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Triangles Which are Bounded Operators on $\mathcal{A}_k$

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Abstract. A lower triangular infinite matrix is called a triangle if there are no zeros on the principal diagonal. The main result of this paper gives a minimal set of sufficient conditions for a triangle $T : \mathcal{A}_k \to \mathcal{A}_k$ for the sequence space $\mathcal{A}_k$ defined as follows:

$$\mathcal{A}_k := \left\{ \{s_n\} \ : \ \sum_{n=1}^{\infty} n^{k-1} \, | \, a_n|^k < \infty \ , \ a_n = s_n - s_{n-1} \right\}.$$

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1. Introduction and background

Let $\sum a_v$ denote a series with partial sums s_n. For an infinite matrix T, t_n, the n th term of the T-transform of $\{s_n\}$ is denoted by

$$t_n = \sum_{v=0}^{\infty} t_{nv} s_v.$$

A series $\sum a_v$ is said to be absolutely T-summable if $\sum_n |\Delta \, t_{n-1}| < \infty$, where Δ is the forward difference operator defined by $\Delta \, t_{n-1} = t_{n-1} - t_n$. Papers dealing with absolute summability date back at least as far as Fekete [2].

A sequence $\{s_n\}$ is said to be of bounded variation (bv) if $\sum_n |\Delta \, s_n| < \infty$. Thus, to say that a series is absolutely summable by a matrix T is equivalent to saying that the T-transform the sequence is in bv. Necessary and sufficient conditions for a matrix $T : bv \to bv$ are known. (See, e.g. Stieglitz and Tietz [7]).

Let σ_n^{α} denote the nth terms of the transform of a Cesáro matrix (C, α) of a sequence $\{s_n\}$. In 1957 Flett [3] made the following definition. A series $\sum a_n$, with

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partial sums s_n, is said to be absolutely (C,α) summable of order $k \geq 1$, written $\sum a_n$ is summable $|C,\alpha|_k$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left|\sigma_{n-1}^{\alpha} - \sigma_n^{\alpha}\right|^k < \infty. \tag{1.1}$$

He then proved the following inclusion theorem. If series $\sum a_n$ is summable $|C,\alpha|_k$, it is summable $|C,\beta|_r$ for each $r \geq k \geq 1$, $\alpha > -1$, $\beta > \alpha + 1/k - 1/r$. It then follows that, if one chooses $r = k$, then a series $\sum a_n$ which is $|C,\alpha|_k$ summable is also $|C,\beta|_k$ summable for $k \geq 1$, $\beta > \alpha > -1$.

Quite recently, Rhoades and Savaş [5] established sufficient conditions for a series $\sum a_n$ summable $|A|_k$ to imply that it is summable $|B|_k$, where A and B are particular lower triangular matrices.

Let $\sum a_n$ be a series with partial sums s_n. Define

$$\mathcal{A}_k := \left\{ \{s_n\} \;:\; \sum_{n=1}^{\infty} n^{k-1} \left| a_n \right|^k < \infty \;,\; a_n = s_n - s_{n-1} \right\}. \tag{1.2}$$

If one sets $\alpha = 0$ in the inclusion statement involving (C,α) and (C,β), then one obtains the fact that $(C,\beta) \in B\left(\mathcal{A}_k\right)$ for each $\beta > 0$, where $B\left(\mathcal{A}_k\right)$ denotes the algebra of all matrices that map $\mathcal{A}_k$ to $\mathcal{A}_k$.

In 1970, using the same definition as Flett, Das [1] defined a matrix T to be absolutely kth power conservative for $k \geq 1$, if $T \in B\left(\mathcal{A}_k\right)$; i.e., if $\{s_n\}$ is a sequence satisfying

$$\sum_{n=1}^{\infty} n^{k-1} \left| s_n - s_{n-1} \right|^k < \infty,$$

then

$$\sum_{n=1}^{\infty} n^{k-1} \left| t_n - t_{n-1} \right|^k < \infty.$$

In that same paper he proved that every conservative Hausdorff matrix $H \in B\left(\mathcal{A}_k\right)$, which contains as a special case the fact that $(C,\beta) \in B\left(\mathcal{A}_k\right)$ for $\beta > 0$. In [6], it is shown that $(C,\alpha) \in B\left(\mathcal{A}_k\right)$ for each $\alpha > -1$. Therefore being conservative is not a necessary condition for a matrix to map $\mathcal{A}_k$ to $\mathcal{A}_k$.

