New Types of Continuous Linear Operator in Probabilistic Normed Space

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Abstract. In this paper, new types of continuous linear operator, such as continuous, strongly continuous, weakly continuous and sequentially continuous linear operators, in probabilistic normed space are introduced. Also, the relation between the boundedness and continuity of these linear operators in probabilistic normed spaces is studied.

1. Introduction and preliminaries

In 1942, Menger introduced the notion of probabilistic metric space as a natural generalization of the notion of a metric space. In complete analogy with the classical case, we then have the notion of a probabilistic normed space. This was introduced by Serstnev in 1963 and later improved by Alsina, Schweizer, and Sklar in 1993.

Before we proceed we must state some definitions, known facts, and, technical results to be used in the sequel; the concepts used are those of [3] and [9]. The space of probability distribution functions (d.f.) which we will consider are

$$\Delta^+ = \{F : [-\infty, \infty] \to [0,1] \mid F \text{ is left-continuous, non-decreasing,}$$

$$F(0) = 0 \text{ and } F(+\infty) = 1\}.$$

In particular for any $a \geq 0$, ε_a is the d.f. defined by

$$\varepsilon_a(x) = \left\{ \begin{array}{l} 0, \text{ if } x \leq a, \\ 1, \text{ if } x > a. \end{array} \right.$$

The space Δ^+ is partially ordered by the usual pointwise ordering of functions, the maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0(x) = \left\{ \begin{array}{l} 0, \text{ if } x \leq 0, \\ 1, \text{ if } x > 0. \end{array} \right.$$

A *triangle function* is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, non-decreasing and which has ε_0 as unit, viz. for all $F, G, H \in \Delta^+$, we have

$$\tau(\tau(F, G), H) = \tau(F, \tau(G, H)),$$
$$\tau(F, G) = \tau(G, F),$$
$$\tau(F, H) \leq \tau(G, H) \quad \text{if} \quad F \leq G,$$
$$\tau(F, \varepsilon_0) = F.$$

Continuity of a triangle function means continuity with respect to the topology of weak convergence in Δ^+. Typical continuous triangle functions are convolution and the operations τ_T and τ_{T^*}, which are given by

$$\tau_T(F, G)(x) = \sup_{s+t=x} T(F(s), G(t)),$$

and

$$\tau_{T^*}(F, G)(x) = \inf_{s+t=x} T^*(F(s), G(t)),$$

for all F, G in Δ^+ and all x in $\Re$ [9; Secs.7.2 and 7.3]. Here T is a continuous *t-norm*, i.e. a continuous binary operation on $[0, 1]$ that is associative, commutative, non-decreasing and has 1 as identity; T^* is a continuous *t-conorm*, namely a continuous binary operation on $[0, 1]$ that is related to continuous t-norm T through

$$T^*(x, y) = 1 - T(1 - x, 1 - y).$$

The notion of a probabilistic normed space was first introduced by Serstnev [9] in 1963. In 1993, Alsina, Schweizer and Sklar gave a new definition of a probabilistic normed space [2].

Definition 1.1. *A probabilistic normed space is a quadruple* (V, v, τ, τ^*), *where* V *is a real vector space,* τ *and* τ^* *are continuous triangle functions, and* v *is a mapping from* V *into* Δ^+ *such that, for all* p, q *in* V, *the following conditions hold:*

(PN1) $v_p = \varepsilon_0$ *if and only if* $p = \theta$, θ *being the null vector in* V;

(PN2) $\nu_{-p} = \nu_p$;

(PN3) $v_{p+q} \geq \tau(\nu_p, \nu_q)$;

(PN4) $\nu_P \leq \tau^*(\nu_{\alpha p}, v_{(1-\alpha)p})$, *for all* α *in* [0,1].

If, instead of (PN1), we only have $v_\theta = \varepsilon_0$, *then we shall speak of a Probabilistic Pseudo Normed space (PPN space). If the inequality (PN4) is replaced by the equality* $\nu_P = \tau_M(\nu_{\alpha p}, v_{(1-\alpha)p})$, *then the PN space is called a Serstnev space. The pair* (V, v) *is said to be a Probabilistic Seminormed space (PSN space) if* $v : V \to \Delta^+$ *satisfies (PN1) and (PN2).*

There is a (ε, λ)-*topology in the PN space* (V, v, τ, τ^*) *which is generated by the family of neighbourhoods,* N_p *of* $p \in V$ *in the following way:*

$$N_p(\varepsilon, \lambda) = \{N_p(\varepsilon, \lambda)\}_{\varepsilon > 0, \lambda \in (0,1)}, \quad N_p(\varepsilon, \lambda) = \{q \in V : v_{q-p}(\varepsilon) > 1 - \lambda\}.$$

2. Continuous linear operator in probabilistic normed spaces

The definition of a bounded linear operator in PN space previously studied by Lafuerza Guillén, Rodríguez Lallena and Sempi [6], Jebril and Ali [3], and Jebril and Noorani [4].

