BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **32***(1) (2009), 1–17*

Barycentric Ramsey Numbers for Small Graphs

[1]SAMUEL GONZÁLEZ, [2]LEIDA GONZÁLEZ AND [2]OSCAR ORDAZ*
[1]Universidad Simón Bolivar Núcleo Litoral, Estado Vargas, Venezuela
[2]Departamento de Matemáticas and Laboratorio LaTecS Centro ISYS,
Facultad de Ciencias, Universidad Central de Venezuela,
Ap. 47567, Caracas 1041-A, Venezuela
flosav@cantv.net*

Abstract. Let G be a finite abelian group of order n. The barycentric Ramsey number $BR(H,G)$ is the minimum positive integer r such that any coloring of the edges of the complete graph K_r by elements of G contains a subgraph H whose assigned edge color constitutes a barycentric sequence, i.e. there exists one edge whose color is the "average" of the colors of its edges. These $BR(H,G)$ are determined for some graphs, in particular for graphs with at most four edges without isolated vertices (i.e. small graphs) and $G = \mathbb{Z}_n$, $2 \leq n \leq 5$. Elementary combinatorial arguments are used for these computations.

2000 Mathematics Subject Classification: 11B50, 11P70, 11B75

Key words and phrases: k-barycentric sequences, k-barycentric Davenport constant, Ramsey barycentric number, small graphs, zero-sum sequences.

1. Introduction

Let G be an abelian group of order n. This research focuses on barycentric sequences, which are introduced by Ordaz in [13] and are a natural extension of zero-sum sequences. In informal words, a sequence in G is barycentric if it contains one element which is the "average" of its terms. Erdős, Ginzburg and Ziv in [15] show that any sequence of length $2n-1$ contains an n-subsequence with zero-sum. This theorem constitutes the beginning of the combinatorics area known as the zero-sum problems. Gao and Geroldinger [17] give a nice and structured survey on zero-sum problems as an update of the first survey on zero-sum theory by Caro [8], appeared in 1996. Zhi Wei Sun in [19] establishes a unified theory among three apparently unrelated areas in combinatorial number theory, zero-sum problems, subset sums and covers of the integers.

Let $H = (V(H), E(H))$ be a graph with $e(H)$ edges. The *Ramsey number* $R(H,n)$ is the smallest positive integer t such that in any coloring of the edges of K_t with n colors there exists a monochromatic copy of H.

Received: October 15, 2006; *Revised:* December 21, 2007.

The *barycentric Ramsey number*, introduced in [14], of the pair (H, G), denoted by $BR(H, G)$, is the minimum positive integer r such that any coloring $f : E(K_r) \to G$ of the edges of K_r by elements of G, contains a subgraph H, with an edge e_0 such that $\sum_{e \in E(H)} f(e) = e(H) f(e_0)$. In this case H is called *a barycentric graph*. It is clear that $BR(H, G) \leq R(H, |G|)$, then $BR(H, G)$ always exists.

Recall that for any graph H whose number of edges $e(H)$ satisfies $e(H) = 0 \pmod{n}$, the *zero-sum Ramsey number* $R(H, G)$ is defined as the minimal positive integer s such that any coloring $f : E(K_s) \to G$ of the edges of K_s by elements of G contains a subgraph H with $\sum_{e \in E(H)} f(e) = 0$, where 0 is the zero element of G. The necessity of the condition $e(H) = 0 \pmod{n}$ for the existence of $R(H, G)$ is clear; it comes from the monochromatic coloring of the edges of H. The zero-sum Ramsey number is introduced by Bialostocki and Dierker in [2] when $e(H) = n$ and the concept is extended to $e(H) = 0 \pmod{n}$ by Caro in [6]. It is clear that $BR(H, G) \geq |V(H)|$ and $BR(H, G) = R(H, G)$ when $e(H) = 0 \pmod{n}$. Notice that when $e(H) = 0 \pmod{n}$ then $R(H, G) \leq R(H, n)$, moreover when $e(H) = n$ then $R(H, 2) \leq R(H, \mathbb{Z}_n)$. If a graph H is not barycentric with exactly two colors, then $R(H, 2) \leq BR(H, G)$.

In [8], Caro gives Table 1 as a survey of the $R(H, \mathbb{Z}_n)$ and $R(H, 2)$ known up to now for small graphs with $n = 2, 3$, or 4, based on results given in [1, 2, 4, 5, 6, 7]. The notation for Table 1 is: $K_{1,k}$ are stars with k edges, $MK_{1,k}$ are modified k-stars, defined as the tree with $k+1$ vertices, k edges and degree sequences $k-1, 2, 1, \ldots, 1$, P_k are paths with k vertices and $k-1$ edges, C_k are circuits with k vertices, mK_2 is an m matching and $K_3 + e$ is a graph with vertices a, b, c, d and edges ab, bc, ca, bd. The graphs union is disjoint.

Table 1. Ramsey number and zero-sum Ramsey number for small graphs

H	$R(H,2)$	$R(H,\mathbb{Z}_2)$	$R(H,\mathbb{Z}_3)$	$R(H,\mathbb{Z}_4)$
P_3	3	3		
$2K_2$	5	5		
C_3	6		11	
P_4	5		5	
$K_{1,3}$	6		6	
$P_3 \cup K_2$	6		6	
$3K_2$	8		8	
C_4	6	4		6
$K_{1,4}$	7	5		7
P_5	6	5		6
$C_3 \cup K_2$	7	6		8
$2P_3$	7	6		7
$P_4 \cup K_2$	8	6		8
$K_{1,3} \cup K_2$	7	7		8
$P_3 \cup 2K_2$	9	7		9
$4K_2$	11	9		11
$MK_{1,4}$	6	5		6
$K_3 + e$	7	4		7

Table 2. Barycentric Ramsey number for stars

k	G	$BR(K_{1,k}, G)$
2	odd order	$n+2$
	even order	$n+1$
k	$\mathbb{Z}_2$	$k+1$
$k = 0(\text{mod})3$	$\mathbb{Z}_3$	$k+3$
$k \neq 0(\text{mod})3$		$k+2$
3	$\mathbb{Z}_p$, p prime ≥ 5	$\leq 2\lceil \frac{p}{3} \rceil + 2$
	$\mathbb{Z}_5$	6
	$\mathbb{Z}_7$	8
	$\mathbb{Z}_{11}$	10
	$\mathbb{Z}_{13}$	10
k	$\mathbb{Z}_p$	$\leq p+k$
$4 \leq k \leq p-1$	$\mathbb{Z}_p$	$\leq p+k-1$
$p-1$	$\mathbb{Z}_p$, p prime ≥ 5	$2p-2$
4	$\mathbb{Z}_7$	9
$tp+4 \leq k \leq tp+p-1$	$\mathbb{Z}_p$, p prime ≥ 5	$\leq p+k-1$
9	$\mathbb{Z}_5$	13
$tp+1$, $t>0$	$\mathbb{Z}_p$	$(t+1)p$
$5t+2$	$\mathbb{Z}_5$	$5(t+1)$

In Table 1, we have $e(H) = 0(\text{mod } n)$ then $BR(H, \mathbb{Z}_n) = R(H, \mathbb{Z}_n)$. The objective of our paper is to complete Table 1 with the values of $BR(H, \mathbb{Z}_n)$ for $e(H) \neq 0$ (mod n).

As usually in case of new subjects in combinatorics, a good idea is to start research with small problems in order to discover strong arguments or general proof techniques to approach more general cases. According to this general idea, in this paper, elementary combinatorial arguments are used to compute the barycentric Ramsey numbers for small graphs.

The barycentric Ramsey number for stars $BR(K_{1,k}, \mathbb{Z}_p)$ is studied and some values or bounds are given in [14]. Table 2 summarizes the values of $BR(K_{1,k}, \mathbb{Z}_p)$ for some p prime known at present [14]. In this table n denotes the order of G.