Given a matrix T one can find a matrix B such that the statement $T \in B\left(\mathcal{A}_k\right)$ is equivalent to $B \in B\left(\ell^k\right)$. Since necessary and sufficient conditions are not known for an arbitrary $B \in B\left(\ell^k\right)$ for $k > 1$, it is not reasonable to expect to find necessary and sufficient conditions for $T \in B\left(\mathcal{A}_k\right)$.

A lower triangular matrix with nonzero principal diagonal entries is called a triangle. In this paper we obtain a minimal set of sufficient conditions for a triangle $T \in B\left(\mathcal{A}_k\right)$. As corollaries we obtain other $A \in B\left(\mathcal{A}_k\right)$ results including that of [5].

We may associate with T two infinite matrices $\bar{T}$ and $\hat{T}$ as follows:

$$\bar{t}_{nv} = \sum_{r=v}^{n} t_{nr} \quad , \quad n, v = 0, 1, 2, ...$$

and

$$\hat{t}_{nv} = \bar{t}_{nv} - \bar{t}_{n-1,v} \quad , \quad n = 1, 2, 3, ...,$$

where $\hat{t}_{00} = \bar{t}_{00} = t_{00}$.

If $T = (t_{nv})$ is a lower triangular matrix, then $\bar{T} = (\bar{t}_{nv})$ and $\hat{T} = \left(\hat{t}_{nv}\right)$ are also lower triangular matrices. If T is a triangle, then for each $n \in \mathbb{N}^0$

$$\hat{t}_{nn} = \bar{t}_{nn} = t_{nn} \neq 0,$$

and $\bar{T}$ and $\hat{T}$ are triangles.

2. Main results

Theorem 2.1. *$T = (t_{nv})$ be a triangle satisfying*

(i) $|t_{nn}| = O(1)$,

(ii) $\sum_{v=0}^{n-1} |t_{vv}| \left|\hat{t}_{nv}\right| = O(|t_{nn}|)$,

and

(iii) $\sum_{n=v+1}^{\infty} (n|t_{nn}|)^{k-1} \left|\hat{t}_{nv}\right| = O(v|t_{vv}|)^{k-1}$.

Then, $T \in B(\mathcal{A}_k)$, $k \geq 1$.

Proof. If y_n denotes the nth term of the T-transform of a sequence $\{s_n\}$, then

$$\begin{aligned}
y_n &= \sum_{v=0}^{n} t_{nv} s_v = \sum_{v=0}^{n} t_{nv} \sum_{i=0}^{v} a_i = \sum_{i=0}^{n} a_i \sum_{v=i}^{n} t_{nv} \\
&= \sum_{i=0}^{n} \bar{t}_{ni} a_i,
\end{aligned}$$

and

$$\begin{aligned}
Y_n := y_n - y_{n-1} &= \sum_{v=0}^{n} \bar{t}_{nv} a_v - \sum_{v=0}^{n-1} \bar{t}_{n-1,v} a_v \\
&= \sum_{v=0}^{n} \left(\bar{t}_{nv} - \bar{t}_{n-1,v}\right) a_v \\
&= \sum_{v=0}^{n} \hat{t}_{nv} a_v \\
&= \sum_{v=0}^{n-1} \hat{t}_{nv} a_v + \hat{t}_{nn} a_n \\
&= T_{n1} + T_{n2}.
\end{aligned}$$

