gamma}-1}$ *and* $g = \dfrac{e^{-s\gamma}-1}{e^{-(k+1)\gamma}-1}$, $1 \leq s \leq k$;
(ii) $f = \dfrac{e^{(k+1)\gamma}-1}{e^{(k+1-s)\gamma}-1}$ *and* $g = \dfrac{e^{-(k+1)\gamma}-1}{e^{-(k+1-s)\gamma}-1}$, $1 \leq s \leq k$;
(iii) $f = \dfrac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1}$ *and* $g = \dfrac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1}$, $1 \leq s \leq k$;

where s and k are positive integers such that s and $k+1$ are relatively prime and γ is a non-constant for entire function.

In 2003, H. X. Yi and Y. H. Li proved the following theorem.

Theorem 1.3. [8] *Let f and g be two non-constant meromorphic functions sharing $0, 1, \infty$ CM. Then*

$$\frac{1}{2} + o(1) \leq \frac{T(r,f)}{T(r,g)} \leq 2 + o(1)$$

and

$$\frac{1}{2} + o(1) \leq \frac{T(r,g)}{T(r,f)} \leq 2 + o(1)$$

as $r \to \infty (r \notin E)$. If, in particular, f and g are entire, then $T(r,f) \sim T(r,g)$ as $r \to \infty (r \notin E)$.

In 2005, Qi Han [3] used the notion of weighted value sharing, introduced in [4], to improve the above results. We now explain the notion of weighted sharing of values which measures how close a shared value is being shared IM or being shared CM.

Definition 1.1. [4] *Let k be a non-negative integer or infinity. For $a \in \{\infty\} \cup \mathbb{C}$, we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f and g share the value a with weight k.*

The definition implies that if f and g share a value a with weight k then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers $p, 0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

We now state the results of Qi Han.

Theorem 1.4. [3] *Let f and g be two non-constant meromorphic functions sharing $(0, 1)$, $(1, \infty)$ and (∞, ∞). If*

$$\limsup_{r\to\infty, r\notin E} \frac{N_0(r)}{T(r, f)} > \frac{1}{2},$$

then the conclusion of Theorem 1.1 holds.

Theorem 1.5. [3] *Let f and g be two distinct non-constant meromorphic functions sharing $(0, 1)$, $(1, \infty)$ and (∞, ∞). If*

$$0 < \limsup_{r\to\infty, r\notin E} \frac{N_0(r)}{T(r, f)} \leq \frac{1}{2},$$

then the conclusion of Theorem 1.2 holds.

Theorem 1.6. [3] *If f and g are two non-constant meromorphic functions sharing $(0, 1)$, $(1, \infty)$ and (∞, ∞), then the conclusion of Theorem 1.3 holds.*

In this paper, we prove the following results which improve the above theorems.

Theorem 1.7. *Let f and g be two non-constant meromorphic functions sharing $(0, k_1), (1, k_2)$ and (∞, k_3), where $k_j (j = 1, 2, 3)$ are positive integers satisfying*

$$k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2 \tag{1.1}$$

If

$$\limsup_{r\to\infty, r\notin E} \frac{N_0(r)}{T(r, f)} > \frac{1}{2},$$

then the conclusion of Theorem 1.1 holds.

Theorem 1.8. *Let f and g be two distinct non-constant meromorphic functions sharing $(0, k_1), (1, k_2)$ and (∞, k_3), where $k_j (j = 1, 2, 3)$ are positive integers satisfying (1.1). If*

$$0 < \limsup_{r\to\infty, r\notin E} \frac{N_0(r)}{T(r, f)} \leq \frac{1}{2},$$

then the conclusion of Theorem 1.2 holds.

Theorem 1.9. *Let f and g be two non-constant meromorphic functions sharing $(0, k_1), (1, k_2)$ and (∞, k_3), where $k_j (j = 1, 2, 3)$ are positive integers satisfying* (1.1). *Then the conclusion of Theorem 1.3 holds.*

We now give some more necessary definitions.

