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The Rupture Degree and Gear Graphs

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Abstract. In a communication network, several vulnerability measures are used to determine the resistance of the network to disruption of operation after the failure of certain stations or communication links. If we think of a graph as modelling a network, the *rupture degree* of a graph is one measure of *graph vulnerability* and it is defined by

$$r(G) = \max\{\omega(G-S) - |S| - m(G-S) : S \subset V(G),\ \omega(G-S) > 1\},$$

where $\omega(G-S)$ is the number of components of $G-S$ and $m(G-S)$ is the order of a largest component of $G-S$. In this paper we give some results on the rupture degree of gear graphs. Also the relationships between the rupture degree and some vulnerability parameters, namely the tenacity and toughness, are given.

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1. Introduction

When investigating the vulnerability of a communication network to disruption, one may want to learn the answer of the following questions (there may be others)[1]:

1. What is the number of elements that are not functioning?
2. What is the number of remaining connected subnetworks?
3. What is the size of a largest remaining group within which mutual communication can still occur?

Many graph theoretical parameters such as connectivity [3], toughness [4], scattering number [6], integrity [1], tenacity [5] and their edge-analogues, have been defined to obtain the answers of these questions. In other words, these parameters have been used to measure the vulnerability of a network. In addition, the *rupture degree* was introduced as a measure of graph vulnerability by Li *et al.* [9]. Formally, the *rupture degree* of an incomplete connected graph G is defined by

$$r(G) = \max\{\omega(G-S) - |S| - m(G-S) : S \subset V(G),\ \omega(G-S) > 1\},$$

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where $\omega(G-S)$ is the number of components of $G-S$ and $m(G-S)$ is the order of a largest components of $G-S$. The rupture degree of K_n is defined as $1-n$.

The connectivity deals with the question (1). The toughness and the scattering number take account of questions (1) and (2). The integrity deals with the questions (1) and (3). The rupture degree is a measure which deals with the questions (1), (2), and (3). Therefore the rupture degree gives us more knowledge about the network to disruption. On the other hand the tenacity is also a measure which deals with the questions (1), (2), and (3) [9]. But Li *et al.* [9] gave examples to show that the rupture degree can reflects the vulnerability of graphs better than the tenacity. Consequently, the rupture degree is a better parameter to measure the vulnerability of a network G. Li *et al.* [9] obtained several results on the rupture degree of a graph.

Let $G=(V,E)$ be a graph. By $\kappa(G)$ we denote the connectivity of G. The symbols $\alpha(G)$ and $\beta(G)$, respectively, denote the independence number and covering number of G. We shall use $\lceil x \rceil$ for the smallest integer not smaller than x.

In Section 2, we give some results on the rupture degree of gear graphs. In Section 3, we consider the relationships between the rupture degree and the tenacity and toughness, respectively.

2. The gear graphs and rupture degree

Geared systems are used in dynamic modelling. These are graph theoretic models that are obtained by using gear graphs. Similarly the cartesian product of gear graphs, the complement of a gear graph, and the line graph of a gear graph can be used to design a gear network. From [9], we know that the rupture degree is a better parameter to measure the vulnerability among the other parameters. Consequently these considerations motivated us to investigate the vulnerability of gear graphs by using rupture degree. Now we give the following definitions.

Definition 2.1. *The* wheel graph *with n spokes, W_n, is the graph that consists of an n-cycle and one additional vertex, say u, that is adjacent to all the vertices of the cycle. In Figure 1, we display W_6.*

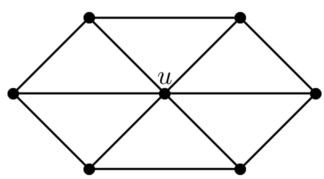

Figure 1. The wheel W_6

Definition 2.2. [2] *The* gear graph *is a wheel graph with a vertex added between each pair adjacent graph vertices of the outer cycle. The gear graph G_n has 2n+1 vertices and 3n edges. In Figure 2 we display G_6.*

In Figure 2, we call the vertex u *center vertex* of G_n. Now we give the rupture degree of a gear graph.

