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On Weakly π-Subcommutative near-Rings

P. NANDAKUMAR
Department of Mathematics
Perunthalaivar Kamarajar Institute of Engineering and Technology (PKIET)
(Government of Puducherry Institution)
Nedungadu (Post), Karaikal-609 603,
Union Territory of Puducherry, India
drpnandakumar@gmail.com

Abstract. In this paper we introduce the concept of weakly π-subcommutative near-rings. We give some characterization of weakly π-subcommutative near-rings. We obtain necessary and sufficient conditions for a near-ring to be π-regular. We have found conditions under which $\sqrt{K}$ is a two-sided N-subgroup, when K is a two-sided N-subgroup. Conditions are obtained for a weakly π-subcommutative near-ring to be subcommutative.

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1. Introduction

A near-ring N is said to be weakly π-subcommutative if whenever a in N there exists a positive integer $n \geq 1$ such that $Na^n \subseteq aN$ and $a^nN \subseteq Na$. Throughout this paper N stands for a near-ring with identity, $Id(N)$ denotes the set of all idempotent elements of N, $C(N)$ denotes the center of N and $Nil(N)$ denotes the set of all nilpotent elements of N. Recall that N is said to be regular (unit regular) if for each a in N, there exists y in N (a unit u in N) such that $xyx = x(xux = x)$. A near-ring N is said to be an S-near-ring if $a \in Na$ for each a in N. The purpose of this paper is to extend the result obtained for rings by A. Badawi [1, Theorem 2] and [2, Theorem 1] to near-rings. We have also obtained the necessary and sufficient condition for a near-ring to be π-regular. If S is any nonempty subset of N, then the left annihilator of S in N is $l(S) = \{x \in N/xs = 0 \text{ for all } s \text{ in } S\}$. For any subset K of N, we write $\sqrt{K} = \{x \in N/x^n \in K, \text{ for some positive integer } n\}$. Very few theorems are available for the structure of $\sqrt{K}$. We have obtained conditions for $\sqrt{K}$ to be two-sided N-subgroup, when K is a two-sided N-subgroup. A near-ring N is said to be subcommutative if $Na = aN$ for each a in N. A natural example of a near-ring is given by the set $M(\Gamma)$ of all mappings of a group Γ into itself with

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componentwise addition and composition. For the basic terminology and notation we refer to Pilz [6].

Definition 1.1. *A near-ring N is said to be weakly π-subcommutative if whenever a in N there exists a positive integer $n \geq 1$ such that $Na^n \subseteq aN$ and $a^nN \subseteq Na$.*

Remark 1.1. A subcommutative near-ring is weakly π-subcommutative. But the converse is not true as the following example shows.

Example 1.1. Consider the near-ring $(N, +, \cdot)$ defined on the Kleins four group $(N, +)$ with $N = \{0, a, b, c\}$, where $\cdot$ is defined as follows (as per scheme 16, p. 408 of Pilz [6]).

·	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	0	a

$(N, +, \cdot)$ is a near-ring which is weakly π-subcommutative but not subcommutative, since $bN = \{0, a\}$ where as $Nb = \{0\}$.

Lemma 1.1. *If N is weakly π-subcommutative, then every idempotent of N is central.*

Proof. Suppose that N is weakly π-subcommutative. Let e be an idempotent of N. Hence $Ne^n \subseteq eN$ and $e^nN \subseteq Ne$. Hence for every x in N, there exist y and z in N such that $xe = ey$ and $ex = ze$. It follows that $ex(= exe) = xe$. ∎

Lemma 1.2. *Let $x \in N$. If x is unit regular, then $x = eu$ for some idempotent e and a unit u.*

Proof. Suppose x is unit regular. Then for some $v \in U(N)$, we have $xvx = x$. Let $e = xv \in Id(N)$ and $u = v^{-1}$. Then $x = eu$. ∎

Theorem 1.1. *Suppose that $Id(N) \subset C(N)$ and $x \in N$. Then x is regular if and only if x is unit regular.*

Proof. Let $x \in N$ and let us assume that x is regular. Hence there exists $y \in N$ such that $xyx = x$. Clearly $xy, yx \in Id(N) \subset C(N)$. Hence $xy = x(yx)y = (xy)yx = y(xy)x = yx$. Let $u = x - 1 + xy$ and $v = xy^2 - 1 + xy$. Then

$$\begin{aligned} uv &= x(xy^2 - 1 + xy) - 1(xy^2 - 1 + xy) + xy(xy^2 - 1 + xy) \\ &= xyx(xy^2 - 1 + xy) - xy + 1 - xy^2 + xyxy^2 - xy + xy \\ &= x(yxxyy - yx + yxxy) - xy + 1 - xy^2 + xy^2 \\ &= 1. \end{aligned}$$

Similarly $vu = 1$. Thus $u, v \in U(N)$. Moreover $xvx = x(xy^2 - 1 + xy)x = x$. Hence x is unit regular.