The paper is structured as follows, besides this introduction and the conclusion: Section 2 contains the tools necessary to develop Section 3, where the main result is given.

2. Tools

In this section we summarize some results used to establish the barycentric Ramsey numbers for some graphs, in particular those with at most four edges.

The following definitions are used:

Definition 2.1. [13] *Let A be a finite set with $|A| \geq 2$ and G a finite abelian group. A sequence $f : A \to G$ is* barycentric *if there exists $a \in A$ such that $\sum_A f = |A| f(a)$. The element $f(a)$ is called a barycenter.*

Moreover in [14] the following definition is introduced:

Definition 2.2. [14] *Let G be an abelian group of order $n \geq 2$. The k-*barycentric Davenport constant *$BD(k,G)$ is the minimal positive integer t such that every t-sequence in G contains a k-barycentric subsequence.*

In [18] Hamidoune shows that $BD(k,G) \leq n+k-1$.

We have the inequality $BR(K_{1,k},G) \leq BD(k,G)+1$: for any vertex in $K_{BD(k,G)+1}$ there is a barycentric star centered on this vertex. The following remark and theorem allow us to establish $BR(H,\mathbb{Z}_2)$.

Remark 2.1. Let H be a graph and $e(H)$ the number of its edges. Then
$$BR(H,\mathbb{Z}_2) = \begin{cases} |V(H)| & \text{if } e(H) \text{ is odd} \\ R(H,\mathbb{Z}_2) & \text{if } e(H) \text{ is even} \end{cases}$$

Theorem 2.1. [4] *Let H be a graph on h vertices and an even number of edges. Then*
$$R(H,\mathbb{Z}_2) = \begin{cases} h+2 & \textit{if } H = K_h, h = 0,1 \pmod 4 \\ h+1 & \textit{if } H = K_p \cup K_q, \binom{p}{2} + \binom{q}{2} = 0 \pmod 2 \\ h+1 & \textit{if all the degrees in } H \textit{ are odd} \\ h & \textit{otherwise.} \end{cases}$$

We have the following results for stars and matching.

Theorem 2.2. [2]

(1) $R(K_{1,m},\mathbb{Z}_m) = R(K_{1,m},2) = \begin{cases} 2m & \textit{if } m \textit{ is odd} \\ 2m-1 & \textit{if } m \textit{ is even} \end{cases}$

(2) $R(mK_2,\mathbb{Z}_m) = R(mK_2,2) = 3m-1$.

Theorem 2.3. [6] *Let $K_{1,m}$ be the stars on m edges with $m = 0 \pmod n$. Then*
$$BR(K_{1,m},\mathbb{Z}_n) = R(K_{1,m},\mathbb{Z}_n) = \begin{cases} m+n-1 & \textit{if } m = n = 0 \pmod 2 \\ m+n & \textit{otherwise} \end{cases}$$

Theorem 2.4. [3] *Let mK_2 be the matching on m edges with $m = 0 \pmod n$. Then* $BR(mK_2,\mathbb{Z}_n) = R(mK_2,\mathbb{Z}_n) = 2m+n-1$.

We use the following lemmas:

Lemma 2.1. [2] *If the edges of K_n, where $n \geq 5$ are colored by at least three colors then there exists a path on three edges each colored differently.*

Let G be a graph with four edges. If f is a $\mathbb{Z}_4$-coloring of $E(K_n)$, where $n \geq 5$, then there exists a $\mathbb{Z}_4$- coloring of $E(K_n)$, say f^1, such that $\sum_{e\in E(H)} f(e) = 0$ if and only if $\sum_{e\in E(H)} f^1(e) = 0$ for all copies H of G in K_n. Moreover there exists a path on three edges in K_n $v_1v_2v_3v_4$ such that: $f^1(v_1v_2) = 0, f^1(v_2v_3) = 1, f^1(v_3v_4) = 2$ or $f^1(v_1v_2) = 0, f^1(v_2v_3) = 1, f^1(v_3v_4) = 3$.

Lemma 2.2. [9] *If the edges of K_5 are colored with any number of colors hence K_5 contains either a path of length 3 using only one color or a path of length 3 using three different colors.*

Remark 2.2. Reminding the definition of Ramsey numbers and since $R(C_4,2) = 6$ then, if the edges of K_n, $n \geq 6$ are colored with exactly two colors then there exists a monochromatic C_4 .

Lemma 2.3. [12] *If the edges of K_n, $n \geq 5$ are colored with at least three colors and contain a C_4 using two colors, one of them repeated three times, then there exists a C_4 using exactly three different colors.*

The following remark is useful to establish the barycentric Ramsey number for small graphs.

Remark 2.3. [12] Let H be a graph with $2 \leq e(H) \leq 4$ edges colored by elements of $\mathbb{Z}_n$ $(2 \leq n \leq 5)$. Table 3 shows all the different coloring for $E(H)$ to be barycentric. For example, in case $e(H) = 3$ and the edges colored by elements from $\mathbb{Z}_4$, H is barycentric when the edges are colored with three different colors a, b, c, or the edges are colored by $a, a, a+2$ for any color a, or the edges are colored monochromatically.

Table 3. Barycentric graph colorings

$e(H)$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_5$
2	monochromatic	monochromatic	monochromatic	monochromatic
3	any coloring	a, b, c	a, b, c	a, b, c
		monochromatic	$a, a, a+2$	monochromatic
			monochromatic	
4	a, a, b, b	a, a, b, c	$a, a, a+2, a+2$	a, a, b, c
	monochromatic	a, a, a, b	$a, a, a+1, a+3$	monochromatic
		monochromatic	monochromatic	

Definition 2.3. [16] *Let $r(n)$ be the smallest number such that any coloration of the edges of $K_{r(n)}$ with n colors induces some K_3 with three colors or with only one color.*

It is shown in [11] that $r(3) = 11$.

Lemma 2.4. [16] $r(n+1) \leq 2 + n(r(n) - 1)$. *Moreover* $r(4) \leq 32$ *and* $r(5) \leq 126$.

3. Main results

This section is dedicated to establish $BR(H, \mathbb{Z}_n)$, $2 \leq n \leq 5$, for the 18 graphs given in Table 1. Some of them are obtained directly from $R(H, \mathbb{Z}_n)$, due to the fact that $BR(H, \mathbb{Z}_n) = R(H, \mathbb{Z}_n)$ when $e(H) = 0 \pmod n$ or from $BR(H, \mathbb{Z}_2)$ using Remark 2.1 and Theorem 2.1.

In what follows Table 4 presents the barycentric Ramsey number for small graphs. The upper bound values were computed manually by cases. Each case with its particular degree of difficulty was treated using the Lemmas and Remarks given in Section 2. For the lower bounds we use ad hoc decomposition of a complete graph to color its edges in order to forbid a particular barycentric graph.

3.1. Barycentric Ramsey numbers for matchings

Theorem 3.1. *Let G be an abelian group of order $n \geq 2$. Then $BR(2K_2, G) = n+3$.*

Table 4. Barycentric Ramsey number for small graphs

H	$BR(H,\mathbb{Z}_2)$	$BR(H,\mathbb{Z}_3)$	$BR(H,\mathbb{Z}_4)$	$BR(H,\mathbb{Z}_5)$
P_3	3	5	5	7
$2K_2$	5	6	7	8
C_3	3	11	6	Th.3.10
P_4	4	5	5	5
$K_{1,3}$	4	6	6	6
$P_3 \cup K_2$	5	6	6	6
$3K_2$	6	8	8	8
C_4	4	5	6	7
$K_{1,4}$	5	6	7	8
P_5	6	5	6	6
$C_3 \cup K_2$	6	5	8	7
$2P_3$	6	6	7	7
$P_4 \cup K_2$	6	6	8	8
$K_{1,3} \cup K_2$	7	6	8	7
$P_3 \cup 2K_2$	7	7	9	9
$4K_2$	9	8	11	11
$MK_{1,4}$	5	5	6	6
$K_3 + e$	4	4	7	7

Proof. The complete graph K_{n+2} can be decomposed into the edge-disjoint union of $n-1$ stars and a complete graph with three vertices. The idea behind this decomposition is to monochromatically color the stars and the complete graph of order three. Notice that the only possible coloring $f : E(K_{n+2}) \to G$ with a monochromatic free $2K_2$ is when f induces a monochromatic K_3.