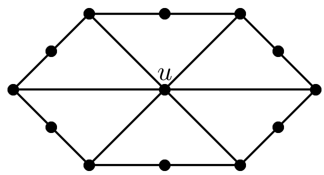

Figure 2. The gear graph G_6

Theorem 2.1. *Let G_n be a gear graph. Then $r(G_n) = 0$.*

Proof. We know that a gear graph G_n can be constructed from a wheel graph W_n by adding the new vertices to W_n. Let S be a subset of $V(G_n)$ such that $w(G_n - S) - |S| - m(G_n - S) = r(G_n)$. Then S must contain the vertices of n-cycle in W_n. It is obvious that S is a covering set of G_n and $|S| = n$. Since S is a covering set of G_n, we have $m(G_n - S) = 1$ and $w(G_n - S) = n + 1$. Consequently $r(G_n) = 0$. The proof is completed. ∎

Theorem 2.2. *Let G_n be a gear graph. Then $r(\bar{G_n}) = 2 - 2n$.*

Proof. We know that a gear graph G_n can be constructed from a wheel graph W_n by adding the new vertices to W_n. Let S' be a set of vertices of n-cycle in W_n, and let S'' be a set of vertices which are added to n-cycle in G_n. Let u be a center vertex. Since S' is an independent set of G_n, these vertices form a complete graph with order n in $\bar{G_n}$. Similarly, since $S'' \bigcup \{u\}$ is an independent set of G_n, these vertices form a complete graph with order $n + 1$ in $\bar{G_n}$. Moreover the graph $\bar{G_n}$ contains some edges joining K_{n+1} to K_n. It is obvious that the vertex u in $\bar{G_n}$ is not adjacent to any vertex in K_n. So we have two cases:

Case 1. If we remove the vertices of S' in $\bar{G_n}$, then we have only one components which is graph K_{n+1}. Then $m(\bar{G_n} - S') = |V(K_{n+1})| = n + 1$ and so

$$w(\bar{G_n} - S') - |S'| - m(\bar{G_n} - S') = -2n. \tag{2.1}$$

Case 2. If we remove the vertices of S'' in $\bar{G_n}$, then we have two components which are graphs K_n and K_1. Then $m(\bar{G_n} - S'') = |V(K_n)| = n$ and so

$$w(\bar{G_n} - S'') - |S''| - m(\bar{G_n} - S'') = 2 - 2n. \tag{2.2}$$

By using (2.1) and (2.2) we have

$$r(\bar{G_n}) = \max\{-2n, 2 - 2n\} = 2 - 2n.$$

The proof is completed. ∎

Now we consider the Cartesian product of two graphs.

Definition 2.3. *The (Cartesian)* product *$G_1 \times G_2$ of graphs G_1 and G_2 also has $V(G_1) \times V(G_2)$ as its vertex set, but here (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1.*

Theorem 2.3. *Let G_n be a gear graph. Then $r(K_2 \times G_n) = -1$.*

Proof. If we remove r vertices from $K_2 \times G_n$, then we have at most r components. Since $1 \leq m((K_2 \times G_n) - S) \leq n - 1$ for every $S \subset V(G_n)$, we have

$$1 - n \leq -m((K_2 \times G_n) - S) \leq -1.$$

Hence

$$w((K_2 \times G_n) - S) - |S| - m((K_2 \times G_n) - S) \leq r - r - 1$$

and so

$$r(K_2 \times G_n) \leq -1. \tag{2.3}$$

On the other hand we can take the covering set of $K_2 \times G_n$ instead of a vertex cut of $K_2 \times G_n$. Then $|S| = \beta(G) = 2n+1$ and $w((K_2 \times G_n) - S) = \alpha(K_2 \times G_n) = 2n+1$. So $m((K_2 \times G_n) - S) = 1$. From the definition of rupture degree we have

$$r(K_2 \times G_n) \geq w((K_2 \times G_n) - S) - |S| - m((K_2 \times G_n) - S)$$

and so

$$r(K_2 \times G_n) \geq -1. \tag{2.4}$$

By using (2.3) and (2.4) we have

$$r(K_2 \times G_n) = -1$$

This completes the proof. ∎

Definition 2.4. *The line graph $L(G)$ of a graph G is a graph such that each vertex of $L(G)$ represents an edge of G, and any two vertices of $L(G)$ are adjacent if and only if their edges are incident, meaning they share a common end vertex, in G.*

