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An Optimal Dimension of Submerged Parallel Bars as a Wave Reflector

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Abstract. This paper studies submerged parallel bars as a wave reflector using the linear Shallow Water Equation (SWE). First, we recall the analytical study from literature about finding an optimal width of a one-bar wave reflector with a specific height. Numerical study is applied using the Lax-Wendroff discretization method with different partition along x-axis in order to avoid numerical diffusion error. Comparison between analytical and numerical solutions shows a good agreement, qualitatively and quantitatively. Using an analogous physical argument as in the case of one-bar wave reflector, we can obtain an optimal dimension of submerged n-bar wave reflector. Numerical computations are made and they confirm this optimal dimension. Finally, we conclude that in order to get the minimal amplitude of transmitted wave, the wavelength of an incident wave and the dimension of bars should be closely matched.

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1. Introduction

When a wave pass over a seabed covered with bars, its amplitude is reduced, because part of it is reflected. Study of wave propagation past a submerged parallel bar has many practical applications in coastal and ocean engineering. This is useful for designing series of bars as breakwaters and bank protection structures. This subject has been studied by many researchers. Most of them study sinusoidal beds as wave breakers related to Bragg resonance phenomenon. See Mei and Liu [2] for a review.

Each time an incoming wave enters a region with a change of depth (which is significant within its wavelength), it will scatter into a transmitted wave and a reflected wave. In the case of an incoming wave pass over a bar, i.e. a depth discontinuities at $x = 0$ and $x = L$, there is a back and forth scattering processes, each corresponds to a change of depth. Hence, on $x > L$ the transmitted wave is actually a superposition of several waves that result from several scattering processes.

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Generally, the amplitude of transmitted wave A_t is smaller than the amplitude of incoming wave A, and it will be minimum if all waves superpose destructively at $x = L$. This happens when $2L$ equals to an odd number times half wavelength. The dependence of $|A_t/A|$ with respect to the width of the bar L is periodic. Meaning that a wider bar is not necessarily a better wave reflector, and so there is an optimal bar width L_{opt}. The above result can be found in Mei [1], see also [3].

In order to simulate the propagation of a monochromatic wave pass over a submerged bar, we construct a numerical code for linear SWE using the Lax-Wendroff discretization method with different partition along x-axis. This is important in order to avoid any diffusion due to numerical error. So that we can then interpret the decrease in transmitted wave amplitude is mainly because of the bars. Comparison between analytical and numerical amplitude of transmitted and reflected waves shows a good agreement qualitatively and quantitatively. Moreover, they both admit energy conservation $|A_t|^2 + |A_r|^2 = |A|$, where A_r denotes the amplitude of reflected wave. Next, using an analogous physical argument as in the case of one bar, we can obtain an optimal dimension of wave reflector that consists of n-bar and the formula of the minimal transmitted wave amplitude. Numerical computations confirm this. When a wave breaker consists of a periodic bar and trough, its optimal dimension will directly relate to the wavelength of a sinusoidal bed that leads to Bragg resonance. For an application of submerged parallel bars as a wave reflector, this research gives its optimal dimension. However, to protect beaches from high amplitude incoming waves, we have to take into account the reflection from the beach, and this is not discussed in here. Yu and Mei in [6] and Pudjaprasetya *et al.* in [5] show that the amplitude of incoming waves may increase when there is reflection from the beach.

The organization of this paper is as follows. In Section 2, we recall shortly the discussion of the optimal dimension of a one-bar wave reflector, for clearance. Numerical discretization is discussed in Section 3. Simulations are given in Section 3. Section 4 contains generalization to the case of two-bar and n-bar wave reflector. Conclusions are given in Section 5.