For the lower bound, we color each star $K_{1,i}$ in K_{n+2} by c_{i-1} for $3 \le i \le n+1$ and the edges of K_3 by c_1.

For the upper bound, we use induction on the order n of G. When $n = 2$, Theorem 2.2 shows that $BR(2K_2,\mathbb{Z}_2) = 5$. Assume that $BR(2K_2, G) = n + 3$ for $|G| = n$. Let $G = \{c_1, c_2, \cdots, c_{n+1}\}$ be an abelian group of order $n + 1$. If for some coloring from G of $E(K_{n+4})$, we have that $K_{n+3} \subseteq K_{n+4}$ is monochromatic free $2K_2$, then $E(K_{n+3})$ must be colored with $n + 1$ colors, as indicated above. Let v_i $(1 \le i \le n)$ be the center of the stars and $v_{n+1}, v_{n+2}, v_{n+3}$ the vertices of K_3. The edges of the stars and K_3 are colored by c_i $(1 \le i \le n+1)$, respectively. Notice that three of the $n+3$ incident edges to $v_{n+4} \in V(K_{n+4}) \setminus V(K_{n+3})$, are also incident to K_3. Then a monochromatic $2K_2$ can be constructed by any coloring from c_i for $1 \le i \le n+1$. ∎

Theorem 3.2. $BR(3K_2,\mathbb{Z}_5) = 8$.

Proof. For the upper bound, we have the following cases:

Case 1. $E(K_8)$ is colored by at least three colors. By Lemma 2.1 it contains a P_4 with three different colors, say a, b, c. Consider the two non-adjacent edges in P_4 with color a and b respectively. Let K_4 be the complete graph built with the vertices in K_8 outside P_4. If some edge in K_4 is colored with $x \notin \{a, b\}$

then we have a $3K_2$ with three different colors. If $E(K_4)$ is colored by only a and b, then we have the following cases:

(i) In K_4 there exists $2K_2$ with color a and b, then with the edge in P_4 colored by c, we have a barycentric $3K_2$.

(ii) Each two independent edges are a-monochromatic or b-monochromatic. Then with the edge in P_4 with color a or b, we have a barycentric $3K_2$.

Case 2. $E(K_8)$ is colored by exactly two colors. Since $R(3K_2, 2) = 8$ hence there exists a monochromatic $3K_2$, therefore barycentric.

The lower bound follows from $8 = R(3K_2, 2) \leq BR(3K_2, \mathbb{Z}_5)$. ∎

Corollary 3.1. $BR(3K_2, \mathbb{Z}_4) = 8$.

Proof. Since $BR(3K_2, \mathbb{Z}_5) = 8$ then for any $f : E(K_8) \to \mathbb{Z}_4$ there exists a monochromatic or with three different colors $3K_2 \subseteq K_8$. Then by Remark 2.3, $3K_2$ is barycentric, so we have the upper bound.

In order to give the lower bound, we color the edges of some $K_5 \subseteq K_7$ by a and the remaining edges from K_7 by $a + 1$. ∎

Theorem 3.3. $BR(4K_2, \mathbb{Z}_3) = 8$.

Proof. Since $BR(3K_2, \mathbb{Z}_3) = R(3K_2, \mathbb{Z}_3) = 8$ then for any $f : E(K_8) \to \mathbb{Z}_3$ there exists a zero-sum $3K_2$ in K_8. Moreover this $3K_2$ is contained in a barycentric perfect matching of K_8. So that we have the upper bound.

The lower bound follows trivially. ∎

Theorem 3.4. $BR(4K_2, \mathbb{Z}_5) = 11$.

Proof. For the upper bound we have two cases:

Case 1. $E(K_{11})$ is colored by a and b. Since $R(4K_2, 2) = 11$, hence there exists a monochromatic and hence barycentric $4K_2$.

Case 2. $E(K_{11})$ is colored by at least three colors. Since $BR(3K_2, \mathbb{Z}_5) = 8$, then for any $f : E(K_8) \to \mathbb{Z}_5$, there exists in $K_8 \subseteq K_{11}$ a-monochromatic $3K_2$ or a $3K_2$ with three different colors, say a, b, c. In both cases we consider the complete graph K_5 built with the five vertices in K_{11} outside $3K_2$. Set $3K_2 = v_1v_2, v_3v_4, v_5v_6$ and $V(K_5) = \{v_7, v_8, v_9, v_{10}, v_{11}\}$. We have the following cases:

(i) $3K_2$ is a-monochromatic then $E(K_5)$ must be x-monochromatic with $x \neq a$ in order to avoid a barycentric $4K_2$. Assuming $E(K_5)$ is b-monochromatic. It is clear that the edges of $E(K_6)$ must be colored by a, or b, else we have the theorem. Let $K_{6,5}$ be the bipartite complete graph from the vertices of K_6 to the vertices of K_5. The edges of $K_{6,5}$ must be colored by a, or b, else we are done. Therefore the edges of K_{11} are colored by two colors, hence by Case 1 we have the theorem.

(ii) When $3K_2$ has colors a, b, c then $E(K_5)$ must be colored by d and e in order to avoid a barycentric $4K_2$. Hence K_5 contains a monochromatic

P_4. Therefore a monochromatic $2K_2$ is derived. So that for any two edges from $3K_2$ we have a barycentric $4K_2$.

The lower bound follows from $11 = R(4K_2, 2) \le BR(4K_2, \mathbb{Z}_5)$. ▮

3.2. Barycentric Ramsey number for paths and circuits

Theorem 3.5. $BR(C_4, \mathbb{Z}_3) = BR(P_5, \mathbb{Z}_3) = 5$.

Proof. Let v_1, v_2, v_3, v_4, v_5 be the vertices of K_5. By Lemma 2.2 for any $f : E(K_5) \to \mathbb{Z}_3$, there exists a path $P_4 = v_1v_2v_3v_4$ colored with three different colors or monochromatic. Hence the circuit $C_4 = v_1v_2v_3v_4v_1$ and the path $P_5 = v_1v_2v_3v_4v_5$ with $f(v_4v_1)$ and $f(v_4v_5)$ in $\mathbb{Z}_3$ respectively are barycentric.

For the lower bound $5 \le BR(C_4, \mathbb{Z}_3)$, we consider the complete graph K_4 of vertices v_1, v_2, v_3, v_4 and edges colored as follows: $f(v_1v_2) = f(v_4v_3) = a$, $f(v_2v_3) = f(v_1v_4) = b$, $f(v_2v_4) = f(v_1v_3) = c$. The lower bound $5 \le BR(P_5, \mathbb{Z}_3)$ is obvious. ▮

Theorem 3.6. $BR(P_4, \mathbb{Z}_4) = 5$.

Proof. The upper bound follows directly from Lemma 2.2. For the lower bound we consider the complete graph K_4 of vertices v_1, v_2, v_3, v_4 and edges colored as follows: $f(v_1v_2) = f(v_4v_3) = a$, $f(v_2v_3) = f(v_1v_4) = a + 1$, $f(v_2v_4) = f(v_1v_3) = a + 2$. ▮

Theorem 3.7. $BR(P_4, \mathbb{Z}_5) = 5$.