Definition 2.1. *Let (V, v, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be PN spaces. A linear map $T : V \to V'$ is said to be bounded if it satisfies one of the following conditions:*

(a) Certainly bounded*: If every certainly bounded set A of the space (V, v, τ, τ^*) has, as image under T a certainly bounded set TA of the space $(V', \mu, \sigma, \sigma^*)$, i.e., if there exists $x_0 \in (0, +\infty)$ such that $v_p(x_0) = 1$ for all $p \in A$, then there exists $x_1 \in (0, +\infty)$ such that $\mu_{Tp}(x_1) = 1$ for all $p \in A$.*

(b) D-bounded*: If it maps every D-bounded set of V into a D-bounded set of V', i.e., if, and only if, it satisfies the implication,*

$$\lim_{x \to +\infty} \inf_{p \in A} v_p(x) = 1 \Rightarrow \lim_{x \to +\infty} \inf_{p \in A} v_{Tp}(x) = 1,$$

for every nonempty subset A of V.

(c) Strongly B-bounded*: if there exists a constant $k > 0$ such that, for every $p \in V$ and for every $x > 0$, $\mu_{Tp}(x) \geq v_p(x/k)$, or equivalently if there exists a constant $h > 0$ such that, for every $p \in V$ and for every $x > 0$,*

$$\mu_{Tp}(hx) \geq v_p(x).$$

(d) Strongly $\boldsymbol{\psi}$-bounded*: if there exists a function $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi(x) < x$, for every $x > 0$ so that the following implication holds for every $p \in V$ and for every $x > 0$,*

$$v_p(x) > 1 - x \Rightarrow \mu_{Tp}(\psi(x)) > 1 - \psi(x).$$

Definition 2.2. *Let $(V, v, \tau, \tau*)$ and $(V', \mu, \sigma, \sigma*)$ be PN spaces. A linear map $T : V \to V'$ is said to be continuous at $p_0 \in V$, if for given $\varepsilon > 0$ and $\alpha \in (0, 1)$, there exist $\beta = \beta(\alpha, \varepsilon) \in (0, 1)$ and $\delta = \delta(\alpha, \beta) > 0$ such that for every p in V,*

$$v_{p-p_0}(\delta) > 1 - \beta \Rightarrow \mu_{Tp-Tp_0}(\varepsilon) > 1 - \alpha.$$

Theorem 2.1. *Let $(V, v, \tau, \tau*)$ and $(V', \mu, \sigma, \sigma*)$ be PN spaces. A linear map $T : V \to V'$ is continuous at a point then it is continuous on V.*

Proof. Since T is continuous at $p_0 \in V$, thus for each given $\varepsilon > 0$ and $\alpha \in (0, 1)$, there exist $\beta = \beta(\alpha, \varepsilon) \in (0, 1)$ and $\delta = \delta(\alpha, \beta) > 0$ such that for every p in V,

$$v_{p-p_0}(\delta) > 1 - \beta \Rightarrow \mu_{Tp-Tp_0}(\varepsilon) > 1 - \alpha.$$

Taking any q in V and replacing p by $p + p_0 - q$, we get

$$v_{p+p_0-q-p_0}(\delta) > 1 - \beta \Rightarrow \mu_{Tp+p_0-q-Tp_0}(\varepsilon) > 1 - \alpha.$$

So that,

$$v_{p-q}(\delta) > 1 - \beta \Rightarrow \mu_{Tp-Tq}(\varepsilon) > 1 - \alpha.$$

Since q is arbitrary, it follows that T is continuous on V. ∎

Definition 2.3. *Let $(V, v, \tau, \tau*)$ and $(V', \mu, \sigma, \sigma*)$ be PN spaces. A linear map $T : V \to V'$ is said to be strongly continuous at $p_0 \in V$, if for each $\varepsilon > 0$, there exist $\delta > 0$ such that for every p in V,*

$$\mu_{Tp-Tp_0}(\varepsilon) \geq v_{p-p_0}(\delta).$$

Theorem 2.2. *Let $(V, v, \tau, \tau*)$ and $(V', \mu, \sigma, \sigma*)$ be PN spaces. A linear map $T : V \to V'$ is strongly continuous at a point then it is strongly continuous on V.*

