Generalized Fuzzy k-Ideals of Semirings with Interval-Valued Membership Functions

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Abstract. The concept of quasi-coincidence of an interval-valued fuzzy set is considered. By using this new idea, the notion of interval-valued (α, β)-fuzzy k-ideals of semirings is introduced, which is a generalization of a fuzzy k-ideal. Also some related properties are studied and in particular, the interval-valued $(\in, \in \vee q)$-fuzzy k-ideals in a semiring will be investigated. Finally, the concept of implication-based interval-valued fuzzy k-ideals and its basic properties in logical view are considered.

1. Introduction

After the introduction of fuzzy sets by Zadeh [22], there have been a number of generalizations of this fundamental concept. In 1975, Zadeh [23] introduced the concept of interval-valued fuzzy subsets, where the values of the membership functions are intervals of numbers instead of the numbers. Such fuzzy sets have some applications in the technological scheme of the functioning of a silo-farm with pneumatic transportation, in a plastic products company and in medicine (see [1]). The fuzzy algebraic structures play a prominent role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic, set theory, group theory, groupoids, real analysis, measure theory etc (for instance see [4–6], [8], [11–14], [16–21], [24] and [25]).

Ideals of hemirings and semirings play a central role in the structure theory and are useful for many purposes. However, they do not in general coincide with the usual ring ideals. Many results in rings apparently have no analogous in hemirings and semirings using only ideals. Henriksen [9] defined in a more restricted class of ideals in semirings, which is called the class of k-ideals, with the property that if the

semiring R is a ring then a complex in R is a k-ideal if and only if it is a ring ideal. Another more restricted, but very important, class of ideals in hemirings, called now h-ideals, has been given and investigated by Izuka [10] and La Torre [15]. It is interesting that the regularity of hemirings can be characterized by fuzzy h-ideals. General properties of fuzzy k-ideals are described in [6], [8] and [14]. Other important results connected with fuzzy h-ideals in hemirings were obtained in [13] and [24]. A new type of fuzzy subgroups($(\in, \in \vee q)$-fuzzy subgroups) was introduced in an earlier paper of Bhakat and Das [3] by using the combined notions of belongingness and quasi-coincidence of fuzzy points and fuzzy sets. In fact, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. This concept has been studied further in [2]. The aim of this paper is to introduce and study new different sorts of interval-valued fuzzy k-ideals of a semiring and to investigate the new aspects of related properties. The combined notions of belongingness and quasi-coincidence (in different cases) of interval-valued fuzzy points and fuzzy sets were used to introduce these sorts of interval-valued fuzzy k-ideals. Also, the definition of implication operator in the Lukasiewicz system of continuous-valued logic for interval-valued fuzzy k-ideals was considered. In particular, the relationship between interval valued fuzzy k-ideals with thresholds and implication-based interval-valued fuzzy k-ideals (under some important implication operators) are investigated.

2. Preliminaries and notations

By a *semiring* we mean an algebraic system $(R, +, .)$ consisting of a nonempty set R together with binary operations on R called add and multiplication such that $(R, +)$ and $(R, .)$ are semigroups and for all $x, y, z \in R$, we have $x.(y + z) = x.y + x.z$ and $(y + z).x = y.x + z.x$, which are called distributive. By a *zero* we mean an element $0 \in R$ such that $0.x = 0 = x.0$ and $0 + x = x = x + 0$ for all $x \in R$. By a *subsemiring* of a semiring R we mean a non-empty subset S of R such that for all $x, y \in S$, we have $x.y \in S$ and $x + y \in S$. By a *left (right) ideal* of a semiring R we mean a subsemiring I of R such that for all $y \in R$ and $x \in I$ we have $y.x \in I$ $(x.y \in I)$. By an *ideal*, we mean a subsemiring of R which both a left and a right ideal of R. A left (right) ideal I of a semiring R is called *left (right) k-ideal* if $y \in I$ and $x + y \in I$ imply that $x \in I$ (see [7]).

Definition 2.1. [7] *Let R_1 and R_2 be two semirings. A mapping $f : R_1 \longrightarrow R_2$ is called a homomorphism if for all $x, y \in R_1$ we have $f(x + y) = f(x) + f(y)$ and $f(x.y) = f(x).f(y)$.*

Lemma 2.1. [7] *Let R_1 and R_2 be two semirings and $f : R_1 \longrightarrow R_2$ an on-to homomorphism.*

   (i) *If I is a k-ideal of R_1, then $f(I)$ is a k-ideal of R_2.*
   (ii) *If J is a k-ideal of R_2, then $f^{-1}(J)$ is a k-ideal of R_1.*

Proof. Straightforward. ∎

   A fuzzy set μ is called a *fuzzy left ideal* of semiring R if for all $x, y \in R$ we have

(I1) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$,
(III1) $\mu(xy) \geq \mu(y)$.

A fuzzy left ideal μ is called a *fuzzy left k-ideal* of a semiring R if for all $x, y \in R$ we have (see [6], [8] and [14])

(III1) $\mu(x) \geq \mu(y) \wedge \mu(x + y)$.

Definition 2.2. [5, 25] *A fuzzy set μ in a set R defined by $\mu(y) = t \neq 0$ if $y = x$; and $\mu(y) = 0$ if $y \neq x$, is said to be a fuzzy point with support x and value t and is denoted x_t. A fuzzy point x_t is said to be belong to (resp. be quasi-coincident with) a fuzzy set μ, written as $x_t \in \mu$ (resp. $x_t q \mu$) if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$). If $x_t \in \mu$ or $x_t q \mu$, then we write $x_t \in \vee q \, \mu$. The symbol $\overline{\in \vee q}$ means neither $\in$ nor q hold.*

By an *interval number* $\widetilde{a}$ we mean [23] an interval $[a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. The set of all interval number is denoted by $D[0, 1]$. The interval $[a, a]$ is identified with the number $a \in [0, 1]$. For interval numbers $\widetilde{a}_i = [a_i^-, a_i^+] \in D[0, 1], i \in I$, we define

$$\inf \widetilde{a}_i = [\bigwedge_{i \in I} a_i^-, \bigwedge_{i \in I} a_i^+], \quad \sup \widetilde{a}_i = [\bigvee_{i \in I} a_i^-, \bigvee_{i \in I} a_i^+]$$

and put

  (i) $\widetilde{a}_1 \leq \widetilde{a}_2 \iff a_1^- \leq a_2^-$ and $a_1^+ \leq a_2^+$,
  (ii) $\widetilde{a}_1 = \widetilde{a}_2 \iff a_1^- = a_2^-$ and $a_1^+ = a_2^+$,
  (iii) $\widetilde{a}_1 < \widetilde{a}_2 \iff \widetilde{a}_1 \leq \widetilde{a}_2$ and $\widetilde{a}_1 \neq \widetilde{a}_2$,
  (iv) $k\widetilde{a} = [ka^-, ka^+]$, whenever $0 \leq k \leq 1$.

It is clear that $(D[0, 1], \leq, \bigvee, \bigwedge)$ is a complete lattice with $0 = [0, 0]$ as the least element and $1 = [1, 1]$ as the greatest element.

By an *interval number fuzzy set F* on X we mean [23] the set

$$F = \{(x, [\mu_F^-(x), \mu_F^+(x)]) \mid x \in X\},$$

where μ_F^- and μ_F^+ are two fuzzy subset of X such that $\mu_F^-(x) \leq \mu_F^+(x)$ for all $x \in X$. Putting $\widetilde{\mu_F}(x) = [\mu_F^-(x), \mu_F^+(x)]$, we see that $F = \{(x, \widetilde{\mu_F}(x)) \mid x \in X\}$, where $\widetilde{\mu_F} : X \longrightarrow D[0, 1]$.

3. Interval-valued (α, β)-fuzzy k-ideals

The concept of quasi-coincidence of fuzzy point can be extended to the concept of quasi-coincidence of a fuzzy interval valued fuzzy set. An interval-valued fuzzy set F of a semiring R of the form

$$\widetilde{\mu_F}(y) = \begin{cases} \widetilde{t}(\neq [0, 0]) & \text{if } y = x, \\ [0, 0] & \text{if } y \neq x, \end{cases}$$

is said to be the *interval-valued fuzzy point* with support x and interval-valued $\widetilde{t}$ and is denoted by $x_{\widetilde{t}}$. An interval-valued fuzzy point $x_{\widetilde{t}}$ is said to *belong to* (resp. be *quasi-coincident with*) an interval-valued fuzzy set F, written as $x_{\widetilde{t}} \in F$ (resp. $x_{\widetilde{t}} q F$) if $\widetilde{\mu_F}(x) \geq \widetilde{t}$ (resp. $\widetilde{\mu_F}(x) + \widetilde{t} > [1, 1]$). If $x_{\widetilde{t}} \in F$ or (resp. and) $x_{\widetilde{t}} q F$, then we write $x_{\widetilde{t}} \in \vee q F$ (resp. $x_{\widetilde{t}} \in \wedge q F$).

In what follows, let R be a semiring. Then we use α and β to denote any one of the $\in, q, \in \vee q$ or $\in \wedge q$ unless otherwise specified. We also emphasis that $\widetilde{\mu_F} = [\mu_F^-, \mu_F^+]$ must satisfy the following conditions:

(i) Any two elements of $D[0,1]$ are comparable,

(ii) $[\mu_F^-(x), \mu_F^+(x)] \leq [0.5, 0.5]$ or $[\mu_F^-(x), \mu_F^+(x)] > [0.5, 0.5]$, for all $x \in R$.