Proof. The upper bound follows directly from Lemma 2.2. For the lower bound, set $f : E(K_4) \to \mathbb{Z}_5$ then we consider the circuit $v_1v_2v_3v_4v_1$ in K_4 and edges colored in the following way: $f(v_1v_2) = f(v_4v_3) = a$ and $f(v_1v_4) = f(v_2v_3) = b$, moreover $f(v_4v_2) = f(v_1v_3) = c$. ▮

Theorem 3.8. $BR(P_5, \mathbb{Z}_5) = 6$.

Proof. Set $V(K_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$. By Lemma 2.2, there exists for every $f : E(K_6) \to \mathbb{Z}_5$ a path $P_4 \subset K_5$ monochromatic or colored with three different colors. We have the following cases:

Case 1. Set $P_4 = v_1v_2v_3v_4$ colored with a, b, c. If $f(v_1v_6) \in \{a, b, c\}$ or $f(v_4v_5) \in \{a, b, c\}$ we have the theorem. Else, if $f(v_6v_5) \in \{a, b, c\}$ we are done. If $f(v_6v_5) = f(v_5v_4)$ or $f(v_1v_6) = f(v_6v_5)$ we are done. Set $f(v_6v_5) \in \{d, e\}$, if $f(v_1v_6) = f(v_5v_4)$ and different from $f(v_6v_5)$ then the path $P_4 = v_2v_1v_6v_5v_4$ is barycentric.

Case 2. The path $P_4 = v_1v_2v_3v_4$ is a-monochromatic. If $f(v_1v_6) = a$ or $f(v_4v_5) = a$ we have the theorem, else we have the following subcases:
(i) If $v_1v_6v_5v_4$ are colored with three different colors, we have Case 1.
(ii) If $v_1v_6v_5v_4$ are colored with two different colors. If $f(v_6v_5) \ne a$ then $f(v_6v_5) \ne f(v_5v_4)$ or $f(v_6v_5) \ne f(v_6v_1)$ hence we have the theorem. Set $f(v_6v_5) = a$ then we have $f(v_1v_6) = f(v_4v_5)$. Set $f(v_1v_6) = x \in \{b, c, d, e\}$, say $f(v_1v_6) = b$, then the colors on the remaining edges of $E(K_6)$ must be a or b else we have Case 1. Moreover, since $R(P_5, 2) = 6$ we have the theorem.

(iii) If $v_1v_6v_5v_4$ is b-monochromatic, then the remaining edge colors must be a, or b, else we have Case 1. Therefore $E(K_6)$ is colored with two colors, then since $R(P_5, 2) = 6$ we are done.

For the lower bound we consider that the circuit $v_1v_2v_3v_4v_5v_1$ has as edge colors $f(v_1v_2) = f(v_2v_3) = f(v_3v_4) = a$ and $f(v_4v_5) = f(v_5v_1) = b$. Moreover the color edges $f(v_3v_5) = f(v_3v_1) = b$, $f(v_4v_1) = f(v_4v_2) = a$ and $f(v_5v_2) = a$. ∎

Theorem 3.9. $BR(C_4, \mathbb{Z}_5) = 7$.

Proof. Since $BR(K_{1,2}, \mathbb{Z}_5) = 7$ then for any $f : E(K_7) \to \mathbb{Z}_5$ there exists a monochromatic star $K_{1,2}$, say $v_1v_3v_2$ colored by a. We have the following cases:

Case 1. $E(K_7)$ is colored by two colors. Then by Remark 2.2 there exists a monochromatic C_4.

Case 2. $E(K_7)$ is colored with at least three colors. We consider the star $v_1v_4v_2$. Then we have the following cases according to the different colors assigned to $v_1v_4v_2$.

(i) $v_1v_4v_2$ has at least one edge colored by a. Hence we have a C_4 with two colors where a is repeated 3 times, then by Lemma 2.3 there exists a C_4 with three different colors, so we are done. In case $v_1v_4v_2$ is a-monochromatic, we have also a monochromatic C_4.

(ii) $v_1v_4v_2$ is colored by two different colors, both of them different from a. Then we have a C_4 with three different colors, hence we are done.

(iii) Consider the bipartite graph $K_{2,5}$ with bipartition $\{v_1v_2\}$ and $\{v_3, v_4, v_5, v_6, v_7\}$. In order to avoid Cases 1 and 2, we color monochromatically the five stars $K_{1,2}$ centered in v_3, v_4, v_5, v_6, v_7 respectively. It is clear that each two stars have different colors, else we have the theorem. Without loss of generality set $f(v_1v_2) = a$, once again without loss of generality set $f(v_6v_7) = b$ in order to avoid a barycentric C_4. Therefore if we color v_5v_6 by any coloring from $\mathbb{Z}_5$ we have a C_4 with three different colors or a C_4 with two colors where one of them is repeated three times. Then by Lemma 2.3 obtaining a subgraph C_4 with three different colors. Hence we are done.

For the lower bound, we consider the complete graph K_6 of vertices $v_1, v_2, v_3, v_4, v_5, v_6$ and edges colored in the following way: $f(v_1v_2) = f(v_2v_3) = f(v_3v_1) = a$, $f(v_1v_6) = f(v_2v_6) = f(v_3v_6) = f(v_4, v_5) = b$, $f(v_1v_5) = f(v_2v_5) = f(v_3v_5) = f(v_4v_6) = c$, $f(v_1v_4) = f(v_2v_4) = f(v_3v_4) = f(v_5v_6) = d$. ∎

Theorem 3.10. $51 \leq BR(C_3, \mathbb{Z}_5) \leq 126$.

Proof. The complete graph with vertices in $\mathbb{Z}_5^2$ with coloration defined in the following manner does not contain a monochromatic or colored with three different colors C_3: The edge between (x, y) and (z, t) is colored by

$$\begin{cases} 0 \text{ if } z - x = \pm 2, \\ 1 \text{ if } z - x = \pm 1, \\ 2 \text{ if } z = x \text{ and } y - t = \pm 2, \\ 3 \text{ if } z = x \text{ and } y - t = \pm 1. \end{cases}$$

The graph made by two copies of the former graph connected by a complete bipartite graph colored by a fifth color, has 50 vertices and is barycentric C_3 free. Moreover by Lemma 2.4 we have $BR(C_3, \mathbb{Z}_5) \leq 126$. ∎

Problem 3.1. *Determine the exact value of* $BR(C_3, \mathbb{Z}_5)$ *or improve the bounds given in Theorem 3.10.*

Theorem 3.11. $BR(C_3, \mathbb{Z}_4) = 6$.

Proof. Since $BR(C_4, \mathbb{Z}_4) = R(C_4, \mathbb{Z}_4) = 6$, then for any $f : E(K_6) \to \mathbb{Z}_4$, there exists a barycentric $C_4 \subseteq K_6$ with edge color sequence: $a, a, a+2, a+2$ or $a, a, a+1, a+3$, or a, a, a, a with $a \in \mathbb{Z}_4$. To obtain the upper bound we consider the following cases:

Case 1. C_4 is colored by $a, a, a+2, a+2$. Then the complete graph K_3 induced by the consecutive edges colored by a and $a+2$, constitutes a barycentric C_3.

Case 2. C_4 is colored by $a, a, a+1, a+3$. Then we have the following subcases:

(i) The edges colored by $a+1$ and $a+3$ are consecutive, then the complete graph induced by theses edges defines a barycentric C_3 with edge color sequence $x, a+1, a+3$ with $x \in \mathbb{Z}_4$.

(ii) The edges colored by $a+1$ and $a+3$ are not consecutive. Set $C_4 = v_1v_2v_3v_4v_1$ and v_2v_3 colored by $a+1$, v_1v_4 colored by $a+3$, v_1v_2, and v_3v_4 colored by a. If edges v_1v_3 and v_2v_4 are not simultaneously colored by a, it is possible to derive a barycentric K_3 from C_4. Otherwise, set $v_5 \in V(K_6 \setminus C_4)$. We have the following subcases:

(a) v_1v_5 is colored by a. If v_2v_5 is colored by $\mathbb{Z}_4 \setminus \{a+1, a+3\}$ we obtain a barycentric C_3; else, color v_3v_5 with any color from $\mathbb{Z}_4$. We are done.