By Minkowski's inequality it is sufficient to prove that

$$\sum_{n=1}^{\infty} n^{k-1} |T_{nr}|^k < \infty, \quad r = 1, 2.$$

Using Hölder's inequality, (ii), and (iii), we get

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{k-1} \left|T_{n1}\right|^k &= \sum_{n=1}^{\infty} n^{k-1} \left|\sum_{v=0}^{n-1} \hat{t}_{nv} a_v\right|^k \\
&\leq \sum_{n=1}^{\infty} n^{k-1} \left\{\sum_{v=0}^{n-1} \frac{\left|t_{vv}\right| \left|\hat{t}_{nv}\right| \left|a_v\right|}{\left|t_{vv}\right|}\right\}^k \\
&\leq \sum_{n=1}^{\infty} n^{k-1} \sum_{v=0}^{n-1} \left|t_{vv}\right|^{1-k} \left|\hat{t}_{nv}\right| \left|a_v\right|^k \times \left\{\sum_{v=0}^{n-1} \left|t_{vv}\right| \ \left|\hat{t}_{nv}\right|\right\}^{k-1} \\
&= O\left(1\right) \sum_{n=1}^{\infty} n^{k-1} \sum_{v=0}^{n-1} \left|t_{vv}\right|^{1-k} \left|\hat{t}_{nv}\right| \ \left|a_v\right|^k \times \left\{\ \left|t_{nn}\right| \ \right\}^{k-1} \\
&= O\left(1\right) \sum_{v=0}^{\infty} \left|t_{vv}\right|^{1-k} \left|a_v\right|^k \sum_{n=v+1}^{\infty} \left(n \left|t_{nn}\right|\right)^{k-1} \left|\hat{t}_{nv}\right| \\
&= O\left(1\right) \sum_{v=1}^{\infty} \left|t_{vv}\right|^{1-k} \left|a_v\right|^k v^{k-1} \left|t_{vv}\right|^{k-1} \\
&= O\left(1\right) \sum_{v=1}^{\infty} v^{k-1} \left|a_v\right|^k = O\left(1\right).
\end{aligned}$$

Using (i),

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{k-1} \left|T_{n2}\right|^k &= \sum_{n=1}^{\infty} n^{k-1} \left|t_{nn} a_n\right|^k \\
&= O\left(1\right) \sum_{n=1}^{\infty} n^{k-1} \left|a_n\right|^k = O\left(1\right),
\end{aligned}$$

since $\{s_n\} \in \mathcal{A}_k$. This completes the proof. ∎

Using an argument similar to that of [4] we now establish another set of sufficient conditions for a nonnegative triangle T with row sums one and decreasing columns to be in $B\left(\mathcal{A}_k\right)$. We then show that the conditions of Theorem 2.2 imply those of Theorem 2.1.

Theorem 2.2. *Let $T = \left(t_{nv}\right)$ be a nonnegative triangle satisfying*

(i) $nt_{nn} \asymp 1$,
(ii) $\bar{t}_{n0} = 1$ *for* $n = 0, 1, 2, \ldots$,
(iii) $t_{n-1,v} \geq t_{nv}$ *for* $n \geq v + 1$,

and

(iv) $\sum_{v=0}^{n-1} t_{vv} \hat{t}_{nv} = O\left(t_{nn}\right)$.

Then, $T \in B\left(\mathcal{A}_k\right), k \geq 1$.

Proof. By $\{x_n\}$ we denote the T-transform of $\{s_n\}$. Then

$$x_n = \sum_{v=0}^{n} t_{nv} s_v,$$

and

$$\begin{aligned} x_n - x_{n-1} &= \sum_{v=0}^{n} \hat{t}_{nv} a_v \\ &= \sum_{v=0}^{n-1} \hat{t}_{nv} a_v + \hat{t}_{nn} a_n \\ &= T_{n1} + T_{n2}, \end{aligned}$$

say. Using Hölder's inequality, (i) and (iv)

$$\begin{aligned} \sum_{n=1}^{m+1} n^{k-1} \left|T_{n1}\right|^k &= \sum_{n=1}^{m+1} n^{k-1} \left|\sum_{v=0}^{n-1} \hat{t}_{nv} a_v\right|^k \\ &\leq \sum_{n=1}^{m+1} n^{k-1} \sum_{v=0}^{n-1} \left|\hat{t}_{nv}\right| \left|t_{vv}\right|^{1-k} \left|a_v\right|^k \times \left\{\sum_{v=0}^{n-1} \left|t_{vv}\right| \left|\hat{t}_{nv}\right|\right\}^{k-1} \\ &= O(1) \sum_{n=1}^{m+1} \left(n t_{nn}\right)^{k-1} \sum_{v=0}^{n-1} \left|\hat{t}_{nv}\right| \left|t_{vv}\right|^{1-k} \left|a_v\right|^k \\ &= O(1) \sum_{v=1}^{m+1} \left|t_{vv}\right|^{1-k} \left|a_v\right|^k \sum_{n=v+1}^{m+1} \left(n \left|t_{nn}\right|\right)^{k-1} \left|\hat{t}_{nv}\right| \\ &= O(1) \sum_{v=1}^{m+1} \left|t_{vv}\right|^{1-k} \left|a_v\right|^k \sum_{n=v+1}^{m+1} \left|\hat{t}_{nv}\right|. \end{aligned}$$