Definition 3.1. *An interval-valued fuzzy set F of R is called an interval-valued (α, β)-fuzzy k-ideal of R if for all $t, r \in (0, 1]$ and $a, x, y \in R$, the following conditions hold:*

(I2) $x_{\widetilde{t}}\alpha F$ and $y_{\widetilde{r}}\alpha F$ imply $(x + y)_{\widetilde{t} \wedge \widetilde{r}}\beta F$,

(II2) $x_{\widetilde{t}}\alpha F$ implies $(xa)_{\widetilde{t}}\beta F$ and $(ax)_{\widetilde{t}}\beta F$,

(III2) $(x + a)_{\widetilde{t}}\alpha F$, and $a_{\widetilde{r}}\alpha F$ imply $x_{\widetilde{t} \wedge \widetilde{r}}\beta F$.

Let F be an interval-valued fuzzy set of R such that $\widetilde{\mu_F}(x) \leq [0.5, 0.5]$, for all $x \in R$. Suppose that $x \in R$ and $t \in (0, 1]$ such that $x_{\widetilde{t}} \in \wedge qF$. Then $\widetilde{\mu_F}(x) \geq \widetilde{t}$ and $\widetilde{\mu_F}(x) + \widetilde{t} > [1, 1]$. It follows that $[1, 1] < \widetilde{\mu_F}(x) + \widetilde{t} \leq \widetilde{\mu_F}(x) + \widetilde{\mu_F}(x) = 2\widetilde{\mu_F}(x)$, which implies that $\widetilde{\mu_F}(x) > [0.5, 0.5]$. This means that $\{x_{\widetilde{t}} \mid x_{\widetilde{t}} \in \wedge qF\} = \varnothing$. Therefore the case $\alpha = \in \wedge q$ in the Definition 3.1 can be removed.

Proposition 3.1. *Every interval-valued $(\in \vee q, \in \vee q)$-fuzzy k-ideal of R is an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R.*

Proof. Let F be an interval-valued $(\in \vee q, \in \vee q)$-fuzzy k-ideal of R. Let $a, x, y \in R$ and $t, r \in (0, 1]$ be such that $x_{\widetilde{t}} \in F$ and $y_{\widetilde{r}} \in F$. Then $x_{\widetilde{t}} \in \vee qF$ and $y_{\widetilde{r}} \in \vee qF$. It follows that $(x + y)_{\widetilde{t} \wedge \widetilde{r}} \in \vee qF, (xa)_{\widetilde{t}} \in \vee qF$ and $(ax)_{\widetilde{t}} \in \vee qF$. Also let $(x + a)_{\widetilde{t}} \in F$, and $a_{\widetilde{r}} \in F$, then $(x + a)_{\widetilde{t}} \in \vee qF$, and $a_{\widetilde{r}} \in \vee qF$. It follows that $x_{\widetilde{t} \wedge \widetilde{r}} \in \vee qF$, which completes the proof. ∎

Proposition 3.2. *Every interval-valued $(\in, \in)$-fuzzy k-ideal of R is an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R.*

Proof. The proof is clear by considering the definitions. ∎

Lemma 3.1. *Let I be a k-ideal of R, then χ_I (the characteristic function of I) is an interval valued $(\in, \in)$-fuzzy k-ideal of R.*

Proof. Let $a, x, y \in R$ and $t, r \in (0, 1]$ be such that $x_{\widetilde{t}} \in \chi_I$ and $y_{\widetilde{r}} \in \chi_I$. Then $\widetilde{\chi_I}(x) \geq \widetilde{t} > [0, 0]$ and $\widetilde{\chi_I}(y) \geq \widetilde{r} > [0, 0]$. These imply $\widetilde{\chi_I}(x) = \widetilde{\chi_I}(y) = [1, 1]$, and so $x, y \in I$, thus $x + y \in I$ and $ax \in I$ and $xa \in I$. It follows that $\widetilde{\chi_I}(x + y) = [1, 1] \geq \widetilde{t} \wedge \widetilde{r}$ and $\widetilde{\chi_I}(ax) = [1, 1] \geq \widetilde{t}$ and $\widetilde{\chi_I}(xa) = [1, 1] \geq \widetilde{t}$, which means $(x + y)_{\widetilde{t} \wedge \widetilde{r}} \in \chi_I$ and $(xa)_{\widetilde{t}} \in \chi_I$. Also let $(x + a)_{\widetilde{t}} \in \chi_I$ and $a_{\widetilde{r}} \in \chi_I$. Then $\widetilde{\chi_I}(x + a) \geq \widetilde{t} > [0, 0]$ and $\widetilde{\chi_I}(a) \geq \widetilde{r} > [0, 0]$. These imply $\widetilde{\chi_I}(x + a) = \widetilde{\chi_I}(a) = [1, 1]$, and so $x + a, a \in I$, thus $x \in I$. It follows that $\widetilde{\chi_I}(x) = [1, 1] \geq \widetilde{t} \wedge \widetilde{r}$, which means $x_{\widetilde{t} \wedge \widetilde{r}} \in \chi_I$. Therefore χ_I is an interval-valued $(\in, \in)$-fuzzy k-ideal of R. ∎

Theorem 3.1. *For any subset I of R, χ_I is an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R if and only if I is a k-ideal of R.*

Proof. Let χ_I be an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R. If $x, y \in I$ and $a \in R$, then $x_{[1,1]} \in \chi_I$ and $y_{[1,1]} \in \chi_I$. These imply $(x + y)_{[1,1]} = (x + y)_{[1,1] \wedge [1,1]} \in \vee q\chi_I$ and $(ax)_{[1,1]} \in \vee q\chi_I$ and $(xa)_{[1,1]} \in \vee q\chi_I$. Hence $\widetilde{\chi_I}(x + y) > [0, 0]$ and $\widetilde{\chi_I}(ax) > [0, 0]$, and $\widetilde{\chi_I}(xa) > [0, 0]$, so $x + y \in I, ax \in I$ and $xa \in I$. Also if $x + a \in I$ and $a \in I$, then $(x + a)_{[1,1]} \in \chi_I$ and $a_{[1,1]} \in \chi_I$. These imply $x_{[1,1]} = x_{[1,1] \wedge [1,1]} \in \vee q\chi_I$. Hence $\widetilde{\chi_I}(x) > [0, 0]$, so $x \in I$. Therefore I is a k-ideal of R. Conversely, if I is a

k-ideal of R, then χ_I is an interval-valued $(\in, \in)$-fuzzy k-ideal of R by Lemma 3.1. Therefore χ_I is an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R by Proposition 3.2. ∎

Theorem 3.2. *Let F be a non-zero interval valued (α, β)-fuzzy k-ideal of R. Then the set* $\mathrm{supp}(\widetilde{\mu_F}) = \{x \in R \mid \widetilde{\mu_F}(x) > [0, 0]\}$ *is a k-ideal of R.*

Proof. Let $x, y \in \mathrm{supp}(\widetilde{\mu_F})$ and $a \in R$, then $\widetilde{\mu_F}(x) > [0, 0]$ and $\widetilde{\mu_F}(y) > [0, 0]$. Now, we assume that $\widetilde{\mu_F}(x + y) = [0, 0]$. If $\alpha \in \{\in, \in \vee q\}$, then $x_{\widetilde{\mu_F}(x)}\alpha F$ and $y_{\widetilde{\mu_F}(y)}\alpha F$, but $(x + y)_{\widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y)}\overline{\beta}F$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Also $x_{[1,1]}qF$ and $y_{[1,1]}qF$, but $(x + y)_{[1,1] \wedge [1,1]} = (x + y)_{[1,1]}\overline{\beta}F$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Hence $\widetilde{\mu_F}(x + y) > [0, 0]$, that is $x + y \in \mathrm{supp}(\widetilde{\mu_F})$. Similarly we can prove that $ax \in \mathrm{supp}(\widetilde{\mu_F})$ and $xa \in \mathrm{supp}(\widetilde{\mu_F})$. Also let $x + b \in \mathrm{supp}(\widetilde{\mu_F})$ and $b \in \mathrm{supp}(\widetilde{\mu_F})$, then $\widetilde{\mu_F}(x + b) > [0, 0]$ and $\widetilde{\mu_F}(b) > [0, 0]$. Assume that $\widetilde{\mu_F}(x) = [0, 0]$. If $\alpha \in \{\in, \in \vee q\}$, then $(x + b)_{\widetilde{\mu_F}(x+b)}\alpha F$ and $b_{\widetilde{\mu_F}(b)}\alpha F$ but $x_{\widetilde{\mu_F}(x+b) \wedge \widetilde{\mu_F}(b)}\overline{\beta}F$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Also $(x + b)_{[1,1]}qF$ and $b_{[1,1]}qF$, but $x_{[1,1] \wedge [1,1]} = x_{[1,1]}\overline{\beta}F$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Hence $\widetilde{\mu_F}(x) > [0, 0]$, that is $x \in \mathrm{supp}(\widetilde{\mu_F})$. Therefore $\mathrm{supp}(\widetilde{\mu_F})$ is a k-ideal of R. ∎

Let F be an interval-valued fuzzy set. Then, for every $t \in [0, 1]$, the set $F_{\widetilde{t}} = \{x \in R \mid \widetilde{\mu_F}(x) \geq \widetilde{t}\}$ is called the *interval-valued level subset* of F. An interval-valued fuzzy set F of R is called *proper* if $\Im F$ contains at least two elements. Two interval-valued fuzzy sets are said to be *equivalent* if they have same family of interval valued level subsets. Otherwise, they are said to be *non-equivalent*.