Definition 1.2. *Let f and g share $(a,0)$ and z be an a-point of f and g with multiplicities $p_f(z)$ and $p_g(z)$ respectively. We put*

$$\overline{\nu}_f(z) = \begin{cases} 1 & \text{if } p_f(z) > p_g(z) \\ 0 & \text{if } p_f(z) \leq p_g(z) \end{cases}$$

and

$$\overline{\mu}_f(z) = \begin{cases} 1 & \text{if } p_f(z) < p_g(z) \\ 0 & \text{if } p_f(z) \geq p_g(z) \end{cases}.$$

Let $\overline{n}(r,a;f > g) = \sum_{|z|\leq r} \overline{nu}_f(z)$ and $\overline{n}(r,a;f < g) = \sum_{|z|\leq r} \overline{\mu}_{f(z)}$. We now denote by $\overline{N}(r,a;f > g)$ and $\overline{N}(r,a;f < g)$ the integrated counting functions obtained from $\overline{n}(r,a;f > g)$ and $\overline{n}(r,a;f < g)$ respectively. Finally, we put

$$\overline{N}_*(r,a;f,g) = \overline{N}(r,a;f > g) + \overline{N}(r,a;f < g)0$$

Definition 1.3. *Let p be a positive integer and $a \in \{\infty\} \cup \mathbb{C}$. By $N(r,a;f \mid \leq p)$, we denote the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $N(r,a;f| \leq p)$, we denote the corresponding reduced counting function. In an analogous manner, we define $N(r,a;f| \geq p)$ and $N(r,a;f| \geq p)$.*

2. Lemmas

In this section, we present some necessary lemmas.

Lemma 2.1. [1] *Let f and g be two non-constant meromorphic functions sharing three values IM. Then*

$$T(r,f) \leq 3T(r,g) + S(r,f)$$

and

$$T(r,g) \leq 3T(r,f) + S(r,g).$$

From Lemma 2.1, we see that $S(r,f) = S(r,g)$, which we denote by $S(r)$ in the sequel.

Lemma 2.2. [9] *Let f and g be two distinct non-constant meromorphic functions sharing $(0,k_1), (1,k_2)$ and (∞,k_3), where $k_j (j = 1,2,3)$ are positive integers satisfying (1.1). Then*

$$\overline{N}(r,a;f| \geq 2) = S(r)$$

and

$$\overline{N}(r,a;g| \geq 2) = S(r)$$

for $a = 0, 1, \infty$.

Lemma 2.3. [5] *Let f and g share $(0,0), (1,0)$ and $(\infty,0)$ and $f \not\equiv g$. If $\alpha = \dfrac{f-1}{g-1}$ and $h = \dfrac{g}{f}$, then*

(i) $\overline{N}(r,0;\alpha) = \overline{N}(r,\infty;f < g) + \overline{N}(r,1;f > g)$,
(ii) $\overline{N}(r,\infty;\alpha) = \overline{N}(r,\infty;f > g) + \overline{N}(r,1;f < g)$,
(iii) $\overline{N}(r,0;h) = \overline{N}(r,0;f < g) + N(r,\infty;f > g)$,

(iv) $\overline{N}(r,\infty;h) = \overline{N}(r,0;f>g) + \overline{N}(r,\infty;f<g)$.

Lemma 2.4. *Let f and g share $(0,k_1),(1,k_2)$ and (∞,k_3) and $f \not\equiv g$, where $k_j(j = 1,2,3)$ are positive integers satisfying* (1.1). *If α and h are defined as in Lemma 2.3, then $\overline{N}(r,a;\alpha) = S(r)$ and $\overline{N}(r,a;h) = S(r)$ for $a = 0,\infty$.*

Proof. The lemma follows from Lemma 2.2 and Lemma 2.3 because

$$\overline{N}_*(r,a;f,g) \le \overline{N}(r,a;f| \ge 2)$$

for $a = 0,1,\infty$. ∎

Using Lemma 2.2 and Lemma 2.4, we can prove the following lemma in the line of Lemma 2.5 [5].

Lemma 2.5. *Let f and g share $(0,k_1),(1,k_2)$ and (∞,k_3) and $f \not\equiv g$, where $k_j(j = 1,2,3)$ are positive integers satisfying* (1.1). *If f is not a bilinear transformation of g, then each of the following equalities holds:*

(i) $T(r,f) + T(r,g) = N(r,0;f| \le 1) + N(r,1;f| \le 1) + N(r,\infty;f| \le 1) + N_0(r) + S(r)$,
(ii) $T(r,f) + T(r,g) = N(r,0;g| \le 1) + N(r,1;g| \le 1) + N(r,\infty;g| \le 1) + N_0(r) + S(r)$,
(iii) $N(r,0;f-g|f=\infty) = S(r)$ *and* $N(r,0;f-g|g=\infty) = S(r)$,
(iv) $N(r,0;f-g| \ge 2) = S(r)$,

where $N(r,0;f-g|f=\infty)$ denotes the counting function of those zeros of $f-g$ which are poles of f.