Since $T = (t_{nv})$ is a positive matrix, from (ii) and (iii),

$$\begin{aligned} \hat{t}_{nv} &= \sum_{r=v}^{n} t_{nr} - \sum_{r=v}^{n-1} t_{n-1,r} \\ &= 1 - \sum_{r=0}^{v-1} t_{nr} - 1 + \sum_{r=0}^{v-1} t_{n-1,r} \\ &= \sum_{r=0}^{v-1} \left(t_{n-1,r} - t_{nr}\right) \geq 0, \end{aligned}$$

and $\hat{T} = \left(\hat{t}_{nv}\right)$ is a positive matrix. Therefore, using (ii)

$$\begin{aligned} \sum_{n=v+1}^{m+1} \left|\hat{t}_{nv}\right| &= \sum_{n=v+1}^{m+1} \left(\bar{t}_{nv} - \bar{t}_{n-1,v}\right) \\ &= \bar{t}_{m+1,v} - t_{vv} = O\left(\bar{t}_{m+1,v}\right) \\ &= O\left(\bar{t}_{m+1,0}\right) = O(1). \end{aligned}$$

Using (i) and since $\{s_n\} \in \mathcal{A}_k$,

$$\sum_{n=1}^{m+1} n^{k-1} |T_{n1}|^k = O(1) \sum_{v=1}^{m+1} |t_{vv}|^{1-k} |a_v|^k$$
$$= O(1) \sum_{v=1}^{m+1} v^{k-1} |a_v|^k = O(1), \text{ as } m \to \infty.$$

Using (i),

$$\sum_{n=1}^{m+1} n^{k-1} |T_{n2}|^k = \sum_{n=1}^{m+1} n^{k-1} |t_{nn} a_n|^k$$
$$= O(1) \sum_{n=1}^{m+1} n^{k-1} |a_n|^k = O(1), \text{ as } m \to \infty,$$

since $\{s_n\} \in \mathcal{A}_k$. Hence the proof is complete. ∎

Corollary 2.1. *The conditions of Theorem 2.2 imply these of Theorem 2.1.*

Proof. Conditions (i) and (iv) of Theorem 2.2 with t_{nv} nonnegative imply conditions (i) and (ii) of Theorem 2.1, respectively.

Since $T = (t_{nv})$ is a positive matrix, from (ii) and (iii) of Theorem 2.2, then $\hat{T} = (\hat{t}_{nv})$ is a positive matrix. Using (i) and (ii) of Theorem 2,

$$\frac{1}{(v |t_{vv}|)^{k-1}} \sum_{n=v+1}^{\infty} (n |t_{nn}|)^{k-1} |\hat{t}_{nv}| = O(1) \sum_{n=v+1}^{\infty} \hat{t}_{nv} = O(1),$$

and condition (iii) of Theorem 2.1 is satisfied. ∎

Setting $A = I$, the identity matrix, in Theorem 1 of [5] gives the following result.

Corollary 2.2. *Let T be a triangle satisfying*

(i) $|t_{nn}| = O(1)$,

(ii) $\sum_{v=0}^{n-1} |\Delta_v \hat{t}_{nv}| = O(|t_{nn}|)$,

(iii) $\sum_{n=v+1}^{\infty} (n |t_{nn}|)^{k-1} |\Delta_v \hat{t}_{nv}| = O\left(v^{k-1} |t_{vv}|^k\right)$,

(iv) $\sum_{v=0}^{n-1} |t_{vv}| |\hat{t}_{n,v+1}| = O(|t_{nn}|)$,

and

(v) $\sum_{n=v+1}^{\infty} (n |t_{nn}|)^{k-1} | \hat{t}_{n,v+1}| = O\left((v \ |t_{vv}|)^{k-1}\right)$.