Theorem 3.3. *Suppose that R contains some proper k-ideals. Then a proper interval-valued $(\in, \in)$-fuzzy k-ideal F of R with $|\Im F| \geq 3$ can be expressed as the union of two proper non-equivalent interval-valued $(\in, \in)$-fuzzy k-ideal of R.*

Proof. Let F be a proper interval-valued $(\in, \in)$-fuzzy k-ideal of R with $\Im F = \{\widetilde{t_0}, \widetilde{t_1}, ..., \widetilde{t_n}\}$, where $\widetilde{t_0} > \widetilde{t_1} > ... > \widetilde{t_n}$ and $n \geq 2$. Then $F_{\widetilde{t_0}} \subseteq F_{\widetilde{t_1}} \subseteq ... \subseteq F_{\widetilde{t_n}} = R$ is the chain of interval-valued level k-ideals of F. Define two interval-valued fuzzy sets G and H in R by

$$\widetilde{\mu_G}(x) = \begin{cases} \widetilde{r_1} & \text{if } x \in F_{\widetilde{t_1}}, \\ \widetilde{t_k} & \text{if } x \in F_{\widetilde{t_k}} \setminus F_{\widetilde{t_{k-1}}} \text{ and } 2 \leq k \leq n \end{cases}$$

$$\widetilde{\mu_H}(x) = \begin{cases} \widetilde{t_0} & \text{if } x \in F_{\widetilde{t_0}}, \\ \widetilde{t_1} & \text{if } x \in F_{\widetilde{t_1}} \setminus F_{\widetilde{t_0}}, \\ \widetilde{r_2} & \text{if } x \in F_{\widetilde{t_3}} \setminus F_{\widetilde{t_1}}, \\ \widetilde{t_k} & \text{if } x \in F_{\widetilde{t_k}} \setminus F_{\widetilde{t_{k-1}}} \text{ and } 4 \leq k \leq n \end{cases}$$

such that $\widetilde{t_1} < \widetilde{r_1} < \widetilde{t_0}$ and $\widetilde{t_3} < \widetilde{r_2} < \widetilde{t_2}$. Then G and H are interval-valued $(\in, \in)$-fuzzy k-ideals of R, where $G_{\widetilde{r_1}} \subseteq G_{\widetilde{t_2}} \subseteq ... \subseteq G_{\widetilde{t_n}} = R$, and $H_{\widetilde{t_0}} \subseteq H_{\widetilde{t_1}} \subseteq ... \subseteq H_{\widetilde{t_n}} = R$ are respectively the chain of interval-valued level k-ideals of R, and $G, H \leq F$. Thus G and H are non-equivalent, and it is obvious that $G \cup H = F$. Therefore F can be expressed as the union of two proper non-equivalent interval-valued $(\in, \in)$-fuzzy k-ideal of R. ∎

4. Interval-valued $(\in, \in \vee q)$-fuzzy k-ideals

Definition 4.1. *An interval-valued fuzzy set F of R is called an interval-valued fuzzy k-ideal of R if for all $a, x, y \in R$, it satisfies the following conditions:*

(I3) $\widetilde{\mu_F}(x+y) \geq \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y)$,

(II3) $\widetilde{\mu_F}(ax) \geq \widetilde{\mu_F}(x)$ *and* $\widetilde{\mu_F}(xa) \geq \widetilde{\mu_F}(x)$,

(III3) $\widetilde{\mu_F}(x) \geq \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a)$.

Theorem 4.1. *An interval-valued fuzzy set F of R is an interval-valued fuzzy k-ideal of R if and only if for any $[0,0] < \widetilde{t} \leq [1,1]$, $F_{\widetilde{t}}(\neq \varnothing)$ is a k-ideal of R.*

Proof. Let F be An interval-valued fuzzy k-ideal of R and $[0,0] < \widetilde{t} \leq [1,1]$ such that $F_{\widetilde{t}}(\neq \varnothing)$. Also let $x, y \in F_{\widetilde{t}}$, then $\widetilde{\mu_F}(x) \geq \widetilde{t}$ and $\widetilde{\mu_F}(y) \geq \widetilde{t}$. So $\widetilde{\mu_F}(x+y) \geq \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) \geq \widetilde{t}$ and hence $x + y \in F_{\widetilde{t}}$. Similarly we can show that $ax \in F_{\widetilde{t}}$ and $xa \in F_{\widetilde{t}}$ for all $a \in R$. Also let $x + b \in F_{\widetilde{t}}$ and $b \in F_{\widetilde{t}}$, then $\widetilde{\mu_F}(x+b) \geq \widetilde{t}$ and $\widetilde{\mu_F}(b) \geq \widetilde{t}$. So $\widetilde{\mu_F}(x) \geq \widetilde{\mu_F}(x+b) \wedge \widetilde{\mu_F}(b) \geq \widetilde{t}$. Therefore $F_{\widetilde{t}}$ is a k-ideal of R. Conversely, suppose that for any $[0,0] < \widetilde{t} \leq [1,1]$, $F_{\widetilde{t}}(\neq \varnothing)$ is a k-ideal of R. Let $a, x, y \in R$ and $\widetilde{\mu_F}(x) = \widetilde{t_1}$ and $\widetilde{\mu_F}(y) = \widetilde{t_2}$. Then $x, y \in F_{\widetilde{t_1} \wedge \widetilde{t_2}}$. So $x+y \in F_{\widetilde{t_1} \wedge \widetilde{t_2}}$, thus $\widetilde{\mu_F}(x+y) \geq \widetilde{t_1} \wedge \widetilde{t_2} = \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y)$. Similarly, we can prove that $\widetilde{\mu_F}(ax) \geq \widetilde{\mu_F}(x)$ and $\widetilde{\mu_F}(xa) \geq \widetilde{\mu_F}(x)$. Also let $b \in R$ and $\widetilde{\mu_F}(x+b) = \widetilde{t_1}$ and $\widetilde{\mu_F}(b) = \widetilde{t_2}$. Then $x + b, b \in F_{\widetilde{t_1} \wedge \widetilde{t_2}}$, so $x \in F_{\widetilde{t_1} \wedge \widetilde{t_2}}$, thus $\widetilde{\mu_F}(x) \geq \widetilde{t_1} \wedge \widetilde{t_2} = \widetilde{\mu_F}(x+b) \wedge \widetilde{\mu_F}(b)$. Therefore F is an interval-valued fuzzy k-ideal of R. ∎

Definition 4.2. *An interval-valued fuzzy set F of R is said to be an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R is if for all $t, r \in (0,1]$ and $a, x, y \in R$, the following conditions hold:*

(I4) $x_{\widetilde{t}} \in F$ *and* $y_{\widetilde{r}} \in F$ *imply* $(x+y)_{\widetilde{t} \wedge \widetilde{r}} \in \vee q F$,

(II4) $x_{\widetilde{t}} \in F$ *implies* $(ax)_{\widetilde{t}} \in \vee q F$ *and* $(xa)_{\widetilde{t}} \in \vee q F$,

(III4) $(x+a)_{\widetilde{t}} \in F$ *and* $a_{\widetilde{r}} \in F$ *imply* $x_{\widetilde{t} \wedge \widetilde{r}} \in \vee q F$.

Note that if F is an interval-valued fuzzy k-ideal of R by Definition 4.1, then it is an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R by Definition 4.2. But the converse, as can be seen in the following examples, is not true in general case.

Example 4.1. We know that $(\mathbb{N}, +, .)$ is a semiring. Define an interval-valued fuzzy set F in $\mathbb{N}$ by

$$\widetilde{\mu_F}(x) = \begin{cases} [0.9, 1] & \text{if } x \in \langle 4 \rangle, \\ [0.5, 0.6] & \text{if } x \in \langle 2 \rangle \setminus \langle 4 \rangle, \\ [0, 0.1] & \text{otherwise,} \end{cases}$$

where $\langle n \rangle$ denotes the set of all integers divide by n. It is routine to calculate that F is an interval-valued fuzzy k-ideal of $(\mathbb{N}, +, .)$ and therefore F is an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of $(\mathbb{N}, +, .)$.

Example 4.2. Consider the semiring $(\mathbb{N}, +, .)$. Define an interval-valued fuzzy set F in $\mathbb{N}$ by

$$\widetilde{\mu_F}(x) = \begin{cases} [0.7, 0.8] & \text{if } 0 < x < 5, \\ [0.8, 0.9] & \text{if } 5 \leq x < 7, \\ [0.9, 1] & \text{if } x \geq 7. \end{cases}$$

It is easy to verify that F is an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of $(\mathbb{N}, +, .)$, but it is not an interval-valued fuzzy k-ideal of $(\mathbb{N}, +, .)$, because $F_{[0.9,1]} = \{7, 8, 9, ...\}$ is not a k-ideal of $(\mathbb{N}, +, .)$.