Converse is obvious. ∎

Corollary 1.1. [2, Theorem 2] *Let K be a ring with 1 such that $Id(K) \subset C(K)$ and $x \in K$ is regular, then x is unit regular.*

Corollary 1.2. *Let N be weakly π-subcommutative. Then N is regular if and only if N is unit regular.*

Theorem 1.2. *Suppose that $Id(N) \subset C(N)$ and $x \in N$. Then x is π-regular if and only if there exists $e \in Id(N)$ such that ex is regular and $(1-e)x \in Nil(N)$.*

Proof. Let $x \in N$ and let us assume that x is π-regular. Since x is π-regular, for some $n \geq 1$, x^n is regular. Hence by Theorem 1.1 and Lemma 1.2, we have $x^n = eu$ for some $e \in Id(N)$ and $u \in U(N)$. Then

$$ex(u^{-1}x^{n-1})ex = ex(u^{-1}x^{n-1})xe = ex(u^{-1}eue) = ex(u^{-1}uee) = exe = eex = ex.$$

Hence ex is regular. Since $(1-e)$ is an idempotent,

$$[(1-e)x]^n = (1-e)x^n = (1-e)eu = 0.$$

For the converse, suppose for some $e \in Id(N)$, ex is regular and $(1-e)x \in Nil(N)$. Then for some $n \geq 1$,

$$[(1-e)x]^n = (1-e)x^n = 0.$$

Hence, $ex^n = x^n$. Since ex is regular, by Lemma 1.2, $ex = cu$ for some $c \in Id(N)$ and $u \in U(N)$. Hence $(ex)^n = (cu)^n = cu^n$. But $(ex)^n = ex^n = x^n$. Thus $x^n = cu^n$. Let $y = cu^{-n}$. Then $x^n y x^n = x^n$ and hence x is π-regular. ∎

Corollary 1.3. [1, Theorem 1] *Let R be a ring. Suppose $Id(R) \subset C(R)$. Let $x \in R$. Then x is π-regular if and only if there exists $e \in Id(R)$ such that ex is regular and $(1-e)x \in Nil(R)$.*

Corollary 1.4. *Let N be weakly π-subcommutative and $x \in N$. Then x is π-regular if and only if there exists $e \in Id(N)$ such that ex is regular and $(1-e)x \in Nil(N)$.*

Theorem 1.3. *Suppose $Id(N) \subset C(N)$. Let $x \in N$ such that x is π-regular. Then for some $e \in Id(N)$ and $u \in U(N)$ we have $ex = eu$.*

Proof. Since x is π-regular, for some $m \geq 1$, we have $x^m = ev$ and ex is regular. Hence, by Theorem 1.1 and Lemma 1.2, $ex = cu$ for some $c \in Id(N)$ and $u \in U(N)$. In fact, $e = c$. For, $e(ex) = e(cu)$. But $e(ex) = ex = cu$. Thus, $e(cu) = cu$ and therefore $ec = c$. Since $e, c \in C(N)$, we have $(ex)^m = ex^m = cu^m$. Since $x^m = ev, ex^m = ev = cu^m$. Hence $e = cu^m v^{-1}$. Thus $ec = cu^m v^{-1} c = cu^m v^{-1}$, since $c \in C(N)$. Hence $ec = e$. Since $ec = c$ and $ec = e, e = c$. Thus $ex = eu$. ∎

Corollary 1.5. [2, Lemma 2] *Let R be a ring. Suppose $Id(R) \subset C(R)$. Let $x \in R$ such that x is π-regular. Then for some $e \in Id(R)$ and $u \in U(R)$ we have $ex = eu$.*