2. One-bar wave reflector

Consider a layer of water above a flat bottom with a submerged bar. The function for the depth is

$$h(x) = \begin{cases} h_1 & \text{for } 0 < x < L, \\ h_0 & \text{elsewhere,} \end{cases} \tag{2.1}$$

with $(h_0 - h_1) > 0$ denotes the height of the bar. The governing equation considered here is the linear Shallow Water Equation (linear SWE)

$$\begin{cases} u_t = -g\eta_x \\ \eta_t = -(h(x)u)_x, \end{cases} \tag{2.2}$$

where $\eta(x,t)$ denotes the surface elevation measured from the undisturbed water level, and $u(x,t)$ denotes the horizontal component of fluid particle velocity along the surface. Variable u in (2.2) can be eliminated to yield

$$\eta_{tt} = (c^2(x)\eta_x)_x \tag{2.3}$$

with $c^2(x) = gh(x)$. For flat bottom h_j, it has the d'Alembert (monochromatic) solution

$$\eta(x,t) = A\exp i(k_j x - \omega t) + cc. + B\exp i(k_j x + \omega t) + cc. \tag{2.4}$$

with frequency ω that satisfies a dispersion relation $\omega = k_j\sqrt{gh_j}, j = 0, 1$.

Imagine that a monochromatic incident wave $A\exp i(k_0 x - \omega t)$ coming from the left, enters a region with shallower depth h_1. Some energy is transmitted beyond the step and some is reflected backward, creating scattered waves: A transmission wave above the depth h_1 and a reflection wave above the depth h_0. When the transmission wave enters an original depth h_0, again it scatters into two waves. The same also with the reflection wave, it will scatter into two waves each time it meets bottom discontinuity. Hence, there are infinitely many scattering processes. Let we assume the general solution $\eta(x,t)$ of (2.3) as

$$\eta(x,t) = \exp(-i\omega t)\nu(x) \tag{2.5}$$

where

$$\nu(x) = \begin{cases} A\exp ik_0 x + A_r \exp -ik_0 x, & x < 0 \\ a\exp ik_1 x + b\exp -ik_1 x, & 0 \le x < L \\ A_t \exp ik_0(x - L), & L \le x, \end{cases} \tag{2.6}$$

with A_r, a, b, A_t have to be determined. Demanding the surface profile η to be continuous at the jump $x = 0$ and $x = L$ implies that $\nu(x)$ should be continuous at $x = 0$ and $x = L$. Another jump condition will be obtain in the following. Substituting η in (2.5) into equation (2.3) yields

$$-\omega^2\nu(x) = (c(x)\nu_x)_x.$$

Integrating the equation above with respect to x from $x = 0_-$ to $x = 0_+$, and using the condition that $\nu(x)$ is continuous at $x = 0$, we obtain

$$\lim_{x\to 0_-} h_0\nu_x = \lim_{x\to 0_+} h_1\nu_x \tag{2.7}$$

or simply $h(x)\nu_x$ should be continuous at $x = 0$. At $x = L$ the jump condition holds analogously.

Hence, the surface profile $\eta(x,t) = \exp(-i\omega t)\nu(x)$ for a bottom with a step is given by (2.6) with A_r, a, b, A_t must be found by matching $\nu(x)$ and $h(x)\nu_x$ at the two edges. Some calculations (see [1] for details) will result

$$\left|\frac{A_t}{A}\right| = \frac{S^2 - D^2}{|S^2E^{-1} - D^2E|}, \tag{2.8}$$

$$\left|\frac{A_r}{A}\right| = \frac{|SD(E^{-1} - E)|}{|S^2E^{-1} - D^2E|}, \tag{2.9}$$

with

$$S = h_1k_1 + h_0k_0, \quad D = h_1k_1 - h_0k_0, \quad E = \exp ik_1L. \tag{2.10}$$

Further, we can also show that $|A_t|^2 + |A_r|^2 = |A|^2$, which is energy conservation. Note that the above results also holds analogously for a wave entering a trough, i.e. $h_0 < h_1$.