Theorem 2.4. *Let G_n be a gear graph with order n. Then*

$$r(L(G_n)) \leq n + 2 - \lceil 2\sqrt{6n} \rceil.$$

Proof. If we remove r vertices from $L(G_n)$, then we have at most $r/2$ components and so

$$m(L(G_n - S)) \leq \frac{3n - r}{r/2}.$$

Since $w(L(G_n - S)) \leq \alpha(L(G_n)) = n$ for every $S \subset V(L(G_n))$, we have

$$r(L(G_n)) \leq \max\left\{ n - r - \frac{3n - r}{r/2} \right\}.$$

The function $n - r - (3n - r)r/2$ takes its maximum value at $r = \sqrt{6n}$ and

$$r(L(G_n)) \leq n + 2 - 2\sqrt{6n}.$$

Since the rupture degree is integer valued, we round this up to get a lower bound and

$$r(L(G_n)) \leq n + 2 - \lceil 2\sqrt{6n} \rceil.$$

The proof is completed. ∎

3. Relationships between rupture degree and vulnerability parameters

In this section, the relationships between the rupture degree and some vulnerability parameters, namely the tenacity and toughness, are established.

We know that the rupture degree and the tenacity deal with the questions (1), (2), and (3) in Section 1. Moreover the rupture degree is additive dual of tenacity. Therefore the relationships between the rupture degree and tenacity are very exciting. Now we consider the tenacity of a graph G.

The concept of tenacity was introduced by Cozzens *et al.*[5]. The tenacity of an incomplete connected graph G, denoted by $T(G)$, is defined as

$$T(G) = \min\left\{\frac{|S| + m(G - S)}{w(G - S)} : S \subset V(G),\ \omega(G - S) > 1\right\}.$$

Theorem 3.1. [10] *Let G be an incomplete connected graph with the tenacity $T(G)$. Then $r(G) \leq \alpha(G)(1 - T(G))$.*

Theorem 3.2. *Let G be a graph with order n. Then*

$$r(G) \leq n\left(\frac{1}{T(G)} - 1\right).$$

Proof. Let S be a vertex cut of G. Then from the definition of $T(G)$ we know that

$$w(G - S) \leq \frac{|S| + m(G - S)}{T(G)}.$$

Hence

$$w(G - S) - |S| - m(G - S) \leq \frac{|S| + m(G - S)}{T(G)} - |S| - m(G - S)$$

and so

$$\max\{w(G - S) - |S| - m(G - S)\} \leq \max\left\{(|S| + m(G - S))\left(\frac{1}{T(G)} - 1\right)\right\}.$$

Since the maximum value of $|S| + m(G - S)$ is the number of vertices of a graph G, we have

$$r(G) \leq n\left(\frac{1}{T(G)} - 1\right).$$

The proof is completed. ∎

Remark 3.1. If $\beta(G) \geq \alpha(G)(T - 1)$ for any graph G, then the result in Theorem 3.2 is better than the result in Theorem 3.1.

Theorem 3.3. *There is no graph G of order n such that $r(G) = T(G)$.*

Proof. Suppose that there is an incomplete connected graph of order n such that $r(G)$=$T(G)$. Since $T(G) > 0$ and $r(G)$ is an integer, we have $T(G) \geq 1$. Then from the definition of $T(G)$ we know that $|S| + m(G - S) \geq w(G - S)$. Therefore

$$w(G - S) - |S| - m(G - S) \leq w(G - S) - w(G - S)$$

and so $r(G) \leq 0$. Hence $T(G) \leq 0$, which is a contradiction. This completes the proof. ∎

Now we consider another vulnerability parameter. The concept of toughness was introduced by Chvátal [4]. The toughness of a graph, denoted by $t(G)$, is defined

$$t(G) = \min\left\{\frac{|S|}{w(G-S)} : S \subset V(G),\ w(G-S) > 1\right\}.$$

The following theorem gives a relation between the rupture degree and toughness.

Theorem 3.4. *Let G be a graph with order n. Then*

$$r(G) \leq \frac{\beta(G)}{t(G)} - \kappa(G) - 1.$$

Proof. Let S be a vertex cut of G. Then from the definition of $t(G)$ we know that

$$w(G-S) \leq \frac{|S|}{t(G)}.$$

Hence

$$w(G-S) - |S| - m(G-S) \leq \frac{|S|}{t(G)} - |S| - m(G-S)$$

and so

$$r(G) \leq \max\left\{\frac{|S|}{t(G)} - |S| - m(G-S)\right\}.$$

Since $\kappa(G) \leq |S| \leq \beta(G)$ for every $S \subset V(G)$, we have

$$r(G) \leq \frac{\beta(G)}{t(G)} - \kappa(G) - 1$$

The proof is completed. ∎

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