Theorem 4.2. *The conditions $(I4)$, $(II4)$ and $III4$ in Definition 4.2, are equivalent to the following conditions, respectively for all $a, x, y \in R$:*

(I5) $\widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) \wedge [0.5, 0.5] \leq \widetilde{\mu_F}(x + y)$,
(II5) $\widetilde{\mu_F}(x) \wedge [0.5, 0.5] \leq \widetilde{\mu_F}(ax)$ *and* $\widetilde{\mu_F}(x) \wedge [0.5, 0.5] \leq \widetilde{\mu_F}(xa)$,
(III5) $\widetilde{\mu_F}(x + a) \wedge \widetilde{\mu_F}(a) \wedge [0.5, 0.5] \leq \widetilde{\mu_F}(x)$.

Proof.
(I4)$\Longrightarrow$(I5): Suppose that $x, y \in R$. Then we consider the following cases:

(a) $\widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) \leq [0.5.0.5]$. In this case, assume that $\widetilde{\mu_F}(x + y) < \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) \wedge [0.5, 0.5]$. Then, it implies that $\widetilde{\mu_F}(x + y) < \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y)$. Choose $\widetilde{t}$ such that $\widetilde{\mu_F}(x + y) < \widetilde{t} < \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y)$. Then $x_{\widetilde{t}} \in F$ and $y_{\widetilde{t}} \in F$, but $(x + y)_{\widetilde{t}} \overline{\in \vee q} F$, which contradicts (I4).
(b) $\widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) > [0.5, 0.5]$. In this case, assume that $\widetilde{\mu_F}(x + y) < [0.5, 0.5]$ Then $x_{[0.5,0.5]} \in F$ and $y_{[0.5,0.5]} \in F$, but $(x + y)_{[0.5,0.5]} \overline{\in \vee q} F$, which is a contradiction. Hence (I5) holds.

(II4)$\Longrightarrow$(II5): The proof is similar to the proof of (I4)$\Longrightarrow$(I5).
(III4)$\Longrightarrow$(III5): Suppose that $a, x \in R$. Then we consider the following cases:

(a) $\widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a) \leq [0.5, 0.5]$. In this case, assume that $\widetilde{\mu_F}(x) < \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a) \wedge [0.5, 0.5]$. Then it implies that $\widetilde{\mu_F}(x) < \widetilde{\mu_F}(x + a) \wedge \widetilde{\mu_F}(a)$. Choose $\widetilde{t}$ such that $\widetilde{\mu_F}(x) < \widetilde{t} < \widetilde{\mu_F}(x + a) \wedge \widetilde{\mu_F}(a)$. Then $(x + a)_{\widetilde{t}} \in F$ and $a_{\widetilde{t}} \in F$, but $x_{\widetilde{t}} \overline{\in \vee q} F$, which contradicts (III4).
(b) $\widetilde{\mu_F}(x + a) \wedge \widetilde{\mu_F}(a) > [0.5, 0.5]$. In this case, assume that $\widetilde{\mu_F}(x) < [0.5, 0.5]$. Then $(x + a)_{[0.5,0.5]} \in F$ and $a_{[0.5,0.5]} \in F$, but $x_{[0.5,0.5]} \overline{\in \vee q} F$, which is a contradiction. Hence (III5) holds.

(I5)$\Longrightarrow$(I4): Let $x_{\widetilde{t}} \in F$ and $y_{\widetilde{r}} \in F$. Then $\widetilde{\mu_F}(x) \geq \widetilde{t}$ and $\widetilde{\mu_F}(y) \geq \widetilde{r}$. We have $\widetilde{\mu_F}(x + y) \geq \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) \wedge [0.5, 0.5] \geq \widetilde{t} \wedge \widetilde{r} \wedge [0.5, 0.5]$. We can consider two following cases:

(a) $\widetilde{t} \wedge \widetilde{r} > [0.5, 0.5]$, then $\widetilde{\mu_F}(x+y) \geq [0.5, 0.5]$, which implies $\widetilde{\mu_F}(x+y)+(\widetilde{t} \wedge \widetilde{r}) > [1, 1]$, or equivalently $(x + y)_{\widetilde{t} \wedge \widetilde{r}} q F$. Thus $(x + y)_{\widetilde{t} \wedge \widetilde{r}} \in \vee q F$.
(b) $\widetilde{t} \wedge \widetilde{r} \leq [0.5, 0.5]$, then $\widetilde{\mu_F}(x + y) \geq \widetilde{t} \wedge \widetilde{r}$, or equivalently $(x+y)_{\widetilde{t} \wedge \widetilde{r}} \in F$. Thus $(x + y)_{\widetilde{t} \wedge \widetilde{r}} \in \vee q F$.

(II5)$\Longrightarrow$(II4): The proof is similar to the proof of (I5)$\Longrightarrow$(I4).
(III5)$\Longrightarrow$(III4): Let $(x + a)_{\widetilde{t}} \in F$ and $a_{\widetilde{r}} \in F$. Then $\widetilde{\mu_F}(x + a) \geq \widetilde{t}$ and $\widetilde{\mu_F}(a) \geq \widetilde{r}$. We have $\widetilde{\mu_F}(x) \geq \widetilde{\mu_F}(x + a) \wedge \widetilde{\mu_F}(a) \wedge [0.5, 0.5] \geq \widetilde{t} \wedge \widetilde{r} \wedge [0.5, 0.5]$. We can consider two following cases:

(a) $\widetilde{t} \wedge \widetilde{r} > [0.5, 0.5]$, then $\widetilde{\mu_F}(x) \geq [0.5, 0.5]$, which implies $\widetilde{\mu_F}(x)+(\widetilde{t} \wedge \widetilde{r}) > [1, 1]$, or equivalently $x_{\widetilde{t} \wedge \widetilde{r}} q F$. Thus $x_{\widetilde{t} \wedge \widetilde{r}} \in \vee q F$.
(b) $\widetilde{t} \wedge \widetilde{r} \leq [0.5, 0.5]$, then $\widetilde{\mu_F}(x) \geq \widetilde{t} \wedge \widetilde{r}$, or equivalently $x_{\widetilde{t} \wedge \widetilde{r}} \in F$. Thus $x_{\widetilde{t} \wedge \widetilde{r}} \in \vee q F$. ∎

Corollary 4.1. *An interval-valued fuzzy set F of R is an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R if and only if conditions* (I5), (II5) *and* (III5) *in Theorem 4.2 hold.*

Theorem 4.3. *Let F be an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R. Then for all $[0,0] < \tilde{t} \leq [0.5, 0.5], F_{\tilde{t}} = \varnothing$ or $F_{\tilde{t}}$ is a k-ideal of R. Conversely, if F is an interval-valued fuzzy set of R such that $F_{\tilde{t}} (\neq \varnothing)$ is a k-ideal of R for all $[0,0] < \tilde{t} \leq [0.5, 0.5]$, then F is an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R.*

Proof. Let F be an interval-valued $(\in, \in \vee q)$- fuzzy k-ideal of R and $[0,0] < \tilde{t} \leq [0.5, 0.5]$. If $x, y \in F_{\tilde{t}}$ and $a \in R$, then $\widetilde{\mu_F}(x) \geq \tilde{t}$ and $\widetilde{\mu_F}(y) \geq \tilde{t}$. Hence $\widetilde{\mu_F}(x+y) \geq \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) \wedge [0.5, 0.5] \geq \tilde{t} \wedge [0.5, 0.5] = \tilde{t}$. Thus $x + y \in F_{\tilde{t}}$. Similarly, we can show that $ax \in F_{\tilde{t}}$ and $xa \in F_{\tilde{t}}$. Also if $x + b \in F_{\tilde{t}}$ and $b \in F_{\tilde{t}}$, then $\widetilde{\mu_F}(x+b) \geq \tilde{t}$ and $\widetilde{\mu_F}(b) \geq \tilde{t}$. Hence $\widetilde{\mu_F}(x) \geq \widetilde{\mu_F}(x+b) \wedge \widetilde{\mu_F}(b) \wedge [0.5, 0.5] \geq \tilde{t} \wedge [0.5, 0.5] = \tilde{t}$. Thus $x \in F_{\tilde{t}}$. Therefore $F_{\tilde{t}}$ is a k-ideal of R. Conversely, Let F be an interval-valued set of R such that $F_{\tilde{t}} (\neq \varnothing)$ is a k-ideal of R, for all $[0,0] < \tilde{t} \leq [0.5, 0.5]$, then for every $a, x, y \in R$, we can say $\widetilde{\mu_F}(x) \geq \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) \wedge [0.5, 0.5] = \tilde{t_0}$ and $\widetilde{\mu_F}(y) \geq \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) \wedge [0.5, 0.5] = \tilde{t_0}$ (if $\tilde{t_0} = [0,0]$, then always $\widetilde{\mu_F}(x+y) \geq [0,0] = \tilde{t_0} = \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) \wedge [0.5, 0.5]$, as desired). Hence $x, y \in F_{\tilde{t_0}}$. So $x + y \in F_{\tilde{t_0}}$. Thus $\widetilde{\mu_F}(x+y) \geq \tilde{t_0} = \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) \wedge [0.5, 0.5]$. Similarly we can show that $\widetilde{\mu_F}(ax) \geq \widetilde{\mu_F}(x) \wedge [0.5, 0.5]$ and $\widetilde{\mu_F}(xa) \geq \widetilde{\mu_F}(x) \wedge [0.5, 0.5]$. Also for every $a, x \in R$, we can say $\widetilde{\mu_F}(x+a) \geq \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a) \wedge [0.5, 0.5] = \tilde{t_1}$ and $\widetilde{\mu_F}(a) \geq \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a) \wedge [0.5, 0.5] = \tilde{t_1}$ (if $\tilde{t_1} = [0,0]$, then always $\widetilde{\mu_F}(a) \geq [0,0] = \tilde{t_1} = \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a) \wedge [0.5, 0.5]$, as desired). Hence $x + a \in F_{\tilde{t_1}}$ and $a \in F_{\tilde{t_1}}$. So $x \in F_{\tilde{t_1}}$. Thus $\widetilde{\mu_F}(x) \geq \tilde{t_1} = \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a) \wedge [0.5, 0.5]$. Therefore F is an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R. ∎