Proof. Since T is strongly continuous at $p_0 \in V$, for each $\varepsilon > 0$, there exist $\delta > 0$ such that for every p in V, we have

$$\mu_{Tp-Tp_0}(\varepsilon) \geq v_{p-p_0}(\delta).$$

Taking any q in V and replacing p by $p + p_0 - q$, we get

$$\mu_{Tp-Tq}(\varepsilon) = \mu_{Tp+p_0-q-Tp_0}(\varepsilon) \geq v_{p+p_0-q-p_0}(\delta) = v_{p-q_0}(\delta).$$

Since q is arbitrary, it follows that T is strongly continuous at on V. $\blacksquare$

Definition 2.4. *Let $(V, v, \tau, \tau*)$ and $(V', \mu, \sigma, \sigma*)$ be PN spaces. A linear map $T : V \to V'$ is said to be weakly continuous at $p_0 \in V$, if for given $\varepsilon > 0$ and $\alpha \in (0, 1)$, there exist $\delta = \delta(\alpha, \varepsilon) > 0$ such that for every p in V,*

$$v_{p-p_0}(\delta) \geq 1 - \alpha \Rightarrow \mu_{Tp-Tp_0}(\varepsilon) \geq 1 - \alpha.$$

Theorem 2.3. *Let $(V, v, \tau, \tau*)$ and $(V', \mu, \sigma, \sigma*)$ be PN spaces. A linear map $T : V \to V'$ is weakly continuous at a point then it is weakly continuous on V.*

Proof. Since T is weakly continuous at $p_0 \in V$, for each given $\varepsilon > 0$ and $\alpha \in (0, 1)$, there exist $\delta = \delta(\alpha, \beta) > 0$ such that for every p in V,

$$v_{p-p_0}(\delta) \geq 1 - \alpha \Rightarrow \mu_{Tp-Tp_0}(\varepsilon) \geq 1 - \alpha.$$

Taking any q in V and replacing p by $p + p_0 - q$, we get

$$v_{p-q}(\delta) = v_{p+p_0-q-p_0}(\delta) > 1 - \alpha \Rightarrow \mu_{Tp+p_0-q-Tp_0}(\varepsilon) = \mu_{Tp-Tq}(\varepsilon) > 1 - \alpha.$$

Since q is arbitrary, it follows that T is weakly continuous on V. $\blacksquare$

Remark 2.1. It is easy to see that if a mapping is strongly continuous then it is weakly continuous.

Definition 2.5. *Let $(V, v, \tau, \tau*)$ and $(V', \mu, \sigma, \sigma*)$ be PN spaces and T is a linear operator from V into V'. T is said to be sequentially continuous at $p_0 \in V$, if for every sequence $(p_n)_{n \in \mathbb{N}}$ of elements of V that converges to p_0 the sequence $(T_{pn})_{n \in \mathbb{N}}$ converges to T_{p_0}, i.e. if for all $\varepsilon > 0$, $\lambda \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,*

$$v_{pn-p_0}(\varepsilon) \geq 1 - \lambda \Rightarrow \mu_{Tp_n-Tp_0}(\varepsilon) \geq 1 - \lambda.$$

Theorem 2.4. *Let $(V, v, \tau, \tau*)$ and $(V', \mu, \sigma, \sigma*)$ be PN spaces. A linear map $T : V \to V'$ is sequentially continuous at a point then it is sequentially continuous on V.*

Proof. Let q be in V, $(q_n)_{n \in \mathbb{N}}$ be a sequence in V converging to q and set $n \in \mathbb{N}$, $p_n := q_n - q + p_0$, so that $(p_n)_{n \in \mathbb{N}}$ converges to p_0; thus, for all $\varepsilon_0 > 0$, there is $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,

$$v_{p_n-p_0}(\varepsilon) = v_{q_n-p_0}(\varepsilon) \geq 1 - \lambda.$$

Since T is sequentially continuous at $p_0 \in V$, the previous inequality yields

$$\mu_{Tq_n - Tp}(\varepsilon) = \mu_{Tp_n - Tp_0}(\varepsilon) \geq 1 - \lambda,$$

which proves the assertion. ∎

Theorem 2.5. *Let $(V, v, \tau, \tau*)$ and $(V', \mu, \sigma, \sigma*)$ be PN spaces. A linear map $T : V \to V'$ is strongly continuous then it is sequentially continuous.*