(b) v_1v_5 is colored by $a+2$. In this case v_2v_5 with any color from $\mathbb{Z}_4$ defines a barycentric C_3.

(c) v_1v_5 is colored by $a+1$. In this case v_4v_5 with any color from $\mathbb{Z}_4$ yields a barycentric C_3.

(d) v_1v_5 is colored by $a+3$. If v_2v_5 is colored from $\mathbb{Z}_4 \setminus \{a, a+3\}$ then we can derive a barycentric C_3. If $f(v_2v_5) = a+3$ then any coloring of edge v_3v_5 as well as $f(v_2v_5) = a$ and any coloring of edge v_4v_5 force a barycentric C_3.

Case 3. C_4 is a-monochromatic. Let $C_4 = v_1v_2v_3v_4v_1$. We can derive a barycentric K_3 from C_4, using the color edges of its diagonal. The complex case is when v_1v_3 and v_2v_4 are colored by $a+1$ or $a+3$. The study of all the cases is similar. For example, let us consider the case when v_1v_3 is colored by $a+1$. Let $v_5 \notin C_4$. Then we consider the color of v_1v_5 and v_3v_5 from $\mathbb{Z}_4$. We have the following subcases:

(i) $f(v_1v_5) \neq f(v_3v_5)$. In case that $f(v_1v_5)$ and $f(v_3v_5)$ are assigned values different from $a+1$ then $v_1v_5v_3v_1$ is a barycentric C_3. Assuming that $f(v_1v_5) = a+1$. If $f(v_3v_5) = a+3$ then $C_3 = v_1v_5v_3v_1$ is barycentric. Set $f(v_3v_5) = a$ then for every $f(v_2v_5) \in \mathbb{Z}_4 \setminus \{a+1\}$ we obtain a barycentric C_3. If $f(v_2v_5) = a+1$ then for every $f(v_4v_5) \in \mathbb{Z}_4$ we have a barycentric C_3. Set $f(v_3v_5) = a+2$ then for every $f(v_2v_5) \in \mathbb{Z}_4$ we are done.

(ii) $f(v_1v_5) = f(v_3v_5)$. Then we have the following subcases:
(a) Set $f(v_1v_5) = f(v_3v_5) = a$ then if $f(v_4v_5) \in \{a, a+2\}$ we obtain a barycentric C_3. If $f(v_4v_5) = a+3$ then the colors of the edges of $C_4 = v_5v_4v_1v_3v_5$ correspond to Case 2. Assuming now $f(v_4v_5) = a+1$, if $f(v_4, v_2) = a+3$ then the colors of the edges of $C_4 = v_1v_2v_4v_5v_1$ constitute Case 2. In case $f(v_4, v_2) = a+1$, each color assigned to $f(v_2, v_5)$ allows to derive a barycentric C_3.
(b) Set $f(v_1v_5) = f(v_3v_5) = a+3$. Then the circuit $C_3 = v_1v_5v_3v_1$ is barycentric.
(c) If $f(v_1v_5) = f(v_3v_5) = a+1$ then $v_5v_1v_3v_5$ is a barycentric C_3.
(d) If $f(v_1v_5) = f(v_3v_5) = a+2$ then the edges of $C_4 = v_5v_1v_4v_3v_5$ constitute Case 1.

The lower bound is derived coloring one of the two edge-disjoint hamiltonian cycles of K_5 by a and the other one by b. ∎

Theorem 3.12. *$BR(2P_3, \mathbb{Z}_3) = 6$.*

Proof. Since $BR(P_3 \cup K_2, \mathbb{Z}_3) = R(P_3 \cup K_2, \mathbb{Z}_3) = 6$, then for any $f : E(K_6) \to \mathbb{Z}_3$ there exists some $P_3 \cup K_2$ with zero-sum in K_6. So that any edge in K_6 vertex-disjoint with P_3 and incident to K_2 forms with the zero-sum $P_3 \cup K_2$ a barycentric $2P_3$. So that we have the upper bound.

The lower bound follows trivially. ∎

Theorem 3.13. *$BR(2P_3, \mathbb{Z}_5) = 7$.*

Proof. Since $BR(C_4, \mathbb{Z}_5) = 7$ then for any $f : E(K_7) \to \mathbb{Z}_5$, there exists in K_7 a barycentric C_4. Set K_4 the complete graph induced by C_4 and K_3 the complete graph with vertices different from C_4. Set $K_{4,3}$ the bipartite complete graph from the vertices of K_4 to the vertices of K_3. We have three cases:

Case 1. C_4 has as edge color sequence $aabc$. Then for any coloring of the edges of K_3, a barycentric $2P_3$ is derived.

Case 2. C_4 has as edge color sequence $abac$. We have the following subcases:

(i) $E(K_3)$ is monochromatic or colored by exactly two colors. If there exists an x-monochromatic $P_3 \subseteq K_3$ with $x \neq a$, we have trivially the theorem. Set now, $x = a$. If in K_4 there exists an a-monochromatic P_3 we are done. Else, there exists in K_4 a P_3 colored with two different colors from a. Hence we have the theorem.

(ii) $E(K_3)$ is colored with three different colors. In this case one of them must be equal to some edge color from $E(K_4)$, say a, b, or c. In every case it easy to see we have a barycentric $2P_3$.

Case 3. C_4 is a-monochromatic. We have the following subcases:

(i) There exists in K_3 a P_3, colored with two different colors from a or a-monochromatic, hence we have the theorem.

(ii) K_3 is x-monochromatic with $x \neq a$. Assuming K_3 is b-monochromatic. If some edge color of $E(K_4)$ is different from a and b we have the theorem. Else, if $K_{4,3}$ is not b-monochromatic we can derive a barycentric $2P_3$. Otherwise $E(K_7)$ is colored with two colors and since $R(2P_3, 2) = 7$ we have the theorem.

(iii) $E(K_3)$ has as edge color sequence axx with $x \in \{b, c, d, e\}$, assuming $x = b$. If some edge color of $E(K_4)$ is different from a and b we have the theorem. Else, if $K_{4,3}$ is not b-monochromatic we can derive a barycentric $2P_3$. Otherwise $E(K_7)$ is colored with two colors and since $R(2P_3, 2) = 7$ we have the theorem.

Since $7 = R(2P_3, 2) \leq BR(2P_3, \mathbb{Z}_5)$, then we have the lower bound. ∎

3.3. Barycentric Ramsey number for path-matching

Theorem 3.14. *$BR(P_3 \cup K_2, \mathbb{Z}_5) = 6$.*

Proof. If $E(K_6)$ is colored by at least two colors, then it contains a P_3 with two colors, say a and b. Let K_3 be built by the vertices in K_6 different from P_3. Then either an edge with color $x \notin \{a, b\}$ appears in $E(K_3)$ hence P_3 with this edge defines a barycentric $P_3 \cup K_2$ or one of a, b, say a, appear at least twice in $E(K_3)$, and then the desired $P_3 \cup K_2$ is a-monochromatic, made from two edges of the $E(K_3)$ and the edge colored a of the first P_3. Hence we have the upper bound.