Then, $T \in B(\mathcal{A}_k), (k \geq 1)$.

Proof. Condition (i) of Corollary 2.2 is condition (i) of Theorem 2.1. Conditions (ii)–(iii) of Theorem 2.1 can be obtained from conditions (i)–(v) of Corollary 2.2 as follows.

Recall that

$$\Delta_v \hat{t}_{nv} = \hat{t}_{nv} - \hat{t}_{n,v+1}.$$

Thus

$$\hat{t}_{nv} = \Delta_v \hat{t}_{nv} + \hat{t}_{n,v+1}.$$

Using (i), (ii) and (iv) of Corollary 2.2,

$$\begin{aligned}
\sum_{v=0}^{n-1} |t_{vv}| \left|\hat{t}_{nv}\right| &= \sum_{v=0}^{n-1} |t_{vv}| \left|\Delta_v \hat{t}_{nv} + \hat{t}_{n,v+1}\right| \\
&\le \sum_{v=0}^{n-1} |t_{vv}| \left|\hat{t}_{n,v+1}\right| + \sum_{v=0}^{n-1} |t_{vv}| \left|\Delta_v \hat{t}_{nv}\right| \\
&= \sum_{v=0}^{n-1} |t_{vv}| \left|\hat{t}_{n,v+1}\right| + O(1) \sum_{v=0}^{n-1} \left|\Delta_v \hat{t}_{nv}\right| \\
&= O(|t_{nn}|) + O(|t_{nn}|) = O(|t_{nn}|),
\end{aligned}$$

and condition (ii) of Theorem 2.1 is satisfied. Using (i), (iii) and (v) of Corollary 2.2,

$$\begin{aligned}
\sum_{n=v+1}^{\infty} (n|t_{nn}|)^{k-1} \left|\hat{t}_{nv}\right| &= \sum_{n=v+1}^{\infty} (n|t_{nn}|)^{k-1} \left|\Delta_v \hat{t}_{nv} + \hat{t}_{n,v+1}\right| \\
&\le \sum_{n=v+1}^{\infty} (n|t_{nn}|)^{k-1} \left|\Delta_v \hat{t}_{nv}\right| \\
&\quad + \sum_{n=v+1}^{\infty} (n|t_{nn}|)^{k-1} \left|\hat{t}_{n,v+1}\right| \\
&= O\left((v|t_{vv}|)^{k-1} |t_{vv}|\right) + O\left((v|t_{vv}|)^{k-1}\right) \\
&= O\left((v|t_{vv}|)^{k-1}\right),
\end{aligned}$$

and condition (iii) of Theorem 2.1 is satisfied. ∎

We remark that conditions (i)–(iv) of Theorem 2.2 imply conditions (i)–(v) of Corollary 2.2. Now we will show the remarks.

Condition (i) of Theorem 2.2 with t_{nv} nonnegative implies condition (i) of Corollary 2.2. Since

$$\Delta_v \hat{t}_{nv} = -\Delta_n (t_{n-1,v}),$$

and using conditions (ii) and (iii) of Theorem 2.2, we have

$$\begin{aligned}
\sum_{v=0}^{n-1} \left|\Delta_v \hat{t}_{nv}\right| &= \sum_{v=0}^{n-1} |t_{nv} - t_{n-1,v}| \\
&= \sum_{v=0}^{n-1} t_{n-1,v} - \sum_{v=0}^{n-1} t_{nv} \\
&= \bar{t}_{n-1,0} - \bar{t}_{n0} + t_{nn} = t_{nn},
\end{aligned}$$

and condition (ii) of Corollary 2.2 is satisfied. By (iii) of Theorem 2.2

$$\begin{aligned}\sum_{n=v+1}^{m+1}\left|\Delta_v\hat{t}_{nv}\right| &= \sum_{n=v+1}^{m+1}\left|t_{nv}-t_{n-1,v}\right| \\ &= \sum_{n=v+1}^{m+1}\left(t_{n-1,v}-t_{nv}\right) \\ &= t_{vv}-t_{m+1,v}\le t_{vv}.\end{aligned}$$