Consider (2.8), $|A_t/A|$ depends on k_0, k_1, E, and hence depends on h_0, h_1, L. For a certain bar height $h_0 - h_1$, the value $|A_t/A|$ depends on L periodically, see Figure 1, and we can find

$$\left|\frac{A_t}{A}\right|_{min} = \frac{S^2 - D^2}{S^2 + D^2} \tag{2.11}$$

when

$$2L = (n + \frac{1}{2})\frac{2\pi}{k_1} = (n + \frac{1}{2})\lambda_1, \ n = 0, 1, 2, \ldots$$

$$\left|\frac{A_t}{A}\right|_{max} = 1 \tag{2.12}$$

when

$$2L = n\frac{2\pi}{k_1} = n\lambda_1, \ n = 0, 1, 2, \ldots$$

The curves of $|A_t/A|$ and $|A_r/A|$ as periodic functions of L are shown on Figure 1.

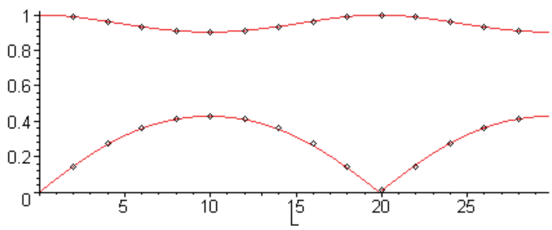

Figure 1. The periodic curves of $|A_t/A|$ (above) and $|A_r/A|$ (below) as functions of L together with their values obtained numerically (dotted). Here, we use $h_0 = 10$ and $h_1 = 0.4 \times h_0$.

Let L_{opt} denotes the minimal width of a bar that gives minimal $|A_t/A|$, hence

$$L_{opt} = 1/4\lambda_1 = \frac{\pi}{2\omega}\sqrt{gh_1}. \tag{2.13}$$

Since we can rewrite $S = \frac{\omega}{\sqrt{g}}(\sqrt{h_1} + \sqrt{h_0})$ and $D = \frac{\omega}{\sqrt{g}}(\sqrt{h_1} - \sqrt{h_0})$, then formula (2.11) can be written explicitly as a function of h_1/h_0, as follows

$$\left|\frac{A_t}{A}\right|_{min} = \frac{(\sqrt{h_1} + \sqrt{h_0})^2 - (\sqrt{h_1} - \sqrt{h_0})^2}{(\sqrt{h_1} + \sqrt{h_0})^2 + (\sqrt{h_1} - \sqrt{h_0})^2} = \frac{2\sqrt{h_1/h_0}}{(1 + h_1/h_0)} \tag{2.14}$$

The plot of the curve $|A_t/A|$ as a monotonically increasing function of h_1/h_0 for $L = L_{opt}$ is given in Figure 2. Note that $|A_t/A| = 1$ when $h_1 = h_0$, and $|A_t/A| = 0$ when $h_1 = 0$, as we expect.

The following is a physical interpretation of the above analytical result, quoted from [1] with some additional remarks. When a wave first strikes the edge at $x = 0$, part of the wave is transmitted into $0 < x < L$, and part is reflected. When the transmitted wave reaches the edge at $x = L$, it undergoes the same scattering process, part of the wave is transmitted to $x > L$, and part is reflected toward the edge $x = 0$. This back-and-forth process of transmission and reflection is repeated indefinitely. The total left going wave in $x < 0$ is the sum of the reflected waves from

$x = 0$ and all the transmitted waves from $0 < x < L$, to $x < 0$. Whereas the total right-going wave in $x > L$ is the sum of all the transmitted waves from $0 < x < L$ to $x > L$. Therefore, all the crest which arrive at $x = L$ at the same time after different numbers of round trips $0, 1, 2, \cdots$ have the same phase if and only if $2kL$ is an even number times π, and they interfere *constructively*, and the total amplitude at $x = L$ is consequently the larger. On the other hand, if $2kL$ is an odd number times π, then after a round trip its a typical wave crest is opposite in phase with the other crest; this interference is *destructive*, resulting the smallest net amplitude at $x = L$.