Proof. Let T be strongly continuous at $p_0 \in V$, if for each $\varepsilon > 0$, there is $\delta > 0$ such that for every p in V,

$$(2.1) \qquad \mu_{Tp - Tp_0}(\varepsilon) \geq v_{p - p_0}(\delta).$$

Let $(p_n)_{n \in \mathbb{N}}$ be a sequence in V such that for all $x > 0$ and $\lambda \in (0, 1)$ we have

$$(2.2) \qquad v_{pn} - p_0(x) \geq 1 - \lambda.$$

Now from (2.1) and (2.2) we have, $\mu_{Tp_n - Tp_0}(\varepsilon) \geq v_{p_n - p_0}(\delta) \geq 1 - \lambda$, for $n \in \mathbb{N}$ so that $(T_{p_n})_{n \in \mathbb{N}}$ converge to (T_{p_0}). ∎

The converse is not true, as can be seen from the following example.

Example 2.1. Let $V = V' = R$ and $v_0 = \mu_0 = \varepsilon_0$, while, if $p \neq 0$, then, for $\varepsilon > 0$, let $v_p(\varepsilon) = G\left(\dfrac{\varepsilon}{|p|}\right)$ and $\mu_p(\varepsilon) = U\left(\dfrac{\varepsilon}{|p|}\right)$, where

$$G(\varepsilon) = \begin{cases} \frac{1}{2}, & 0 < \varepsilon \leq 1, \\ 1, & 1 < \varepsilon \leq \infty, \end{cases}$$

and U is d.f. of the uniform low on $(0, 1)$,

$$U(\varepsilon) = \begin{cases} \varepsilon, & 0 < \varepsilon \leq 1, \\ 1, & 1 < \varepsilon \leq \infty. \end{cases}$$

Consider now the identity map $I : (R, |.|, G, v) \to (R, |.|, U, \mu)$.

(1) I is not strongly continuous, because, such that for every δ and for every $p - p_0 \neq 0$, one has, for $\varepsilon < |p - p_0| \min\{\frac{1}{2}, M\}, \forall M > 0$, then

$$\mu_{Ip - Ip_0}(\varepsilon) = \mu_{p - p_0}(\varepsilon) = U\left(\frac{\varepsilon}{|p - p_0|}\right) = \frac{\varepsilon}{|p - p_0|}$$

$$< \frac{1}{2} = G\left(\frac{\varepsilon}{M|p - p_0|}\right) = v_{p - p_0}(\frac{\varepsilon}{M}) = v_{p - p_0}(\delta),$$

where $\delta = \dfrac{\varepsilon}{M}$.

(2) I is a sequentially continuous at $p_0 \in V$, if for any sequence $\{p_n\}_{n=1}^{\infty}, p_n \in V$ converge to $\{p_0\}$ that implies $v_{p_n - p}(\varepsilon) > 1 - \lambda$ satisfied for every $\varepsilon > 0$ and $\lambda \in (0, 1)$,

$$\mu_{Ip_n - Ip_0}(\varepsilon) = \mu_{p - p_0}(\varepsilon) = \varepsilon_0(x) > 1 - x,$$

so that I is a sequentially continuous.

Theorem 2.6. *Let $(V, v, \tau, \tau*)$ and $(V', \mu, \sigma, \sigma*)$ be PN spaces. A linear map $T : V \to V'$ is continuous if and only if it is sequentially continuous.*

Proof. Let T be continuous at $p_0 \in V$. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence in V such that it is converge to p_0. Let $\varepsilon > 0$ be given, choose $\alpha \in (0,1)$. Since T is continuous at $p_0 \in V$, there exist $\beta = \beta(\alpha, \varepsilon) \in (0,1)$ and $\delta = \delta(\alpha, \beta) > 0$ such that for every p in V,

$$v_{p-p_0}(\delta) > 1 - \beta \Rightarrow \mu_{Tp-Tp_0}(\varepsilon) > 1 - \alpha.$$

Since $\{p_n\}_{n=1}^{\infty}$ converge to $\{p_0\}$ in V, there exist a positive integer n_0 such that, $\forall n \geq n_0$

$$v_{p_n-p_0}(\delta) > 1 - \beta \Rightarrow \mu_{Tp_n-Tp_0}(\varepsilon) > 1 - \alpha.$$

This implies $\{T_{p_n}\}_{n=1}^{\infty}$ is converge to $\{T_{p_0}\}$ in $(V, v, \tau, \tau*)$.