Theorem 4.4. *Let F be an interval-valued fuzzy set of R. Then $F_{\tilde{t}} (\neq \varnothing)$ is a k-ideal of R for all $[0.5, 0.5] < \tilde{t} \leq [1, 1]$ if and only if for all $a, x, y \in R$ the following conditions hold:*

   (I6) $\widetilde{\mu_F}(x+y) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y)$,
  (II6) $\widetilde{\mu_F}(ax) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x)$ *and* $\widetilde{\mu_F}(xa) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x)$,
 (III6) $\widetilde{\mu_F}(x) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a)$.

Proof. Assume that $F_{\tilde{t}} (\neq \varnothing)$ is a k-ideal of R for all $[0.5, 0.5] < \tilde{t} \leq [1, 1]$. If there exist $x, y \in R$ such that $\widetilde{\mu_F}(x+y) \vee [0.5, 0.5] < \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) = \tilde{t}$, then we have $[0.5, 0.5] < \tilde{t} \leq [1,1]$, $\widetilde{\mu_F}(x+y) < \tilde{t}$ and $x, y \in F_{\tilde{t}}$. So $x + y \in F_{\tilde{t}}$, which implies that $\widetilde{\mu_F}(x+y) \geq \tilde{t}$. This is a contradiction. Thus $\widetilde{\mu_F}(x+y) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y)$ for all $x, y \in R$. Similarly we can prove that $\widetilde{\mu_F}(ax) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x)$ and $\widetilde{\mu_F}(xa) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x)$. Therefore (I6) and (II6) hold for all $a, x, y \in R$. Also if there exist $a, x \in R$ such that $\widetilde{\mu_F}(x) \vee [0.5, 0.5] < \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a) = \tilde{t}'$, then we have $[0.5, 0.5] < \tilde{t}' \leq [1,1]$, $\widetilde{\mu_F}(x) < \tilde{t}'$ and $a, x + a \in F_{\tilde{t}'}$. So $x \in F_{\tilde{t}'}$, which implies that $\widetilde{\mu_F}(x) \geq \tilde{t}'$. This is a contradiction. Thus $\widetilde{\mu_F}(x) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a)$ for all $a, x \in R$. Therefore (III6) holds. Conversely, suppose that conditions (I6), (II6) and (III6) hold. Let $[0.5, 0.5] < \tilde{t} \leq [1, 1]$, $x, y \in F_{\tilde{t}}$ and $a \in R$. We have $[0.5, 0.5] < \tilde{t} \leq \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) \leq \widetilde{\mu_F}(x+y) \vee [0.5, 0.5]$, which implies $\widetilde{\mu_F}(x+y) \geq \tilde{t}$.

Thus $x + y \in F_{\widetilde{t}}$. Similarly we can prove that $ax \in F_{\widetilde{t}}$ and $xa \in F_{\widetilde{t}}$. Also if $x + b \in F_{\widetilde{t}}$ and $b \in F_{\widetilde{t}}$, then we have $[0.5, 0.5] < \widetilde{t} \leq \widetilde{\mu_F}(x + b) \wedge \widetilde{\mu_F}(b) \leq \widetilde{\mu_F}(x) \vee [0.5, 0.5]$, which implies that $\widetilde{\mu_F}(x) \geq \widetilde{t}$. Thus $x \in F_{\widetilde{t}}$. Therefore $F_{\widetilde{t}}$ is a k-ideal of R. ∎

Theorem 4.5. *Let $\{F_i\}_{i \in I}$ be a family of interval-valued $(\in, \in \vee q)$-fuzzy k-ideals of R. Then $\bigcap_{i \in I} F_i$ is an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R, where $\bigcap_{i \in I} F_i = \bigwedge_{i \in I} F_i$.*

Proof. Let $x, y \in R$, then we have

$$(\bigcap_{i \in I} \widetilde{\mu_{F_i}})(x + y) = \bigwedge_{i \in I} \widetilde{\mu_{F_i}}(x + y) \geq \bigwedge_{i \in I} (\widetilde{\mu_{F_i}}(x) \wedge \widetilde{\mu_{F_i}}(y) \wedge [0.5, 0.5])$$

$$= (\bigwedge_{i \in I} \widetilde{\mu_{F_i}}(x)) \wedge (\bigwedge_{i \in I} \widetilde{\mu_{F_i}}(y)) \wedge [0.5, 0.5]$$

$$= (\bigcap_{i \in I} \widetilde{\mu_{F_i}})(x) \wedge (\bigcap_{i \in I} \widetilde{\mu_{F_i}})(y) \wedge [0.5, 0.5].$$

Thus

$$(\bigcap_{i \in I} \widetilde{\mu_{F_i}})(x + y) \geq (\bigcap_{i \in I} \widetilde{\mu_{F_i}})(x) \wedge (\bigcap_{i \in I} \widetilde{\mu_{F_i}})(y) \wedge [0.5, 0.5].$$

Similarly, if $a, x \in R$ we can show that

$$(\bigcap_{i \in I} \widetilde{\mu_{F_i}})(ax) \geq (\bigcap_{i \in I} \widetilde{\mu_{F_i}})(x) \wedge [0.5, 0.5],$$

$$(\bigcap_{i \in I} \widetilde{\mu_{F_i}})(xa) \geq (\bigcap_{i \in I} \widetilde{\mu_{F_i}})(x) \wedge [0.5, 0.5]$$

and

$$(\bigcap_{i \in I} \widetilde{\mu_{F_i}})(x) \geq (\bigcap_{i \in I} \widetilde{\mu_{F_i}})(x + a) \wedge (\bigcap_{i \in I} \widetilde{\mu_{F_i}})(a) \wedge [0.5, 0.5].$$

Therefore $\bigcap_{i \in I} F_i$ is an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R. ∎

Example 4.3. In the semiring $(\mathbb{Z}, +, .)$, $2\mathbb{Z}$ and $3\mathbb{Z}$ are k-ideals, but $2\mathbb{Z} \cup 3\mathbb{Z}$ is not even an ideal (and consequently is not a k-ideal). So by Theorem 3.1, $\chi_{2\mathbb{Z}}$ and $\chi_{3\mathbb{Z}}$ are interval-valued $(\in, \in \vee q)$-fuzzy k-ideals of $\mathbb{Z}$, but $\chi_{2\mathbb{Z}} \cup \chi_{3\mathbb{Z}} = \chi_{2\mathbb{Z} \cup 3\mathbb{Z}}$ is not an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of $\mathbb{Z}$.

Theorem 4.6. *Let $\{F_i\}_{i \in I}$ be a family of interval-valued $(\in, \in \vee q)$-fuzzy k-ideals of R such that $F_i \subseteq F_j$ or $F_j \subseteq F_i$ for all $i, j \in I$. Then $\bigcup_{i \in I} F_i$ is an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R, where $\bigcup_{i \in I} F_i = \bigvee_{i \in I} F_i$.*

Proof. For all $x, y \in R$, we have

$$(\bigcup_{i \in I} \widetilde{\mu_{F_i}})(x + y) = \bigvee_{i \in I} \widetilde{\mu_{F_i}}(x + y) \geq \bigvee_{i \in I} (\widetilde{\mu_{F_i}}(x) \wedge \widetilde{\mu_{F_i}}(y) \wedge [0.5, 0.5])$$

$$= (\bigcup_{i \in I} \widetilde{\mu_{F_i}})(x) \wedge (\bigcup_{i \in I} \widetilde{\mu_{F_i}})(y) \wedge [0.5, 0.5].$$

It is clear that

$$\bigvee_{i \in I} (\widetilde{\mu_{F_i}}(x) \wedge \widetilde{\mu_{F_i}}(y) \wedge [0.5, 0.5]) \leq (\bigcup_{i \in I} \widetilde{\mu_{F_i}})(x) \wedge (\bigcup_{i \in I} \widetilde{\mu_{F_i}})(y) \wedge [0.5, 0.5].$$

Assume that

$$\bigvee_{i\in I}(\widetilde{\mu_{F_i}}(x)\wedge\widetilde{\mu_{F_i}}(y)\wedge[0.5,0.5])\neq(\bigcup_{i\in I}\widetilde{\mu_{F_i}})(x)\wedge(\bigcup_{i\in I}\widetilde{\mu_{F_i}})(y)\wedge[0.5,0.5].$$