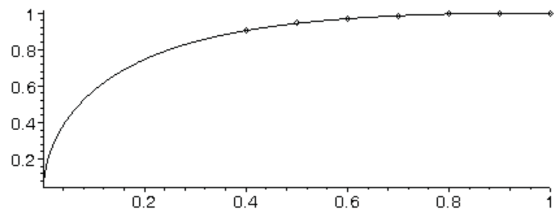

Figure 2. The curve $|A_t/A|$ as a function of h_1/h_0 together with its values obtained numerically (dotted), when the bar width is $L = L_{opt} = \frac{1}{4}\lambda_1$.

3. Discretization and simulation

Consider the linear SWE (2.2) with a piecewise flat bottom (2.1) in a domain $(x, t) \in [a, b] \times [0, T]$, with $a < 0 < L < b$. The discretization that will be used is the Lax-Wendroff method, which is of order $\mathcal{O}(\triangle x^2, \triangle t^2)$ in accuracy. The von Neumann stability analysis as in [4] yields the Courant number $C_j \equiv gh_j \left(\frac{\triangle t}{\triangle x}\right)^2$ should be less than one. Numerical diffusion error will not appear if and only if $C_j = 1$, for $j = 0, 1$. In this numerical discretization, numerical diffusion should be avoided completely in order to make a quantitative comparison. Hence, for a certain $\triangle t$, we take $\triangle x_1 = \sqrt{gh_1}\triangle t$ for $0 < x < L$, and $\triangle x_0 = \sqrt{gh_0}\triangle t$, elsewhere. Hence, the descritized linear SWE equation for each flat depth h_j, $j = 0, 1$ reads

$$\eta_i^{n+1} = \eta_i^n - h_j \frac{\triangle t}{2\triangle x_j}(u_{i+1}^n - u_{i-1}^n) + \frac{1}{2} g h_j \left(\frac{\triangle t}{\triangle x_j}\right)^2 (\eta_{i+1}^n - 2\eta_i^n + \eta_{i-1}^n),$$

$$u_i^{n+1} = u_i^n - g \frac{\triangle t}{2\triangle x_j}(\eta_{i+1}^n - \eta_{i-1}^n) + \frac{1}{2} g h_j \left(\frac{\triangle t}{\triangle x_j}\right)^2 (u_{i+1}^n - 2u_i^n + u_{i-1}^n).$$

We will simulate a monochromatic wave coming from the left, enters a domain with a submerged bar. So, we use the still-water initial condition $\eta(x, 0) = 0$ and $u(x, 0) = 0$ and the left boundary condition $\eta(a, t) = A \sin \omega t$ and $u(a, t) = \sqrt{\frac{g}{h_0}}\eta(a, t)$. Note that these boundary conditions should be satisfied by the right running wave solution of the linear SWE (2.2). For the right boundary $u(b, t)$ and $\eta(b, t)$, we apply the Forward Time Backward Space method of the order $\mathcal{O}(\triangle x, \triangle t)$, that yields a right absorbing boundary condition. This lower order accuracy will directly leave the computation domain following the right going characteristic, and will not effect the higher accuracy $\mathcal{O}(\triangle x^2, \triangle t^2)$ in the interior domain. As a check, simulation for flat bottom h_0 using the above initial and boundary condition will show a monochromatic wave with amplitude A running to the right with velocity $c_0 = \sqrt{gh_0}$, as we expect.