Next we suppose that T is sequentially continuous at $p_0 \in V$. If possible suppose that T is not continuous at p_0. Thus there exist $\varepsilon > 0$ and $\alpha > 0$ such that for all $\delta > 0$ and $\beta > 0$, there exist $q = q(\delta, \beta)$ such that $v_{p_0-q}(\delta) > 1 - \beta$ but

(2.3) $$\mu_{Tp_0-Tq}(\varepsilon) \leq 1 - \alpha,$$

let $\delta = \beta = \frac{1}{n+1}$, $n = 1, 2, \ldots\ldots$, there exist $\{q_n\}$ such that

$$v_{p_0-q_n}\left(\frac{1}{n+1}\right) > 1 - \frac{1}{n+1},$$

but $\mu_{Tp_0-Tq_n}(\varepsilon) \leq 1 - \alpha$, taking $\delta > 0$, there exist n_0 such that $\delta = \frac{1}{n+1}, \forall n \geq n_0$. Then,

$$v_{p_0-q_n}(\delta) = v_{p_0-q_n}\left(\frac{1}{n+1}\right) > 1 - \frac{1}{n+1}, \quad \forall n \geq n_0.$$

That means $\{q_n\}_{n=1}^{\infty}$ is converge to p_0, where $n \to \infty$. But from (2.3) we have,

$$\mu_{Tp_0-Tq_n}(\varepsilon) \leq 1 - \alpha, \quad \text{where} \quad n \to \infty.$$

Thus $(T_{q_n})_{n \in \mathbb{N}}$ does not converge to our assumption. Hence T is continuous at p_0. ∎

Theorem 2.7. *Let $(V, v, \tau, \tau*)$ and $(V', \mu, \sigma, \sigma*)$ be PN spaces. A linear map $T : V \to V'$ is strongly continuous if and only if is strongly B-bounded.*

Proof. First we suppose that T is strongly B-bounded. Thus there is exist $M > 0$ such that

$$\mu_{Tp}(\varepsilon) \geq v_p\left(\frac{\varepsilon}{M}\right), \ \forall p \in V, \ \forall \varepsilon > 0.$$

i.e.

$$\mu_{Tp-T_\theta}(\varepsilon) \geq v_{p-\theta}\left(\frac{\varepsilon}{M}\right), \ \forall p \in V, \ \forall \varepsilon > 0,$$

so that,

$$\mu_{Tp-T_\theta}(\varepsilon) \geq v_{p-\theta}(\delta) \text{ where } \delta = \frac{\varepsilon}{M}.$$

This implies that T is strongly continuous at θ and hence it is strongly continuous on V.

Conversely, suppose that T is strongly continuous on V. Using the strongly continuity of T at $p = \theta$, for $\varepsilon = \delta M, \forall M > 0$, there exist $\delta > 0$ such that

$$\mu_{Tp-Tp_\theta}(\varepsilon) \geq v_{p-\theta}(\delta) = v_{p-\theta}\left(\frac{\varepsilon}{M}\right)$$

so that T is strongly bounded. ∎

Theorem 2.14

Strongly B-bounded $\Longleftrightarrow$ Strongly B-continuous

Continuous   Theorem 2.11

Theorem 2.18   Theorem 2.13

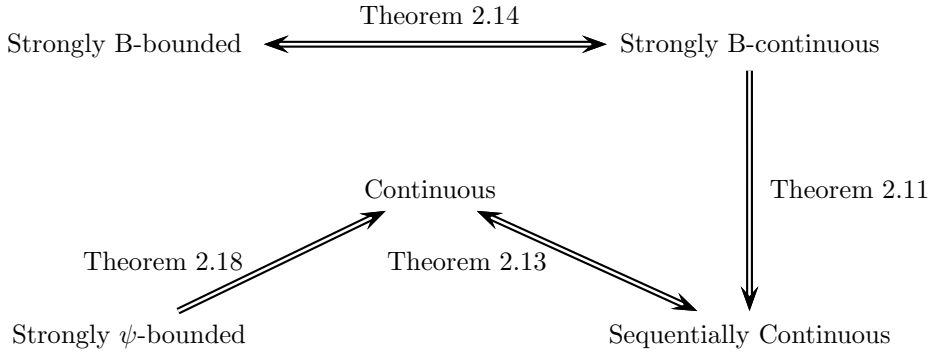Strongly ψ-bounded   Sequentially Continuous

Figure 1. Relations between continuous and bounded types

From Theorem 2.5, Theorem 2.6, and Theorem 2.7 the next result is immediate (see Figure 1).