Corollary 1.6. *Let N be a weakly π-subcommutative. Let $x \in N$ such that x is π-regular. Then for some $e \in Id(N)$ and $u \in U(N)$ we have $ex = eu$.*

Theorem 1.4. *N is π-regular if and only if for each a in N there exists a positive integer $n \geq 1$ and an idempotent e such that $a^n \in Na^n = Ne$.*

Proof. Let N be a π-regular near-ring. Then for every a in N there exists x in N and an integer t such that $a^t = a^t x a^t$. Let $xa^t = e$, then e is clearly an idempotent and $Na^t = Ne$. Since $a^t = a^t x a^t$, we have $a^t \in Na^t$. Therefore $a^t \in Na^t = Ne$.

Conversely, let $a \in N$, and let us assume that there exists an idempotent b and an integer t such that $a^t \in Na^t = Nb$. This gives $a^t = ub$ for some u in N. Also $b \in Nb = Na^t$ gives $b = ya^t$ for some y in N. Therefore

$$a^t y a^t = ubya^t = ubb = ub^2 = ub = a^t.$$

Hence N is a π-regular near-ring. ∎

Lemma 1.3. *If N is an S-near-ring, then the following are equivalent:*

(a) $l(a) = l(a^2)$ *for every a in N,*

(b) $x^3 = x^2$ *implies* $x^2 = x$ *for every x in N.*

Proof. (a) $\Rightarrow$ (b) : Suppose $x^3 = x^2$ for some x in N. Since N is an S-near-ring, $x = ex$ for some e in N. Now $(e - x)x^2 = 0$. Hence by (a), $(e - x)x = 0$. Thus $x = ex = x^2$.

(b) $\Rightarrow$ (a) : Let $a \in N$. Clearly $l(a) \subseteq l(a^2)$. Suppose $ya^2 = 0$ for some y in N. Now $(aya)^2 = ay(aa)ya = aya^2ya = a0$ so that $(aya)^3 = (aya)^2 aya = a0(aya) = a0$. Thus $(aya)^3 = (aya)^2$. Hence by (b), $aya = (aya)^2 = a0$. Now $(ya)^2 = y(aya) = ya0$ so that $(ya)^3 = (ya)^2 ya = (ya0)ya = ya0$. Thus $(ya)^3 = (ya)^2$. Again by (b), $(ya) = (ya)^2 = ya0$. Now $0 = ya^2 = (ya)a = (ya0)a = ya0 = ya$. Hence $y \in l(a)$ that is $l(a^2) \subseteq l(a)$. Thus $l(a) = l(a^2)$. ∎

Definition 1.2. *A near-ring N is said to be left bipotent if $Na = Na^2$ for all $a \in N$.*

Definition 1.3. *A near-ring N is strongly regular if for each a in N, there exists an element b in N such that $a = ba^2$.*

Lemma 1.4. *Let N be a left bipotent S-near-ring. Then the following are true:*

(a) $x^3 = x^2$ *implies* $x^2 = x$ *for every x in N,*

(b) $l(a) = l(a^2)$ *for every a in N.*

Proof. (a) Suppose $x^3 = x^2$ for some x in N. Since N is a left bipotent S-near-ring, $x \in Nx = Nx^2$. Hence $x = yx^2$ for some y in N so that $x^2 = yx^3 = yx^2 = x$.
(b) follows from Lemma 1.3 and from (a). ∎

Theorem 1.5. *If N is zero-symmetric and regular near-ring then the following are equivalent:*

(a) *N is subcommutative,*

(b) *N is left bipotent,*

(c) $N = l(a) \oplus Na$ *for each a in N,*

(d) *N is weakly π-subcommutative.*

Proof. (a) $\Rightarrow$ (b): Clearly $Na^2 \subseteq Na$. Let $a \in N$. Now $a = axa$ for some x in N. Since N is subcommutative, $ax = ya$ for some y in N. Hence $Na = Naxa = Nyaa = Nya^2 \subseteq Na^2$. These implies that N is left bipotent.