Lemma 2.6. [7] *Let f be a non-constant meromorphic function and*

$$R(f) = \frac{\sum_{i=0}^{m} a_i f^i}{\sum_{j=0}^{n} b_j f^j}$$

be a non-constant irreducible rational in f with constant coefficient $\{a_i\}$ and $\{b_j\}$ satisfying $a_m \ne 0$ and $b_n \ne 0$. Then

$$T(r,R(f)) = \max\{m,n\}T(r,f) + O(1).$$

In particular, if f is a bilinear transformation of g, then we have $T(r,f) \sim T(r,g)$ as $r \to \infty$.

Lemma 2.7. [6] *Let f_1 and f_2 be two distinct non-constant meromorphic functions satisfying*

$$\overline{N}(r,0;f_i) + \overline{N}(r,\infty;f_i) = S(r;f_1,f_2)$$

for $i = 1,2$. If $f_1^s f_2^t - 1$ is not identically zero for all integers s and $t(|s|+|t|>0)$, then for any positive ε we have

$$\overline{N}_0(r,1;f_1,f_2) \le \varepsilon T(r;f_1,f_2) + S(r;f_1,f_2),$$

where $\overline{N}_0(r,1;f_1,f_2)$ denotes the reduced counting function of f_1 and f_2 related to the common 1 -points and $T(r;f_1,f_2) = T(r,f_1)+T(r,f_2)$, $S(r;f_1,f_2) = o\{T(r;f_1,f_2)\}$ as $r \to \infty (r \notin E)$.

3. Proof of the theorems

Proof of Theorems 1.7 and 1.8. Suppose that f is not a bilinear transformation of g, otherwise, we obtain that f and g share $(0,\infty),(1,\infty),(\infty,\infty)$ and hence Theorem 1.7 and Theorem 1.8 follow from Theorem 1.1 and Theorem 1.2 respectively.

From the definitions of $N_0^*(r)$ and $N_0(r)$, we see that $N_0^*(r)-N_0(r)$ is the counting function of those zeros of $f-g$ which are the poles of f. So by (iii) of Lemma 2.5 we get

$$N_0^*(r)-N_0(r)=S(r). \tag{3.1}$$

Now we prove that

$$\overline{N}_0(r,1;\alpha,h)=N_0(r)+S(r). \tag{3.2}$$

We consider the following cases.

Case 1. Let z_0 be a common simple zero of f and g such that $\alpha(z_0)=h(z_0)=1$. Since $h-1=\dfrac{g-f}{f}$, it follows that z_0 is a multiple zero of $f-g$. Hence by Lemma 2.2 and (iv) of Lemma 2.5 we see that the reduced counting function of the zeros of f and g for which $\alpha(z)=h(z)=1$ is $S(r)$.

Case 2. Let z_1 be a common simple 1-point of f and g such that $\alpha(z_1)=h(z_1)=1$. Since $\alpha-1=\dfrac{f-g}{g-1}$, it follows that z_1 is a multiple zero of $f-g$. Hence by Lemma 2.2 and (iv) of Lemma 2.5 we see that the reduced counting function of the 1-points of f and g for which $\alpha(z)=h(z)=1$ is $S(r)$.

Case 3. Let z_2 be a common simple pole of f and g such that $\alpha(z_2)=h(z_2)=1$. Since $\alpha-1=\dfrac{f-g}{g-1}$ and $h-1=\dfrac{g-f}{f}$, it follows that z_2 is not a pole of $f-g$. Hence, z_2 is a zero of $\alpha+h-2=\dfrac{(f-g)(f-g+1)}{f(g-1)}$ with multiplicity ≥ 2 and so z_2 is a zero of $\alpha'+h'$. Also

$$\frac{\alpha'}{\alpha}+\frac{h'}{h}=(\alpha'+h')\frac{\alpha+h-1}{\alpha h}-\frac{(\alpha-1)\alpha'+(h-1)h'}{\alpha h}.$$

Since f is not a bilinear transformation of $g, \dfrac{\alpha'}{\alpha}+\dfrac{h'}{h}\not\equiv 0$. From the preceding identity, we see that z_2 is a zero of $\dfrac{\alpha'}{\alpha}+\dfrac{h'}{h}$. Now by Lemma 2.4, we get

$$\begin{aligned}
N\left(r,0;\frac{\alpha'}{\alpha}+\frac{h'}{h}\right) &\leq T\left(r,\frac{\alpha'}{\alpha}+\frac{h'}{h}\right)\\
&= N\left(r,\frac{\alpha'}{\alpha}\right)+N(r,\frac{h'}{h})+S(r)\\
&= \overline{N}(r,0;\infty)+\overline{N}(r,\infty,\alpha)+\overline{N}(r,0;h)+\overline{N}(r,\infty;h)+S(r)\\
&= S(r)
\end{aligned}$$

Therefore, by Lemma 2.2 we see that the reduced counting function of the poles of f and g for which $\alpha(z) = h(z) = 1$ is $S(r)$.