The lower bound follows from $6 = R(P_3 \cup K_2, 2) \leq BR(P_3 \cup K_2, \mathbb{Z}_5)$. ∎

Corollary 3.2. *$BR(C_3 \cup K_2, \mathbb{Z}_5) = 7$.*

Proof. Since $BR(P_3 \cup K_2, \mathbb{Z}_5) = 6$ there exists in K_7, for any $f : E(K_7) \to \mathbb{Z}_5$, a barycentric $P_3 \cup K_2$ i.e. monochromatic or with three different colors. Say $P_3 : v_0v_1v_2$ and $K_2 : v_3v_4$. Set the complete graphs $K_3 \subseteq K_7$ induced by $v_0v_1v_2$ and $K_4 \subseteq K_7$ built with vertices v_3, v_4, v_5, v_6. We have two cases:

Case 1. $E(P_3 \cup K_2)$ is colored as follows: v_0v_1 by a, v_1v_2 by b and v_3v_4 by c. If v_0v_2 is colored by a, b, or c we are done. Assuming v_0v_2 is colored by d. Then the edges of K_4 must be colored by c or e, else we are done. If some $K_3^1 \subseteq K_4$ is no monochromatic we are done, else K_4 is c-monochromatic. Now, we study the following case: set $K_{1,4} = v_1v_3, v_1v_4, v_1v_5, v_1v_6$ if some edge, say v_1v_i is colored from $\{a, b, c, e\}$ then for each $K_3^1 : v_1v_iv_jv_1$ with $v_j \in V(K_4)$, $j \neq i$ there exists an appropriate K_2 such that $K_3^1 \cup K_2$ is barycentric. In case that star $K_{1,4}$ centered in v_1 and with end vertices in $\{v_3, v_4, v_5, v_6\}$ is d-monochromatic, then for any color of edge v_2v_6, we obtain a barycentric $K_3 \cup K_2$.

Case 2. $E(P_3 \cup K_2)$ is a-monochromatically colored as follows: set v_0v_1, v_1v_2 and v_3v_4 colored by a. If v_0v_2 is colored by a we are done. Assume now v_0v_2 colored by some $x \in \{b, c, d, e\}$ say b. If some edge in K_4 is colored by

$x \in \{c, d, e\}$, we are done. Then the edges of K_4 are colored by a and b. Notice that each complete graph K_3^1 induced by three vertices of K_4 must be no monochromatic, else we are done. Moreover, the color of the edges from each vertex of $K_4 \setminus K_3^1$ to K_3 is a or b. Hence $E(K_7)$ is colored by two colors and since $R(K_3 \cup K_2, 2) = 7$, we are done.

The lower bound follows from $7 = R(K_3 \cup K_2, 2) \leq BR(K_3 \cup K_2, \mathbb{Z}_5)$. ∎

Theorem 3.15. $BR(P_4 \cup K_2, \mathbb{Z}_3) = 6$.

Proof. Since $BR(P_4, \mathbb{Z}_3) = R(P_4, \mathbb{Z}_3) = 5$ then for any $f : E(K_6) \to \mathbb{Z}_3$, there exists in K_5 a zero-sum P_4. This path with a vertex-disjoint edge from it, in K_6, defines a barycentric $P_4 \cup K_2$.

The lower bound $6 \leq BR(P_4 \cup K_2, \mathbb{Z}_3)$ is obvious. Thus we have the theorem. ∎

Theorem 3.16. $BR(P_3 \cup K_2, \mathbb{Z}_4) = 6$.

Proof. If $E(K_4) \subseteq E(K_5)$ is colored by a and the remaining edges from K_5 by $a+1$ then K_5 is P_3 barycentric free. Hence the lower bound is obtained.

The upper bound follows directly from the fact that $BR(P_3 \cup K_2, \mathbb{Z}_5) = 6$. ∎

Theorem 3.17. $BR(P_3 \cup 2K_2, \mathbb{Z}_3) = 7$.

Proof. Since $BR(P_3 \cup K_2, \mathbb{Z}_3) = R(P_3 \cup K_2, \mathbb{Z}_3) = 6$ then there exists in K_7, for any $f : E(K_7) \to \mathbb{Z}_3$, a zero-sum $P_3 \cup K_2$. This graph with a vertex-disjoint edge from it in K_7, defines a barycentric $P_3 \cup 2K_2$.

Moreover, since the lower bound must be 7, we are done. ∎

Theorem 3.18. $BR(P_4 \cup K_2, \mathbb{Z}_5) = 8$.

Proof. By Lemma 2.1, K_8 with the edges colored with at least three colors, contains a $P_4 \subseteq K_8$ colored by a, b and c. Set K_4 the complete graph induced by P_4 and K_4^1 built with the vertices in K_8 different from K_4. Set $K_{4,4}$ the bipartite graph from K_4 to K_4^1. If some edge in K_4^1 is colored from $\{a, b, c\}$ we are done. Else K_4^1 is colored from $\{d, e\}$. If K_4^1 contains a monochromatic C_4 then with any edge of $K_{4,4}$, we can define a barycentric $P_4 \cup K_2$. Else in K_4^1, there exists a P_4^1 with edge color sequence dde, or eed, or ded; hence with some edge in P_4 a barycentric $P_4 \cup K_2$ is obtained. Assume that the edges of K_8 are colored by a and b. Then, since $R(P_4 \cup K_2, 2) = 8$ (see Table 1) there exists a monochromatic $P_4 \cup K_2$. Hence we have the upper bound.

The lower bound is obtained from $8 = R(P_4 \cup K_2, 2) \leq BR(P_4 \cup K_2, \mathbb{Z}_5)$. ∎

Theorem 3.19. $BR(P_3 \cup 2K_2, \mathbb{Z}_5) = 9$.

Proof. Since $BR(3K_2, \mathbb{Z}_5) = 8$ there exists, for any $f : E(K_8) \to \mathbb{Z}_5$, a monochromatic or with three different colors $3K_2 \subseteq K_8$. Set $3K_2 = v_1v_2, v_3v_4, v_5v_6$. Let K_6 be the complete graph induced by $3K_2$ and K_3 the complete graph in K_9 built with vertices different from K_6, set $V(K_3) = \{v_7, v_8, v_9\}$. Assume $E(3K_2)$ is colored by a, b and c. Then for any color of $E(K_3)$ we obtain a barycentric $P_3 \cup 2K_2$. Assume $3K_2$ is a-monochromatic. Consider the bipartite graph $K_{3,6}$ from $V(K_3)$ to $V(K_6)$, and the nine edge-disjoint stars $v_1v_iv_2$, $v_3v_iv_4$, $v_5v_iv_6$ with $7 \leq i \leq 9$. If some edge in $K_{3,6}$ is colored by a, we are done. Hence these stars are colored from

$\{b, c, d, e\}$. If there exists a no monochromatic star we have a barycentric $P_3 \cup 2K_2$, else all are monochromatic; in this case we can also derive a barycentric $P_3 \cup 2K_2$. In consequence we have the upper bound.

The lower bound follows from $9 = R(P_3 \cup 2K_2, 2) \leq BR(P_3 \cup 2K_2, \mathbb{Z}_5)$. ∎

3.4. Barycentric Ramsey number for modified stars, circuit-matching and stars-matching

Theorem 3.20. $BR(MK_{1,4}, \mathbb{Z}_5) = 6$.

Proof. Since $BR(MK_{1,4}, \mathbb{Z}_4) = 6$, there exists in K_6, for any $f : E(K_6) \to \mathbb{Z}_4$ a barycentric $MK_{1,4}$, monochromatic or with edge colors $a, a, a+1, a+3$ or $a, a, a+2, a+2$. In the first two cases $MK_{1,4}$ is also barycentric with respect to $\mathbb{Z}_5$. Consider now the third case where $MK_{1,4}$ has as edge colors $a, a, a+2, a+2$. In this case, the only way to avoid a barycentric $MK_{1,4}$ with colors from $\mathbb{Z}_5$ is to have each $K_{1,5} \subseteq K_6$ colored by a or $a+2$. Then since $R(MK_{1,4}, 2) = 6$ we are done.

It is clear that $6 = R(MK_{1,4}, 2) \leq BR(MK_{1,4}, \mathbb{Z}_5)$. Therefore we have the lower bound. ∎

Theorem 3.21. $BR(MK_{1,4}, \mathbb{Z}_3) = 5$.