Remark 2.2. Every strongly B-bounded linear operator T is continuous. If T is strongly continuous then it is continuous. If T is strongly B-bounded it is sequentially continuous on V.

Theorem 2.8. *Let* $(V, v, \tau, \tau*)$ *and* $(V', \mu, \sigma, \sigma*)$ *be PN spaces. A linear map* $T : V \to V'$ *is strongly ψ-bounded linear operator then T is continuous.*

Proof. Let T is ψ-bounded, if there exists a $\psi : [0, \infty) \to [0, \infty)$ such that $\psi(\delta) < \delta$, $\forall \delta > 0$ so that the following implication holds for every $p \in V$ for every $\delta > 0$:

$$v_p(\delta) > 1 - \delta \Rightarrow \mu_{Tp}(\psi(\delta)) > 1 - \psi(\delta),$$

i.e.

$$v_{p-\theta'}(\delta) > 1 - \delta \Rightarrow \mu_{Tp-T_{\theta'}}(\psi(\delta)) > 1 - \psi(\delta).$$

So that, let $\varepsilon > 0$, be an arbitrary neighborhood of θ' and $\lambda > 0$ are given, we choose $\delta > 0$ such that $0 < \psi(\delta) < \min\{\varepsilon, \lambda\}$, then

$$v_{p-\theta'}(\delta) > 1 - \delta \Rightarrow \mu_{Tp-T_{\theta'}}(\psi(\delta)) > 1 - \psi(\delta)$$
$$\Rightarrow \mu_{Tp-T_{\theta'}}(\varepsilon) > 1 - \psi(\delta) > 1 - \lambda$$

This implies that T is ψ-bounded at θ' and it is continuous. ∎

The following example shows that the converse need not be true.

Example 2.2. Let $V = V' = \Re$ and $v_0 = \mu_0 = \varepsilon_0$, while, if $p \neq 0$, then, for $x > 0$, let $v_p(x) = G\left(\frac{x}{|p|}\right)$, $\mu_p(x) = U\left(\frac{x}{|p|}\right)$, where

$$G(x) = \begin{cases} \frac{9}{10}, & 0 < x \leq 1, \\ 1, & 1 < x \leq \infty, \end{cases} \quad U(x) = \begin{cases} \frac{1}{10}, & 0 < x \leq 1, \\ 1, & 1 < x \leq \infty. \end{cases}$$

Consider now the identity map $I : (R, |.|, G, v) \to (R, |.|, U, \mu)$.

(1) I is a continuous, such that for every $\varepsilon > 0$ and every $\alpha \in (0,1)$, let $\delta > \max\{\varepsilon, |p - p_0|\}$ and $\beta \in (0,1)$ such that the following condition will satisfied, $\forall p, p_0 \in \Re$, $v_p(\delta) > 1 - \beta$. Since

$$\delta > \max\{\varepsilon, |p - p_0|\} \Rightarrow \frac{\delta}{|p - p_0|} > \frac{\max\{\varepsilon, |p - p_0|\}}{|p - p_0|} \geq \frac{\varepsilon}{|p - p_0|}.$$

Therefore

$$\mu_{Ip - Ip_0}(\varepsilon) = \mu_{p - p_0}(\varepsilon) = U\left(\frac{\varepsilon}{|p - p_0|}\right) < U\left(\frac{\delta}{|p - p_0|}\right) = 1 > 1 - \alpha.$$

(2) I is not strongly ψ-bounded, such that for every mapping $\psi(x) < x \ \forall x > 0$. Let $p \in (x, \frac{9}{10})$, $x \in (\frac{1}{10}, \frac{8}{10})$, the condition $v_p(x) > 1 - x$ is satisfied, but we note that

$$\mu_{Ip}(\psi(x)) = U\left(\frac{\psi(x)}{|p|}\right) \leq U\left(\frac{x}{|p|}\right) = \frac{1}{10} < 1 - x < 1 - \psi(x).$$

Hence I is continuous, but not strongly ψ-bounded.

From Example 2.1, Theorem 2.6, Theorem 2.7 and Theorem 2.8, the next result is immediate (see Figure 1).

Remark 2.3. If T is strongly ψ-bounded then it is sequentially continuous. If T is strongly ψ-bounded then it is not strongly continuous on V. If T is strongly ψ-bounded then it is not strongly B-bounded on V.

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