Also by (iv) of Lemma 2.5 we have

$$N_0^*(r) = \overline{N}_0^*(r) + S(r) \text{ and } N_0(r) = N_0(r) + S(r).$$

Hence from above we get by (3.1)

$$\overline{N}_0(r, 1; \alpha, h) = N_0^*(r) + S(r) = N_0(r) + S(r),$$

which is (3.2). From the definitions of α and h and from Lemma 2.1 we get

$$\begin{aligned} T(r,\alpha) + T(r,h) &\leq 2T(r,f) + 2T(r,g) + O(1) \\ &\leq 8T(r,f) + S(r). \end{aligned} \tag{3.3}$$

Hence by (3.2) we obtain

$$\limsup_{r\to\infty, r\notin E} \frac{\overline{N}_0(r,1;\alpha,h)}{T(r;\alpha,h)} \geq \frac{1}{8} \limsup_{r\to\infty, r\notin E} \frac{N_0(r)}{T(r,f)} > 0.$$

If we put

$$a = \limsup_{r\to\infty, r\notin E} \frac{\overline{N}_0(r,1;\alpha,h)}{T(r;\alpha,h)},$$

we see that the following inequality does not hold for any $\varepsilon (0 < \varepsilon < a)$

$$\overline{N}_0(r,1;\alpha,h) \leq \varepsilon T(r;\alpha,h) + S(r;\alpha,h)$$

as $r \to \infty (r \notin E)$, where $T(r;\alpha,h) = T(r,\alpha) + T(r,h)$.

Since $f = \dfrac{1-\alpha}{1-\alpha h}$ and $g = \dfrac{h(1-\alpha)}{1-\alpha h}$, we get

$$T(r,f) \leq 2T(r,\alpha) + 2T(r,h) + O(1)$$

and

$$T(r,g) \leq 2T(r,\alpha) + 2T(r,h) + O(1).$$

This together with (3.3) implies that $S(r) = S(r;\alpha,h)$. Hence by Lemma 2.4 and Lemma 2.7, there exist two integers s and $t (|s| + |t| > 0)$ such that

$$\alpha^t h^s \equiv 1.$$

Hence

$$\left(\frac{f-1}{g-1}\right)^t \equiv \left(\frac{f}{g}\right)^s \tag{3.4}$$

and so

$$\left(\frac{1-\frac{1}{f}}{1-\frac{1}{g}}\right)^t \equiv \left(\frac{f}{g}\right)^{s-t}. \tag{3.5}$$

If $st = 0$ or $s - t = 0$, then from (3.4) and (3.5) we see that f is bilinear transformation of g, which is a contradiction. Therefore st and $s - t$ are not equal to zero. Consequently, (3.4) and (3.5) imply that f and g share $(0,\infty), (1,\infty), (\infty,\infty)$. Now, Theorem 1.7 and Theorem 1.8 follow respectively from Theorem 1.1 and Theorem 1.2. ∎

Proof of Theorem 1.9. We consider the following cases.

Case 1. Let

$$\limsup_{r\to\infty, r\notin E} \frac{N_0(r)}{T(r,f)} > 0.$$

If f is a bilinear transformation of g, then one of the relations of Theorem 1.7 holds. So by Lemma 2.6 we get $T(r,f) \sim T(r,g)$ as $r \to \infty$. If f is not a bilinear transformation of g, then one of the relations of Theorem 1.8 holds. Since s and $k+1$ are positive integers which are relatively prime, by Lemma 2.6 we get $T(r,f) \sim T(r,g)$ as $r \to \infty$.

Case 2. Let

$$\limsup_{r\to\infty, r\notin E} \frac{N_0(r)}{T(r,f)} = 0.$$

If f is a bilinear transformation of g, by Lemma 2.6 we have $T(r,f) \sim T(r,g)$ as $r \to \infty$.

If f is not a bilinear transformation of g by Lemma 2.5(i) and (ii) we get

$$T(r,g) \leq T(r,f) + N(r,\infty; f| \leq 1) + S(r) \tag{3.6}$$

and

$$T(r,f) \leq T(r,g) + N(r,\infty; g| \leq 1) + S(r). \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$T(r,g) \leq 2T(r,f) + S(r)$$

and

$$T(r,f) \leq 2T(r,g) + S(r).$$

If, in particular, f and g are entire from (3.6) and (3.7) we get $T(r,f) \sim T(r,g)$ as $r \to \infty (r \notin E)$. This proves the theorem. ∎

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