Proof. Let v_i with $1 \leq i \leq 5$ be the vertices of K_5. Since $R(K_{1,3}, \mathbb{Z}_3) = 6$ then there exists some coloring of $E(K_5)$ with a zero-sum free $K_{1,3}$. Let $K^1_{1,3}$ be the zero-sum free star centered in v_1 and edges v_1v_2, v_1v_3, and v_1v_4; the coloring of its edges must be a, a, b respectively. Let $K^2_{1,3}$ be the star centered in v_5 and edges v_5v_2, v_5v_3 and v_5v_4. Then for any coloring of the edges of $K^2_{1,3}$, we obtain a barycentric $MK_{1,4}$.

The lower bound is obvious. ∎

Corollary 3.3. $BR(C_3 \cup K_2, \mathbb{Z}_3) = 5$.

Proof. Since $BR(MK_{1,4}, \mathbb{Z}_3) = 5$ then for any $f : E(K_5) \to \mathbb{Z}_3$ there exists a barycentric $MK_{1,4}$ in K_5. Let v_1, v_2, v_3, v_4, v_5 be the vertices of K_5. We have the following barycentric $MK_{1,4}$ types: A : constituted by star $K^1_{1,3}$ centered in v_1 and edges v_1v_2, v_1v_3 and v_1v_4 colored by a, a, b respectively and the edge v_2v_5 colored by c. B : constituted by star $K^2_{1,3}$ centered in v_1 and edges v_1v_2, v_1v_3 and v_1v_4 colored by a, a, a respectively and the edge v_2v_5 colored by a. C : constituted by star $K^3_{1,3}$ centered in v_1 and edges v_1v_2, v_1v_3 and v_1v_4 colored by a, a, a respectively and the edge v_2v_5 colored by b. It is easy to see that from types A and B we can obtain a barycentric $C_3 \cup K_2$ with the different colors assigned to edge v_3v_4. In case of type C we can obtain a barycentric $C_3 \cup K_2$ with $f(v_3v_4) \in \{a, c\}$.

Set $f(v_3v_4) = b$ hence, if $f(v_1v_5) = c$ then circuit $C_3 = v_1v_5v_2v_1$ and edge v_3v_4 constitute a barycentric $C_3 \cup K_2$. If $f(v_1v_5) = b$ then the circuit $C_3 = v_1v_5v_2v_1$ and the edge v_3v_4 constitute a barycentric $C_3 \cup K_2$. Assuming now that $f(v_1v_5) = a$.

Case 1. If $f(v_5v_4) = a$ then it easy to see that from the different values of $f(v_2v_3)$ we obtain a barycentric $C_3 \cup K_2$.

Case 2. If $f(v_5v_4) = b$ when $f(v_2v_3) \in \{a, c\}$ we can obtain a barycentric $C_3 \cup K_2$. Set $f(v_2v_3) = b$. If $f(v_5v_3) \in \{b, c\}$ we obtain a barycentric $C_3 \cup K_2$. Set $f(v_5v_3) = a$

If $f(v_2v_4) \in \{b, c\}$ then circuit $C_3 = v_2v_3v_4v_2$ and the edge v_5v_1 constitute a barycentric $C_3 \cup K_2$. Set $f(v_2v_4) = a$ then the circuit $C_3 = v_3v_1v_5v_3$ and edge v_2v_4 constitute a barycentric $C_3 \cup K_2$.

Case 3. If $f(v_5v_4) = c$ then for each value of $f(v_3v_5) \in \mathbb{Z}_3$, circuit $C_3 = v_3v_5v_4v_3$ and edge v_2v_1 constitute a barycentric $C_3 \cup K_2$.

The lower bound is obvious. ∎

Theorem 3.22. *$BR(K_{1,3} \cup K_2, \mathbb{Z}_3) = 6$.*

Proof. Since $BR(K_{1,3}, \mathbb{Z}_3) = R(K_{1,3}, \mathbb{Z}_3) = 6$, there exists a zero-sum $K_{1,3} \subseteq K_6$, for any $f : E(K_6) \to \mathbb{Z}_3$. Then this star with some edge in K_6, vertex-disjoint with $K_{1,3}$, defines a barycentric $K_{1,3} \cup K_2$. In consequence we have the upper bound.
The lower bound follows trivially. ∎

Corollary 3.4. *$BR(K_{1,3} \cup K_2, \mathbb{Z}_5) = 7$.*

Proof. Since $BR(C_3 \cup K_2, \mathbb{Z}_5) = 7$ there exists in K_7, for any $f : E(K_7) \to \mathbb{Z}_5$, a barycentric $C_3 \cup K_2$ i.e. monochromatic or colored by three different colors. Set $C_3 : v_0v_1v_2v_0$, $K_2 : v_3v_4$ and $K_4 \subseteq K_7$ the complete graph built with vertices v_3, v_4, v_5, v_6. We have two cases:

Case 1. $E(C_3 \cup K_2)$ is colored as follows: v_0v_1 by a, v_1v_2 by b, v_0v_2 by c, and v_3v_4 by a. Then for any color of $f(v_2v_5)$ from $\{a, b, c\}$ we are done. Set $f(v_2v_5) = d$, if $f(v_4v_6) \in \{b, c, d\}$ we have the corollary. Else, when $f(v_4v_6) = a$ or $f(v_4v_6) = e$ we have for any value of $f(v_5v_4)$ from $\mathbb{Z}_5$ a barycentric $K_{1,3} \cup K_2$. Now, set $f(v_5v_2) = e$, if $f(v_4v_6) \in \{b, c, e\}$ we have the corollary. Else, when $f(v_4v_6) = a$ or $f(v_4v_6) = d$ we have for any value of $f(v_5v_4)$ from $\mathbb{Z}_5$ a barycentric $K_{1,3} \cup K_2$. Let us consider now the case: $f(v_0v_1) = a$, $f(v_1v_2) = a$, $f(v_0v_2) = b$, and $f(v_3v_4) = c$. If $f(v_2v_5) \in \{a, b, c\}$ we are done. Set $f(v_2v_5) = d$ then for any value of $f(v_4v_6) \in \{a, b, d\}$ we have the corollary. Set $f(v_4v_6) = c$ then for any value of $f(v_2v_6) \in \{a, b, c, d\}$ we are done. Set $f(v_2v_6) = e$ then for any value of $f(v_2v_3) \in \mathbb{Z}_5$. we have the result. Set $f(v_4v_6) = e$ then we must have $f(v_2v_6) = e$ else we are done. Therefore, for any value of $f(v_2v_3) \in \mathbb{Z}_5$. we have the corollary. The case, when $f(v_2v_5) = e$ follows in a similar way.

Case 2. $E(C_3 \cup K_2)$ is a-monochromatically colored as follows: v_0v_1, v_1v_2, v_0v_2 and v_3v_4 colored by a. Assuming v_4v_5 is colored by $x \in \{b, c, d, e\}$, say c. If v_3v_5 is colored by some $x \in \{b, d, e\}$ then, since v_0v_1 is colored by a, we have case 1. Coloring edge v_3v_5 by a then edges v_6v_0, v_6v_1, v_6v_2 must be also colored by a. Else we have case 1. Hence star $K_{1,3}$ centered in v_0 with end vertices in $\{v_6, v_1, v_2\}$, and edge v_3v_5 define an a-monochromatic $C_3 \cup K_2$. Assuming now that v_3v_5 is colored by c. Hence from any coloring given to star $K_{1,3}$, centered in v_6 with end vertices in $\{v_o, v_1, v_2\}$, we can derive a barycentric $K_{1,3} \cup K_2$.
Therefore v_4v_5 must be colored by a. Hence by coloring edge v_3v_5 from

$\{b, c, d, e\}$ we are done. Assuming v_3v_5 is colored by a, hence for any color of edge v_5v_6 we can derive the theorem.