Then there exists $\widetilde{r}$ such that

$$\bigvee_{i\in I}(\widetilde{\mu_{F_i}}(x)\wedge\widetilde{\mu_{F_i}}(y)\wedge[0.5,0.5])<\widetilde{r}<(\bigcup_{i\in I}\widetilde{\mu_{F_i}})(x)\wedge(\bigcup_{i\in I}\widetilde{\mu_{F_i}})(y)\wedge[0.5,0.5].$$

Since $F_i\subseteq F_j$ or $F_j\subseteq F_i$ for all $i,j\in I$, there exists $k\in I$ such that $\widetilde{r}<\widetilde{\mu_{F_k}}(x)\wedge\widetilde{\mu_{F_k}}(y)\wedge[0.5,0.5]$. On the other hand, $\widetilde{\mu_{F_i}}(x)\wedge\widetilde{\mu_{F_i}}(y)\wedge[0.5,0.5]<\widetilde{r}$ for all $i\in I$, which is a contradiction. Hence

$$\bigvee_{i\in I}(\widetilde{\mu_{F_i}}(x)\wedge\widetilde{\mu_{F_i}}(y)\wedge[0.5,0.5])=(\bigcup_{i\in I}\widetilde{\mu_{F_i}})(x)\wedge(\bigcup_{i\in I}\widetilde{\mu_{F_i}})(y)\wedge[0.5,0.5].$$

Similarly, if $a,x\in R$ we can show that

$$(\bigcup_{i\in I}\widetilde{\mu_{F_i}})(ax)\geq(\bigcup_{i\in I}\widetilde{\mu_{F_i}})(x)\wedge[0.5,0.5],$$

$$(\bigcup_{i\in I}\widetilde{\mu_{F_i}})(xa)\geq(\bigcup_{i\in I}\widetilde{\mu_{F_i}})(x)\wedge[0.5,0.5]$$

and

$$(\bigcup_{i\in I}\widetilde{\mu_{F_i}})(x)\geq(\bigcup_{i\in I}\widetilde{\mu_{F_i}})(x+a)\wedge(\bigcup_{i\in I}\widetilde{\mu_{F_i}})(a)\wedge[0.5,0.5].$$

Therefore $\bigcup_{i\in I}F_i$ is an interval-valued $(\in,\in\vee q)$-fuzzy k-ideal of R. ∎

Definition 4.3. *Let $f:X\longrightarrow Y$ be a mapping and F and G be interval-valued fuzzy sets of X and Y, respectively. Then $f(F)$ is an interval-valued fuzzy set of Y defined by*

$$f(\widetilde{\mu_F})(y)=\begin{cases}\bigvee_{z\in f^{-1}(y)}\widetilde{\mu_F}(z) & \text{if } f^{-1}(y)\neq\varnothing,\\ [0,0] & \text{otherwise}\end{cases}$$

for all $y\in Y$. Also the inverse image $f^{-1}(G)$ is an interval-valued fuzzy set of X defined by $f^{-1}(\widetilde{\mu_G})(x)=\widetilde{\mu_G}(f(x))$ for all $x\in X$.

Theorem 4.7. *Let R_1 and R_2 be two semirings and $f:R_1\longrightarrow R_2$ be an on-to homomorphism.*

(i) *If F is an interval-valued $(\in,\in\vee q)$-fuzzy k-ideal of R_1, then $f(F)$ is an interval-valued $(\in,\in\vee q)$-fuzzy k-ideal of R_2.*

(ii) *If G is an interval-valued $(\in,\in\vee q)$-fuzzy k-ideal of R_2, then $f^{-1}(G)$ is an interval-valued $(\in,\in\vee q)$-fuzzy k-ideal of R_1.*

Proof.

(i) Let F be an interval-valued $(\in,\in\vee q)$-fuzzy k-ideal of R_1 and $[0,0]<\widetilde{t}\leq[0.5,0.5]$. By Theorem 4.3, $F_{\widetilde{t}}$ is a k-ideal of R_1. Therefore, by Lemma 2.1, $f(F_{\widetilde{t}})$ is a k-ideal of R_2. But $(f(F))_{\widetilde{t}}=f(F_{\widetilde{t}})$, so the interval-valued level subset $(f(F))_{\widetilde{t}}$ is a k-ideal of R_2. Therefore $f(F)$ is an interval-valued $(\in,\in\vee q)$-fuzzy k-ideal of R_2.

(ii) For any $x, y \in R_1$, we have

$$f^{-1}(\widetilde{\mu_G})(x+y) = \widetilde{\mu_G}(f(x+y)) = \widetilde{\mu_G}(f(x) + f(y))$$
$$\geq \widetilde{\mu_G}(f(x)) \wedge \widetilde{\mu_G}(f(y)) \wedge [0.5, 0.5]$$
$$= f^{-1}(\widetilde{\mu_G})(x) \wedge f^{-1}(\widetilde{\mu_G})(y) \wedge [0.5, 0.5].$$

Similarly, if $a, x \in R_1$, we can prove that

$$f^{-1}(\widetilde{\mu_G})(ax) \geq f^{-1}(\widetilde{\mu_G})(x) \wedge [0.5, 0.5],$$
$$f^{-1}(\widetilde{\mu_G})(xa) \geq f^{-1}(\widetilde{\mu_G})(x) \wedge [0.5, 0.5]$$

and

$$f^{-1}(\widetilde{\mu_G})(x) \geq f^{-1}(\widetilde{\mu_G})(x+a) \wedge f^{-1}(\widetilde{\mu_G})(a) \wedge [0.5, 0.5].$$

Therefore $f^{-1}(\widetilde{\mu_G})$ is an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R_1. ∎

By Theorem 4.1, it is well known that an interval-valued fuzzy set F of R is an interval-valued fuzzy k-ideal if and only if $F_{\widetilde{t}}(\neq \varnothing)$ is a k-ideal of R for all $[0,0] < \widetilde{t} \leq [1,1]$. In Theorem 4.3, we prove that F is an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R if and only if the set $F_{\widetilde{t}}(\neq \varnothing)$ is a k-ideal of R for all $[0,0] < \widetilde{t} \leq [0.5, 0.5]$. Naturally, a corresponding result should be considered when $F_{\widetilde{t}}$ is a k-ideal of R for all $[0.5, 0.5] < \widetilde{t} \leq [1,1]$.

Definition 4.4. *An interval-valued fuzzy set F of R is said to be an interval-valued $(\overline{\in}, \overline{\in \wedge q})$-fuzzy k-ideal if for all $[0,0] < \widetilde{t}, \widetilde{r} \leq [1,1]$ and $a, x, y \in R$ the following conditions hold:*

(I7) $(x+y)_{\widetilde{t} \wedge \widetilde{r}} \overline{\in} F$ *implies* $x_{\widetilde{t}} \overline{\in \wedge q} F$ *or* $y_{\widetilde{r}} \overline{\in \wedge q} F$,
(II7) $(ax)_{\widetilde{t}} \overline{\in} F$ *or* $(xa)_{\widetilde{t}} \overline{\in} F$ *implies* $x_{\widetilde{t}} \overline{\in \wedge q} F$,
(III7) $x_{\widetilde{t} \wedge \widetilde{r}} \overline{\in} F$ *implies* $(x+a)_{\widetilde{t}} \overline{\in \wedge q} F$ *or* $a_{\widetilde{r}} \overline{\in \wedge q} F$.

Example 4.4. On a four element semiring $R = \{0, 1, 2, 3\}$ defined by the following two tables:

+	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	2

.	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	1
3	0	1	1	1

Consider an interval-valued fuzzy set F defined by $\widetilde{\mu_F}(0) = [0.4, 0.5]$ and $\widetilde{\mu_F}(x) = [0.2, 0.3]$ for all $x \neq 0$. Then it is easy to see that F is an interval-valued $(\overline{\in}, \overline{\in \wedge q})$-fuzzy k-ideal of $(R, +, .)$.

Theorem 4.8. *Let F be an interval-valued fuzzy set of R. Then F is an interval-valued $(\overline{\in}, \overline{\in \wedge q})$-fuzzy k-ideal of R if and only if for all $a, x, y \in R$ the following conditions hold:*

(I8) $\widetilde{\mu_F}(x+y) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y)$,
(II8) $\widetilde{\mu_F}(ax) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x)$ *and* $\widetilde{\mu_F}(xa) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x)$,
(III8) $\widetilde{\mu_F}(x) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a)$.