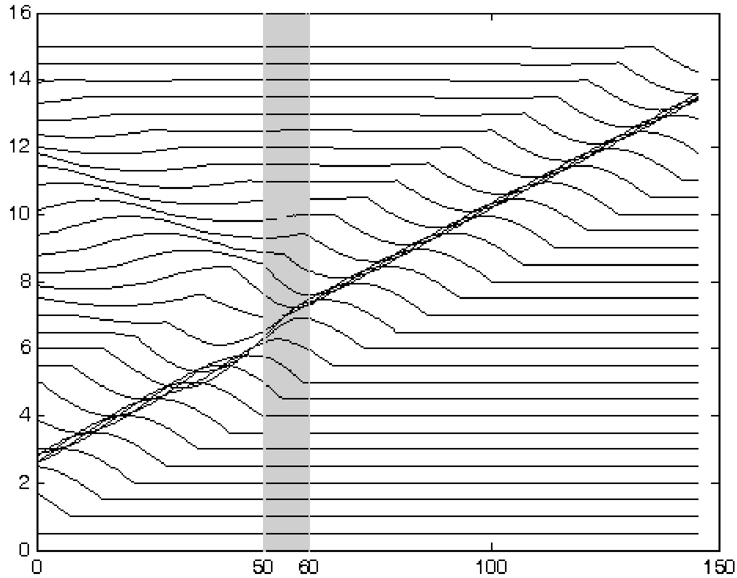

Figure 3. The surface wave profiles $\eta(x,t)$ at several times. The grey area indicates the position of a submerged bar.

For simulation, we use spatial interval $[0, 150]$, time interval $[0, 22]$, gravitation $g = 10$, frequency $\omega = 1$, and a bottom topography

$$h(x) = \begin{cases} 4, & \text{for } 50 < x < 50 + (L_{opt} \approx 10) \\ 10, & \text{for others,} \end{cases} \tag{3.1}$$

Figure 3 shows the result of numerical computation when a monochromatic wave running above topography (3.1). It clearly shows that a wave scatters into a transmission wave running to the right and a reflected waves running to the left. The numerical values of $|A_t/A|$ and $|A_r/A|$ computed for several values of L are given in Figure 1. We can hardly see the difference between the analytical and numerical results. For the choices of $\triangle t = 0.01$ the results agree up to $\mathcal{O}(10^{-4})$ as expected. So we can conclude that numerical computation also admits energy conservation $|A_t|^2 + |A_r|^2 = |A|^2$.

4. Two-bar wave reflector and its generalization

Now we consider a 2-bar wave reflector that consist of two identical bars with height $(h_1 - h_0)$ width L_1, and separate at distance L_0. Hence, the bottom topography is

$$h(x) = \begin{cases} h_1 & \text{for } 0 < x < L_1 \text{ or } (L_0 + L_1) < x < (L_0 + 2L_1), \\ h_0 & \text{elsewhere,} \end{cases} \tag{4.1}$$

What is the optimal dimension (to be precise, L_0 and L_1) of this wave reflector? Analytical approach will involve very lengthy calculation. But we can generalize the physical argument in Section 2 about destructive interference that result the smallest net amplitude of the total right-going wave in $x > L$. Imagine an incidence wave above a flat depth h_0, running through the first bar with depth h_1. When the width of the first bar $L_1 = (n + \frac{1}{2})\frac{1}{2}\lambda_1$, $n = 0, 1, \cdots$, the amplitude of the total right going wave in $x > L_1$ is minimum, and from (2.11), it equals to $(S^2 - D^2)/(S^2 + D^2)A$. The back and forth process of transmitted and reflected waves in $L_1 < x < L_1 + L_0$ will also give a minimal net amplitude of total right going wave in $x > L_1 + L_0$ when $L_0 = (n + \frac{1}{2})\frac{1}{2}\lambda_0$, $n = 0, 1, \cdots$. The minimal amplitude is now $(S^2 - D^2)^2/(S^2 + D^2)^2 A$. The same thing, the width of the second bar must be $L_1 = (n + \frac{1}{2})\frac{1}{2}\lambda_1$, $n = 0, 1, \cdots$, in order to get a minimal amplitude of transmission wave in $x > (L_0 + 2L_1)$. And the minimal transmitted wave amplitude is now $(S^2 - D^2)^3/(S^2 + D^2)^3 A$. Hence, the most optimal 2-bar wave reflector with height $h_0 - h_1$ has width $L_{1opt} = \frac{1}{4}\lambda_1$, and the two bars are separated at distance $L_{0opt} = \frac{1}{4}\lambda_0$.