The lower bound follows from $7 = R(K_{1,3} \cup K_2, 2) \leq BR(K_{1,3} \cup K_2, \mathbb{Z}_5)$. ∎

Corollary 3.5. $BR(K_3 + e, \mathbb{Z}_5) = 7$.

Proof. Since $BR(K_{1,3} \cup K_2, \mathbb{Z}_5) = 7$, there exists in K_7, for any $f : E(K_7) \to \mathbb{Z}_5$, a barycentric $K_{1,3} \cup K_2$, i.e. monochromatic or colored by a, a, b, c. We have the following cases:

Case 1. $K_{1,3} = v_1v_2, v_1v_3, v_1v_4$ and $K_2 = v_5v_6$ are a-monochromatic coloring: set $f(v_2v_3) \in \{b, c, d, e\}$, else we have the corollary. Without loss of generality, set $f(v_2v_3) = b$ then $f(v_3v_4) = b$, otherwise we are done. Hence $f(v_2v_4) = b$, else we have the corollary. Therefore $f(v_4v_5) = a$, else we have the result. Finally for each value of $f(v_4v_6)$ in $\mathbb{Z}_5$ we have a barycentric $K_3 + e$.

Case 2. $K_{1,3} = v_1v_2, v_1v_3, v_1v_4$ and $K_2 = v_5v_6$ are colored as: $f(v_1v_2) = f(v_1v_3) = a$, $f(v_1v_4) = b$, and $f(v_5v_6) = c$: if $f(v_2v_3) \in \{c, d, e\}$ or $f(v_3v_4) \in \{c, d, e\}$ we are done. If $f(v_3v_4) \in \{a, b\}$ then for any values of $f(v_4v_5)$ and $f(v_3, v_5)$ we have also the corollary. Set $f(v_1v_2) = f(v_5v_6) = a$, $f(v_1v_3) = b$ and $f(v_1v_4) = c$: then if $f(v_2v_3) \in \{a, b, c\}$ or $f(v_3v_4) \in \{a, b, c\}$ we are done. Otherwise, if $f(v_2v_3) = f(v_3v_4)$ we have the corollary, else with any color in $\mathbb{Z}_5$ given to $f(v_3v_6)$ we have a barycentric $K_3 + e$.

The lower bound is derived coloring the edges of two vertices-disjoint complete graphs $K_3 \subseteq K_6$ by a and the remaining edges of K_6 by b. ∎

Corollary 3.6. $BR(K_3 + e, \mathbb{Z}_3) = 4$.

Proof. Set $f : E(K_4) \to \mathbb{Z}_3$ and let $K_{1,3}$ be defined by v_1v_2, v_1v_3, v_1v_4. It is easy to see that we have the following alternative cases:

Case 1. $K_{1,3}$ is a-monochromatic. Set $f(v_1v_2) = f(v_1v_3) = f(v_1v_4) = a$. Then for any value of $f(v_3v_4)$ in $\mathbb{Z}_3$ we have the corollary.

Case 2. $K_{1,3}$ is colored with two different colors. Set $f(v_1v_2) = f(v_1v_3) = a$, hence $f(v_1v_4) \in \{b, c\}$. Set $f(v_1v_4) = b$, when $f(v_3v_4) \in \{a, c\}$ we have the corollary. Assuming $f(v_3v_4) = b$ then for any value of $f(v_4v_2)$ in $\mathbb{Z}_3$ we are done. Set now $f(v_1v_4) = c$, when $f(v_3v_4) \in \{a, b\}$ we are also done. Set $f(v_3v_4) = c$ then for any value of $f(v_4v_2)$ in $\mathbb{Z}_3$ we have a barycentric $K_3 + e$.

Case 3. $K_{1,3}$ is colored with three different colors. Set $f(v_1v_2) = a$, $f(v_1v_3) = b$ and $f(v_1v_4) = c$. Therefore for any value of $f(v_1v_4) \in \mathbb{Z}_3$ we have the result.

The lower bound is obvious. ∎

Corollary 3.7. $BR(K_{1,3}, \mathbb{Z}_4) = 6$.

Proof. The lower bound is derived coloring one of the two edge-disjoint hamiltonian cycles of K_5 by a and the other one by $a+1$. The upper bound follows directly from the fact that $BR(K_{1,3}, \mathbb{Z}_5) = 6$ (see Table 2). ∎

4. Conclusion

The combinatorial arguments used in this paper are really elementary. Having in mind a possible automation, we have detailed them. The computation of $BR(H, \mathbb{Z}_n)$, $n \geq 6$ for the graphs given in Table 4, are open problems. We expect that their degree of difficulty will increase, as in case of the computation of the classic Ramsey numbers and the zero-sum Ramsey numbers involving many colors.

Acknowledgement. Heartfelt thanks go to the referees for the patience to indicate precise weaknesses and the useful remarks that allowed us to improve some of the proofs.

References

[1] N. Alon and Y. Caro, On three zero-sum Ramsey-type problems, *J. Graph Theory* **17** (1993), no. 2, 177–192.
[2] A. Bialostocki and P. Dierker, On zero sum Ramsey numbers: small graphs, *Ars Combin.* **29** (1990), A, 193–198.
[3] A. Bialostocki and P. Dierker, On the Erdős-Ginzburg-Ziv theorem and the Ramsey numbers for stars and matchings, *Discrete Math.* **110** (1992), no. 1-3, 1–8.
[4] Y. Caro, A complete characterization of the zero-sum (mod 2) Ramsey numbers, *J. Combin. Theory Ser. A* **68** (1994), no. 1, 205–211.
[5] Y. Caro, Zero-sum subsequences in abelian non-cyclic groups, *Israel J. Math.* **92** (1995), no. 1-3, 221–233.
[6] Y. Caro, On zero-sum Ramsey numbers—stars, *Discrete Math.* **104** (1992), no. 1, 1–6.
[7] Y. Caro, Exact cuts and a simple proof of the zero graphs theorem, *Ars Combin.* **41** (1995), 123–128.
[8] Y. Caro, Zero-sum problems—a survey, *Discrete Math.* **152** (1996), no. 1-3, 93–113.
[9] A. Cammaroto, *Estrellas con cero sumas en grafos*, MSc. UCV. Caracas Venezuela 3 (1996).
[10] Y. Caro, On zero-sum delta-systems and multiple copies of hypergraphs, *J. Graph Theory* **15** (1991), no. 5, 511–521.
[11] F. R. K. Chung and R. L. Graham, Edge-colored complete graphs with precisely colored subgraphs, *Combinatorica* **3** (1983), no. 3-4, 315–324.
[12] L. Cordero, *Determinación del número Ramsey baricéntrico de algunos grafos*, MSc. UCV. Caracas Venezuela 3 (1999).
[13] C. Delorme et al., Existence conditions for barycentric sequences, *Discrete Math.* **281** (2004), no. 1-3, 163–172.
[14] C. Delorme et al., Barycentric sequences and barycentric Ramsey numbers stars, *Discrete Math.* **277** (2004), no. 1-3, 45–56.
[15] P. Erdös, A. Ginzburg, and A. Ziv, Theorem in the additive number theory, *Bull. Res. Council Israel 10F* (1961), 41–43.
[16] R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey theory*, Wiley, New York, 1980.
[17] W. Gao and A. Geroldinger, Zero-sum problems in finite abelian groups: a survey, *Expo. Math.* **24** (2006), no. 4, 337–369.
[18] Y. O. Hamidoune, On weighted sums in abelian groups, *Discrete Math.* **162** (1996), no. 1-3, 127–132.
[19] Z. W. Sun, Unification of zero-sum problems, subset sums and covers of $\mathbb{Z}$, *Electron. Res. Announc. Amer. Math. Soc.* **9** (2003), 51–60 (electronic).