Proof. Let F be an interval-valued $(\overline{\in}, \overline{\in \wedge q})$-fuzzy k-ideal of R. If there exist $x, y \in R$ such that $\widetilde{\mu_F}(x+y) \vee [0.5, 0.5] < \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) = \widetilde{t}$, then $[0.5, 0.5] < \widetilde{t} \leq [1, 1]$, $(x+y)_{\widetilde{t}}\overline{\in}F$, $x_{\widetilde{t}} \in F$ and $y_{\widetilde{t}} \in F$. It follows that $x_{\widetilde{t}}\overline{q}F$ or $y_{\widetilde{t}}\overline{q}F$. Then $\widetilde{\mu_F}(x) + \widetilde{t} \leq [1, 1]$ or $\widetilde{\mu_F}(y) + \widetilde{t} \leq [1, 1]$. It follows that $\widetilde{t} \leq [0.5, 0.5]$, which is a contradiction. Hence (I8) holds. Similarly we can show that (II8) holds. Also if there exist $a, x \in R$ such that $\widetilde{\mu_F}(x) \vee [0.5, 0.5] < \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a) = \widetilde{t'}$ then $[0.5, 0.5] < \widetilde{t'} \leq [1, 1]$, $x_{\widetilde{t'}}\overline{\in}F$, $(x+a)_{\widetilde{t'}} \in F$ and $a_{\widetilde{t'}} \in F$. It follows that $(x+a)_{\widetilde{t'}}\overline{q}F$ or $a_{\widetilde{t'}}\overline{q}F$. Then $\widetilde{\mu_F}(x+a) + \widetilde{t'} \leq [1, 1]$ or $\widetilde{\mu_F}(a) + \widetilde{t'} \leq [1, 1]$. It follows that $\widetilde{t'} \leq [0.5, 0.5]$, which is a contradiction. Hence (III8) holds. Conversely, let (I8), (II8) and (III8) hold. Also let $x, y \in R$ such that $(x+y)_{\widetilde{t} \wedge \widetilde{r}}\overline{\in}F$, then $\widetilde{\mu_F}(x+y) < \widetilde{t} \wedge \widetilde{r}$. Then we have the following cases:

(a) If $\widetilde{\mu_F}(x+y) \geq \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y)$, then $\widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) < \widetilde{t} \wedge \widetilde{r}$, and so $\widetilde{\mu_F}(x) < \widetilde{t}$ or $\widetilde{\mu_F}(y) < \widetilde{r}$. It follows that $x_{\widetilde{t}}\overline{\in}F$ or $y_{\widetilde{r}}\overline{\in}F$, which implies that $x_{\widetilde{t}}\overline{\in \wedge q}F$ or $y_{\widetilde{r}}\overline{\in \wedge q}F$.

(b) If $\widetilde{\mu_F}(x+y) < \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y)$, then we have $[0.5, 0.5] \geq \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y)$ (since $\widetilde{\mu_F}(x+y) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y)$). Now if $x_{\widetilde{t}} \in F$ and $y_{\widetilde{r}} \in F$, then $\widetilde{t} \leq \widetilde{\mu_F}(x) \leq [0.5, 0.5]$ or $\widetilde{r} \leq \widetilde{\mu_F}(y) \leq [0.5, 0.5]$. It follows that $x_{\widetilde{t}}\overline{q}F$ or $y_{\widetilde{r}}\overline{q}F$, which implies that $x_{\widetilde{t}}\overline{\in \wedge q}F$ or $y_{\widetilde{r}}\overline{\in \wedge q}F$. Similarly, if $a, x, y \in R$, then $(ax)_{\widetilde{t}}\overline{\in}F$ or $(xa)_{\widetilde{t}}\overline{\in}F$ implies that $x_{\widetilde{t}}\overline{\in \wedge q}F$. Also let $a, x \in R$ such that $x_{\widetilde{t} \wedge \widetilde{r}}\overline{\in}F$, then $\widetilde{\mu_F}(x) < \widetilde{t} \wedge \widetilde{r}$.

Then we have the following cases:

(a) If $\widetilde{\mu_F}(x) \geq \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a)$, then $\widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a) < \widetilde{t} \wedge \widetilde{r}$, and so $\widetilde{\mu_F}(x+a) < \widetilde{t}$ or $\widetilde{\mu_F}(a) < \widetilde{r}$. It follows that $(x+a)_{\widetilde{t}}\overline{\in}F$ or $a_{\widetilde{r}}\overline{\in}F$ which implies that $(x+a)_{\widetilde{t}}\overline{\in \wedge q}F$ or $a_{\widetilde{r}}\overline{\in \wedge q}F$.

(b) If $\widetilde{\mu_F}(x) < \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a)$, then we have $[0.5, 0.5] \geq \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a)$ (since $\widetilde{\mu_F}(x) \vee [0.5, 0.5] \geq \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a)$). Now if $(x+a)_{\widetilde{t}} \in F$ and $a_{\widetilde{r}} \in F$, then $\widetilde{t} \leq \widetilde{\mu_F}(x+a) \leq [0.5, 0.5]$ or $\widetilde{r} \leq \widetilde{\mu_F}(a) \leq [0.5, 0.5]$. It follows that $(x+a)_{\widetilde{t}}\overline{q}F$ or $a_{\widetilde{r}}\overline{q}F$, which implies that $(x+a)_{\widetilde{t}}\overline{\in \wedge q}F$ or $a_{\widetilde{r}}\overline{\in \wedge q}F$. Therefore F is an interval-valued $(\overline{\in}, \overline{\in \wedge q})$-fuzzy k-ideal of R. $\blacksquare$

Corollary 4.2. *An interval-valued fuzzy set F of R is an $(\overline{\in}, \overline{\in \wedge q})$-fuzzy k- ideal of R if and only if $F_{\widetilde{t}}(\neq \varnothing)$ is a k-ideal of R for all $[0.5, 0.5] < \widetilde{t} \leq [1, 1]$.*

Proof. The result is immediately followed by Theorems 4.4 and 4.8. $\blacksquare$

In [21], Yuan et al. gave the definition of fuzzy subgroup with thresholds which is a generalization of the fuzzy subgroup of Rosenfeld and also the fuzzy subgroup of Bhakat and Das. Based on [21], we can extend the concept of a fuzzy subgroup with thresholds to the concept of interval-valued fuzzy k-ideal with thresholds in the following way.

Definition 4.5. *Let $[0, 0] \leq \widetilde{s} < \widetilde{t} \leq [1, 1]$. Then an interval-valued fuzzy set F of R is called an interval-valued fuzzy k-ideal with thresholds $(\widetilde{s}, \widetilde{t})$ of R if for all $a, x, y \in R$, the following conditions hold:*

(I9) $\widetilde{\mu_F}(x+y) \vee \widetilde{s} \geq \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) \wedge \widetilde{t}$,

(II9) $\widetilde{\mu_F}(ax) \vee \widetilde{s} \geq \widetilde{\mu_F}(x) \wedge \widetilde{t}$ and $\widetilde{\mu_F}(xa) \vee \widetilde{s} \geq \widetilde{\mu_F}(x) \wedge \widetilde{t}$,

(III9) $\widetilde{\mu_F}(x) \vee \widetilde{s} \geq \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a) \wedge \widetilde{t}$.

Remark 4.1. If F is an interval-valued fuzzy k-ideal with thresholds of R, then we can conclude that F is an ordinary interval-valued fuzzy k-ideal when $\widetilde{s} = [0,0]$ and $\widetilde{t} = [1,1]$. Also F is an interval-valued $(\in, \in \vee q)$-fuzzy (resp. $(\overline{\in}, \overline{\in} \wedge q)$-fuzzy) k-ideal of R when $\widetilde{s} = [0,0]$ and $\widetilde{t} = [0.5, 0.5]$ (resp. $\widetilde{s} = [0.5, 0.5]$ and $\widetilde{t} = [1,1]$).

Theorem 4.9. *An interval-valued fuzzy set F of R is an interval-valued fuzzy k-ideal with thresholds $(\widetilde{s}, \widetilde{t})$ of R if and only if $F_{\widetilde{\alpha}}(\neq \varnothing)$ is a k-ideal of R for all $\widetilde{s} < \widetilde{\alpha} \leq \widetilde{t}$.*

Proof. Let F be an interval-valued fuzzy k-ideal with thresholds $(\widetilde{s}, \widetilde{t})$ of R and $\widetilde{s} < \widetilde{\alpha} \leq \widetilde{t}$. Let $x, y \in F_{\widetilde{\alpha}}$, then $\widetilde{\mu_F}(x) \geq \widetilde{\alpha}$ and $\widetilde{\mu_F}(y) \geq \widetilde{\alpha}$. Now we have

$$\widetilde{\mu_F}(x+y) \vee \widetilde{s} \geq \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) \wedge \widetilde{t} \geq \widetilde{\alpha} \wedge \widetilde{t} \geq \widetilde{\alpha} > \widetilde{s},$$

which implies that $\widetilde{\mu_F}(x+y) \geq \widetilde{\alpha}$, and so $x + y \in F_{\widetilde{\alpha}}$. Similarly, we can prove that $ax \in F_{\widetilde{\alpha}}$ and $xa \in F_{\widetilde{\alpha}}$ for all $x \in F_{\widetilde{\alpha}}$ and $a \in R$. Also let $x + b, b \in F_{\widetilde{\alpha}}$, then $\widetilde{\mu_F}(x+b) \geq \widetilde{\alpha}$ and $\widetilde{\mu_F}(b) \geq \widetilde{\alpha}$. Now we have

$$\widetilde{\mu_F}(x) \vee \widetilde{s} \geq \widetilde{\mu_F}(x+b) \wedge \widetilde{\mu_F}(b) \wedge \widetilde{t} \geq \widetilde{\alpha} \wedge \widetilde{t} \geq \widetilde{\alpha} > \widetilde{s},$$