(b) $\Rightarrow$ (c): Let a be in N. Since N is left bipotent, $Na = Na^2$. Hence for any $n \in N$, $na = xa^2$ for some x in N. Since $n = (n - xa) + xa$ and $(n - xa) \in l(a)$ we hav $N = l(a) + Na$. Hence by [5, Corollary 2.7], $l(a)$ is an ideal and since N is regular, $a = axa$ for some x in N. Let $e = xa$ then clearly $Na = Ne$ and also we can easily shows that $Na = Ne = l(1 - e)$. These implies that Na is an ideal. Suppose $y \in l(a) \cap Na$. Then $ya = 0$ and $y = za$ for some z in N. Hence

$0 = ya = (za)a = za^2$ so that $z \in l(a^2)$. By Lemma 1.4, we have $z \in l(a)$. Thus $0 = za = y$. Hence the result.

(c) $\Rightarrow$ (d): Since $N = l(a) \oplus Na$ for each a in N, $1 = y + na$ for some n in N. Then $a = ya + na^2$ and hence $a = na^2$ for some n in N. This implies that N is strongly regular. Hence by [6, 9.158] idempotent elements are central. Let $a = axa$ for some x in N. For any $n \in N$, $na^2 = (na)a = n(axa)a = axnaa \in aN$ and so we have $Na^2 \subseteq aN$. Next $a^2n = a(axa)n = aanxa \subseteq Na$ and so we have $a^2N \subseteq Na$. These implies that N is weakly π-subcommutative.

(d) $\Rightarrow$ (a): Let a be in N. Since N is regular, $a = axa$ for some x in N. Since ax and xa are idempotents by Lemma 1.1, $Na = Naxa = axNa \subseteq aN$ and $aN = axaN = aNxa \subseteq Na$. Hence N is subcommutative. ∎

Remark 1.2. The above theorem will fail if N is not zero-symmetric. Consider the near-ring $(N, +, \cdot)$ defined on the Klein's four group $(N, +)$ with $N = \{0, a, b, c\}$, where $\cdot$ is defined as follows (as per scheme 20, p.408 of Pilz [6]).

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	a	b	c
c	a	0	c	b

$(N, +, \cdot)$ is a near-ring which is regular with identity and left bipotent but not subcommutative, since $aN = \{a\}$ where as $Na = \{0, a\}$.

Definition 1.4. *A near-ring N is said to be Expansion Possible near-ring (EP near-ring) if given a, b in N for all $k \geq 1$, $(a+b)^{2k} = \sum n_i a^k n_j + \sum n_i' b^k n_j' + \sum a^k b^k$ where each summation contains only a finite number of terms.*

Commutative rings, rings which are subcommutative and strongly regular near-rings are EP near-rings.

Theorem 1.6. *If N is an EP near-ring and N is weakly π-subcommutative, then for any two-sided N-subgroup K, $\sqrt{K}$ is also a two-sided N-subgroup of N.*

Proof. Let $a, b \in \sqrt{K}$ there is a natural number k_1, k_2 such that $a^{k_1}, b^{k_2} \in K$. Let $k = \max\{k_1, k_2\}$. Therefore $a^k, b^k \in K$. Since N is an EP near-ring, $(a+b)^{2k} = \sum n_i a^k n_j + \sum n_i' b^k n_j' + \sum a^k b^k \in K$ where each summation contains only a finite number of terms. Therefore $(a+b) \in \sqrt{K}$. Hence $\sqrt{K}$ is a subgroup of N. Let $g \in \sqrt{K}$ and let $n \in N$. Since N is weakly π-subcommutative, there exists a positive integer t such that $N(gn)^t \subseteq (gn)N$. By induction on k, we will prove $(gn)^{kt} \in g^k N$. When $k = 1$ the assertion is clearly true. Suppose that the assertion is true for k that is $(gn)^{kt} = g^k h$ for some $h \in N$. Since $N(gn)^t \subseteq (gn)N$, $h(gn)^t = (gn)n_1$, for some $n_1 \in N$. Then $(gn)^{(k+1)t} = (gn)^{kt}(gn)^t = g^k h(gn)^t = g^k(gn)n_1 \in g^{k+1}N$. This completes the induction. Hence we obtain $gn \in \sqrt{K}$. Since N is weakly π-subcommutative, there exists a positive integer m such that $Ng^m \subseteq gN$. Then $(ng)^{mt+1} = n(gn)^{mt}g \in Ng^m N \subseteq gN \subseteq \sqrt{K}$. Hence $ng \in \sqrt{K}$. These proves that $\sqrt{K}$ is a two-sided N-subgroup of N. ∎

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