Several numerical computations finding $|A_t/A|$ using L_{0opt}, for several values of L_1 is made, in order to confirm the optimal width $L_{1opt} = \frac{1}{4}\lambda_1$. The result is shown on Figure 4. The periodic curve of

$$\left|\frac{A_t}{A}\right| = \left(\frac{S^2 - D^2}{|S^2 E^{-1} - D^2 E|}\right)^3 \tag{4.2}$$

as function of L_1 for L_{0opt} is also plotted. Hence, the identical 2-bar wave reflector height $h_0 - h_1 = 0.6 \times h_0$ with optimal width will reduce the amplitude up to $|A_t|_{min} = (S^2 - D^2)^3/(S^2 + D^2)^3 A$ or 69% of the incoming wave amplitude.

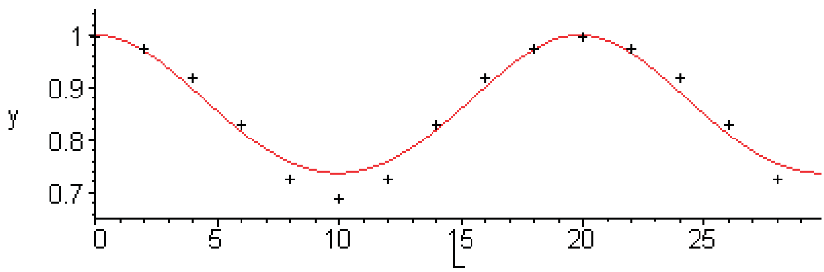

Figure 4. The periodic curve of $|A_t/A|$ given in (4.2) as function of L_1 together with their values obtained numerically (dotted). We use $h_0 = 10, h_1 = 0.4 \times h_0$ and $L_{0opt} = \frac{1}{4}\lambda_0 = \frac{\pi}{2}\sqrt{gh_0} \approx 15.7$. This confirms the optimal value $L_{1opt} = \frac{1}{4}\lambda_1 = \frac{\pi}{2}\sqrt{gh_1} \approx 10$.

In the case of n-bar wave reflector, following the physical argument we can easily find its optimal dimension. The main point is that above each bar, and above a space between the two adjacent bars, all waves should interfere destructively. The destructive interference will be obtained when the width of each piece of the piecewise constant depth function is one forth of the wavelength, or formula (2.13). And the transmitted wave amplitude can be obtained analogous to formula (4.2). In the case $h_1 = 0.4 \times h_0$, numerical calculations for the three-bar and four bar wave reflector reduce the amplitude up to 74% and 60%, respectively. The above argument can be generalized directly to get the optimal dimension of the nonidentical n-bar wave reflector.

Further, we relate this optimal dimension with the Bragg resonance condition. This phenomenon happens when a monochromatic wave enters a region of sinusoidal bar with wavelength half of the wavelength of the incident wave. When this phenomenon occurs, the amplitude of the reflected wave is maximal, and the amplitude of the transmitted wave is minimal. Suppose that on a flat bottom with depth h_0 there is a series of bar and trough with amplitude $\varepsilon D h_0$. The optimal length of this bar is $\frac{\pi}{2\omega}\sqrt{gh_0(1-\varepsilon D)}$ and length of the trough is $\frac{\pi}{2\omega}\sqrt{gh_0(1+\varepsilon D)}$. So, the wavelength of this periodic bar and trough is $\frac{\pi}{2\omega}\sqrt{gh_0}\left(\sqrt{1-\varepsilon D}+\sqrt{1+\varepsilon D}\right)$. In the limit $\varepsilon D \to 0$, this wavelength tends to $\frac{\pi}{\omega}\sqrt{gh_0}$ which is a half times the wavelength of the incident wave. This is exactly the Bragg resonance condition.

5. Conclusions

The Lax-Wendroff discretization method with different partition along x-axis was applied to solve the linear SWE with a bottom topography of a submerged n-bar. Numerical computations confirmed the optimal dimension of the one-bar wave reflector: The width of the bar should be one forth times the wavelength of the incident wave (obtained analytically by C. C. Mei in [1]). The optimal dimension of a wave reflector that consists of a submerged nonidentical n-bar and the formula of the transmitted wave amplitude were obtained. Numerical computations were used to confirm these results.

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