which implies that $\widetilde{\mu_F}(x) \geq \widetilde{\alpha}$, and so $x \in F_{\widetilde{\alpha}}$. Therefore $F_{\widetilde{\alpha}}$ is a k-ideal of R. Conversely, let F be an interval-valued fuzzy set of R such that $F_{\widetilde{\alpha}}(\neq \varnothing)$ is a k-ideal of R for all $\widetilde{s} < \widetilde{\alpha} \leq \widetilde{t}$. If there exist $x, y \in R$ such that $\widetilde{\mu_F}(x+y) \vee \widetilde{s} < \widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y) \wedge \widetilde{t} = \widetilde{\alpha}$, then $\widetilde{s} < \widetilde{\alpha} \leq \widetilde{t}$, $\widetilde{\mu_F}(x+y) < \widetilde{\alpha}$ and $x, y \in F_{\widetilde{\alpha}}$. So we have $x + y \in F_{\widetilde{\alpha}}$, which implies that $\widetilde{\mu_F}(x+y) \geq \widetilde{\alpha}$. This contradicts $\widetilde{\mu_F}(x+y) < \widetilde{\alpha}$. Similarly we can show that $\widetilde{\mu_F}(ax) \vee \widetilde{s} \geq \widetilde{\mu_F}(x) \wedge \widetilde{t}$, and $\widetilde{\mu_F}(xa) \vee \widetilde{s} \geq \widetilde{\mu_F}(x) \wedge \widetilde{t}$. Also if there exist $a, x \in R$ such that $\widetilde{\mu_F}(x) \wedge \widetilde{s} < \widetilde{\mu_F}(x+a) \wedge \widetilde{\mu_F}(a) \wedge \widetilde{t} = \widetilde{\alpha}'$, then $\widetilde{s} < \widetilde{\alpha}' \leq \widetilde{t}$, $\widetilde{\mu_F}(x) < \widetilde{\alpha}'$ and $a, x + a \in F_{\widetilde{\alpha}'}$. So we have $x \in F_{\widetilde{\alpha}'}$, which implies that $\widetilde{\mu_F}(x) \geq \widetilde{\alpha}'$. This is a contradiction. Therefore F is an interval-valued fuzzy k-ideal with thresholds $(\widetilde{s}, \widetilde{t})$ of R. ∎

Note that similar to the Theorem 4.7, the homomorphic properties of the interval-valued fuzzy k-ideals with thresholds can be investigated. Also, however interval-valued fuzzy k-ideals and interval-valued $(\in, \in \vee q)$-fuzzy k-ideals are interval-valued fuzzy k-ideals with thresholds, the following example shows that an interval-valued fuzzy k-ideal with thresholds can not be an interval-valued fuzzy k-ideal and can not be an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal.

Example 4.5. On a four element semiring $R = \{0, 1, 2, 3\}$ defined by the following two tables:

+	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	2

.	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	1
3	0	1	1	1

Define $\widetilde{\mu_F}(0) = [0.1, 0.2]$ and $\widetilde{\mu_F}(x) = [0.2, 0.3]$ for all $x \neq 0$. It is easy to verify that F is an interval-valued fuzzy k-ideal with thresholds $(\widetilde{s} = [0.4, 0.5], \widetilde{t} = [0.9, 1])$ of R. In other hand F is not an interval-valued $(\in, \in \vee q)$-fuzzy k-ideal of R (and consequently

F is not an interval-valued fuzzy k-ideal of R). Because $[0,0] < [0.2, 0.3] \leq [0.5, 0.5]$ and $F_{[0.2,0.3]} = \{1,2,3\}$ is not a k-ideal of R.

5. Implication-based interval-valued fuzzy k-ideals

Set theoretic multi-valued logic is a special case of fuzzy logic such that the truth values are linguistic variables (or terms of the linguistic variables truth). By using extension principle some operator like $\wedge$, $\vee$, $\neg$, $\rightarrow$ can be applied in fuzzy logic. In fuzzy logic, $[P]$ means the truth value of fuzzy proposition P. In the following, we show a correspondence between fuzzy logic and set-theoretical notions.

$$[x \in F] = \widetilde{\mu_F}(x),$$
$$[x \notin F] = [1,1] - \widetilde{\mu_F}(x),$$
$$[P \wedge Q] = \min\{[P], [Q]\},$$
$$[P \vee Q] = \max\{[P], [Q]\},$$
$$[P \rightarrow Q] = \min\{[1,1], [1,1] - [P] + [Q]\},$$
$$[\forall x \ P(x)] = \inf[P(x)],$$
$$\models P \ \text{if and only if} \ [P] = [1,1] \ \text{for all valuations.}$$

We show some of important implication operators, where α denotes the degree of membership of the premise and β is the degree of membership of the consequence and I the resulting degree of truth for the implication.

Early Zadeh $I_m(\alpha, \beta) = \max\{1 - \alpha, \min\{\alpha, \beta\}\},$

Lukasiewicz $I_a(\alpha, \beta) = \min\{1, 1 - \alpha + \beta\},$

Standard Star (Godel) $I_g(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta & \text{otherwise} \end{cases}$

Contraposition of Godel $I_{cg}(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ 1 - \alpha & \text{otherwise} \end{cases}$

Gaines-Rescher $I_{gr}(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \leq \beta \\ 0 & \text{otherwise} \end{cases}$

Kleene-Dienes $I_b(\alpha, \beta) = \max\{1 - \alpha, \beta\}.$

Definition 5.1. *An interval-valued fuzzy set F of R is called an interval-valued fuzzifying k-ideal of R if it satisfies the following conditions:*

(I10) *for any $x, y \in R$, $\models [\{[x \in F] \wedge [y \in F]\} \longrightarrow [x + y \in F]]$,*

(II10) *for any $x, a \in R$, $\models [\{[x \in F] \longrightarrow [ax \in F]]$ and $\models [\{[x \in F] \longrightarrow [xa \in F]]$,*

(III10) *for any $x, b \in R$, $\models [\{[x + b \in F] \wedge [b \in F]\} \longrightarrow [x \in F]]$.*

Obviously the Definition 5.1 and 4.1 are equivalent. Therefore there is no difference between interval-valued fuzzifying k-ideals and ordinary interval-valued fuzzy k-ideals.

Now, we have the concept of interval-valued $\widetilde{t}$-tautology, in fact $\models_{\widetilde{t}} P$ if and only if $[P] \geq \widetilde{t}$, for all valuations. Based on [23], we can extend the concept of implication-based fuzzy k-ideals.

Definition 5.2. *Let F be an interval-valued fuzzy set of R and $[0,0] < \widetilde{t} \leq [1,1]$. Then F is called a $\widetilde{t}$-implication-based interval-valued fuzzy k-ideal of R if it satisfies the following conditions:*

(I11) *for any $x, y \in R$, $\models_{\widetilde{t}} [\{[x \in F] \wedge [y \in F]\} \longrightarrow [x + y \in F]]$,*

(II11) *for any $x, a \in R$, $\models_{\widetilde{t}} [[x \in F] \longrightarrow [ax \in F]]$ and $\models_{\widetilde{t}} [[x \in F] \longrightarrow [xa \in F]]$,*

(III11) *for any $x, b \in R$, $\models_{\widetilde{t}} [\{[x + b \in F] \wedge [b \in F]\} \longrightarrow [x \in F]]$.*

Corollary 5.1. *An interval-valued fuzzy set F of R is a $\widetilde{t}$-implication-based interval-valued fuzzy k-ideal of R if for all $a, x, y \in R$ we have:*

(I12) $I(\widetilde{\mu_F}(x) \wedge \widetilde{\mu_F}(y), \widetilde{\mu_F}(x + y)) \geq \widetilde{t}$,

(II12) $I(\widetilde{\mu_F}(x), \widetilde{\mu_F}(ax)) \geq \widetilde{t}$, and $I(\widetilde{\mu_F}(x), \widetilde{\mu_F}(xa)) \geq \widetilde{t}$,

(III12) $I(\widetilde{\mu_F}(x + a) \wedge \widetilde{\mu_F}(a), \widetilde{\mu_F}(x)) \geq \widetilde{t}$.

where I is an implication operator.

Proof. The result is clear by considering the definitions. ∎

Theorem 5.1.

(i) *Let $I = I_{gr}$ (Gaines-Rescher.) Then F is an $[0.5, 0.5]$-implication-based interval valued fuzzy k-ideal of R if and only if F is an interval valued fuzzy k-ideal with thresholds $(\widetilde{r} = [0, 0], \widetilde{s} = [1, 1])$ of R.*

(ii) *Let $I = I_g$ (Godel). Then F is an $[0.5, 0.5]$-implication-based interval-valued fuzzy k-ideal of R if and only if F is an interval-valued fuzzy k-ideal with thresholds $(\widetilde{r} = [0, 0], \widetilde{s} = [0.5, 0.5])$ of R.*

(iii) *Let $I = I_{cg}$ (Contraposition of Godel). Then F is an $[0.5, 0.5]$-implication-based interval-valued fuzzy k-ideal of R if and only if F is an interval-valued fuzzy k-ideal with thresholds $(\widetilde{r} = [0.5, 0.5], \widetilde{s} = [1, 1])$ of R.*

Proof. The proof is straightforward by considering the definitions. ∎

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