Thus

$$\sum_{n=v+1}^{\infty}\left|\Delta_v\hat{t}_{nv}\right| = O\left(\left|t_{vv}\right|\right).$$

Using condition (i) of Theorem 2.2

$$\sum_{n=v+1}^{\infty}\left(n\left|t_{nn}\right|\right)^{k-1}\left|\Delta_v\hat{t}_{nv}\right| = O\left(1\right)\sum_{n=v+1}^{\infty}\left|\Delta_v\hat{t}_{nv}\right| = O\left(v^{k-1}\left|t_{vv}\right|^k\right),$$

and condition (iii) of Corollary 2.2 is satisfied.

Obviously, conditions (i)–(iv) of Theorem 2.2 imply conditions (iv) and (v) of Corollary 2.2.

Corollary 2.3. *If $\{p_n\}$ is a positive sequence satisfying*

$$np_n \asymp P_n, \tag{2.1}$$

then $\left(\bar{N},p_n\right)\in B\left(\mathcal{A}_k\right)$.

Proof. In Theorem 2.1 set $T=\left(\bar{N},p_n\right)$. Condition (i) of Theorem 2.1 is clearly satisfied.

$$\begin{aligned}\frac{1}{\left|t_{nn}\right|}\sum_{v=0}^{n-1}\left|t_{vv}\right|\ \left|\hat{t}_{nv}\right| &= \frac{P_n}{p_n}\sum_{v=0}^{n-1}\frac{p_v}{P_v}\frac{p_nP_{v-1}}{P_nP_{n-1}} \\ &= O\left(1\right)\frac{1}{P_{n-1}}\sum_{v=0}^{n-1}p_v = O\left(1\right)\end{aligned}$$

and condition (ii) of Theorem 2.1 is satisfied.

$$\begin{aligned}\frac{1}{v\left|t_{vv}\right|^{k-1}}\sum_{n=v+1}^{\infty}\left(n\left|t_{nn}\right|\right)^{k-1}\left|\hat{t}_{nv}\right| &= \left(\frac{P_v}{vp_v}\right)^{k-1}\sum_{n=v+1}^{\infty}\left(\frac{np_n}{P_n}\right)^{k-1}\frac{p_nP_{v-1}}{P_nP_{n-1}} \\ &= O\left(1\right)P_{v-1}\sum_{n=v+1}^{\infty}O\left(1\right)\frac{p_n}{P_nP_{n-1}} \\ &= O\left(1\right)\frac{P_{v-1}}{P_{v-1}} = O\left(1\right)\end{aligned}$$

and condition (iii) of Theorem 2.1 is satisfied. ∎

From Das [1], $(C,1)\in B\left(\mathcal{A}_k\right)$. For completeness, we show that Corollary 2.4 follows from Corollary 2.3.

Corollary 2.4. $(C,1) \in B(\mathcal{A}_k)$.

Proof. Set each $p_n = 1$ in Corollary 2.3. Then the condition (2.1) is satisfied. ▮

Since $(\bar{N}, p_n)$ and $(C,1)$ are nonnegative and have each row sum equal to one, we also obtain Corollary 2.3 and Corollary 2.4 from Theorem 2.2.

Remark 2.1. Condition (i) of Theorem 2.1 is necessary. To see this, let $e^{(j)}$ denote the jth coordinate sequence; i.e., the sequence with 1 in position j and zeros elsewhere. Then $Y_n = \sum_{v=0}^{n} \hat{t}_{nv} a_v$ satisfies

$$Y_n = \begin{cases} 0 \;, & n < j\,, \\ \hat{t}_{nn} \;, & n = j\,, \\ \hat{t}_{nj} \;, & n > j\,. \end{cases}$$

Since $T \in B(\mathcal{A}_k)$ there exists a positive number m such that

$$\begin{aligned} m &\geq \sum_{n=1}^{\infty} n^{k-1} |Y_n|^k = \sum_{n=j}^{\infty} n^{k-1} |Y_n|^k \\ &\geq j^{k-1} |Y_j|^k = j^{k-1} |t_{jj}|^k\,, \end{aligned}$$

which implies that $|t_{jj}| = O(1)$, and condition (i) is necessary.

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